

Robust Finite-time Control of Robot Manipulators

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Abstract: This paper considers finite-time position control of robot manipulators. The robot manipulators are modeled by discontinuous differential equations. In this paper, we prove that the Nakamura's local homogeneous controller based on a control Lyapunov function is valid to the position control of the robot manipulators, and show the effectiveness of the controller by experiments. Moreover, we compare the controller with other nonlinear controllers and show advantages of the controller.

Keywords: finite-time stability, robot control, discontinuous nonlinear system, local homogeneity, control Lyapunov function.

1. INTRODUCTION

Finite-time control [7] attracts much attention in recent years [9], [10]. This paper considers finite-time position control of robot manipulators. The robot manipulators are modeled by discontinuous differential equations [4].

Nakamura et al. proposed a robust controller with a sector margin based on a control Lyapunov function [13]. The controller guarantees local convergence rates by utilizing local homogeneity, and becomes a finite-time controller in some cases. However, the method requires continuity of the control system.

In this paper, we prove that the controller is valid to position control of the robot manipulators, and show the effectiveness of the controller by experiments. Moreover, we compare the controller with other nonlinear controllers and show advantages of the controller.

2. PROBLEM STATEMENT

A robot manipulator (see Fig. 1) is modeled by the following equation of motion of an one-link robot manipulator (see Fig. 2):

$$J\ddot{\theta} = mag \sin(\theta) - D\dot{\theta} - f \operatorname{sgn}(\dot{\theta}) - F\delta(\dot{\theta}) \operatorname{sgn}(u) + u, \quad (1)$$

where θ , $\dot{\theta}$ and $\ddot{\theta}$ denote an angle of the joint, angular velocity and angular acceleration, respectively. Table 1 represents meanings of the parameters. A function sgn is defined as follows: if $\forall x \in \mathbb{R} \setminus \{0\}$, $\operatorname{sgn}(x) := x/|x|$ and $\operatorname{sgn}(0) := 0$. A function δ is defined as follows: if $\forall x \in \mathbb{R} \setminus \{0\}$, $\delta(x) := 0$ and $\delta(0) := 1$.

Let state $x = (x_1, x_2)^T := (\theta, \dot{\theta})^T$. Then, we can obtain the following input affine nonlinear system with input u :

$$\Sigma : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{J} \left(mag \sin x_1 - Dx_2 - f \operatorname{sgn}(x_2) - F\delta(x_2) \operatorname{sgn}(u) \right) + \frac{u}{J}. \end{cases} \quad (2)$$

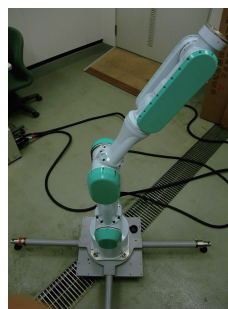


Fig. 1. robot manipulator PA-10

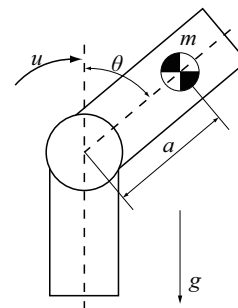


Fig. 2. model of one-link robot manipulator

Table 1. meanings of parameters of robot manipulator

J	inertial moment of the joint
m	mass
a	distance from the joint to the center of the gravity
g	gravitational acceleration
D	coefficient of viscosity
f	coefficient of Coulomb friction
F	coefficient of maximum static friction
u	input torque
I	inertial moment of the center of the gravity

The problem of the paper is robust finite-time stabilization at the origin of (2).

3. PRELIMINARIES

We show definitions and previous results, in this section.

3.1 Autonomous system

Definition 1. (stability[2]). Consider the following differential equation:

$$\dot{x} = f(x), \quad (3)$$

where $x \in \mathbb{R}^n$ is a state, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is measurable mapping, and $f(0) = 0$.

The origin of system (3) is said to be

- (1) stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq 0$;
- (2) globally asymptotically stable if it is stable and all solutions $x(t)$ satisfy $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$.

Definition 2. (convergence rate [1, 6]). The origin of system (3) is said to be

- (1) finite-time stable if it is stable and there exists a function $T : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$ such that the following statements hold:
 - (a) for every $x_0 \in \mathbb{R}^n \setminus \{0\}$, the solution $x(t) \in \mathbb{R}^n \setminus \{0\}$ with $x(0) = x_0$ is defined on $[0, T(x_0))$, $\forall t \in [0, T(x_0))$;
 - (b) $\lim_{t \rightarrow T(x_0)} x(t) = 0$;
- (2) exponentially stable if there exist positive constants δ, b_1 and b_2 such that for any $x_0 \in \mathbb{R}^n \setminus \{0\}$, the solution $x(t)$ with $x(0) = x_0$ is defined on $[0, +\infty)$ and satisfies

$$\|x(t)\| \leq b_1 e^{b_2 t} \|x_0\|, \forall t \geq 0;$$

- (3) rational stable if there exist positive constants $b_1, b_2 > 0$ and $0 < \eta \leq 1$ such that for any $x_0 \in \mathbb{R}^n \setminus \{0\}$, the solution $x(t)$ with $x(0) = x_0$ is defined on $[0, +\infty)$ and satisfies

$$\|x(t)\| \leq b_1 (1 + \|x_0\|^{b_2 t})^{-1/b_2} \|x_0\|^\eta, \forall t \geq 0.$$

3.2 Control system

Definition 3. (control Lyapunov function [16]). Consider the following input-affine nonlinear system:

$$\dot{x} = f(x) + g(x)u, \quad (4)$$

where $x \in \mathbb{R}^n$ is a state, $u \in \mathbb{R}^m$ is an input, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ are measurable mappings, and $f(0) = 0$. Then, a C^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a global control Lyapunov function (CLF) for system (4) such that the following properties hold:

- (1) V is proper, that is, $\{x \in \mathbb{R}^n | V(x) \leq L\}$ is compact for each $L > 0$;
- (2) V is positive definite on \mathbb{R}^n :
 $V(0) = 0$, and $V(x) > 0$ for each $x \in \mathbb{R}^n \setminus \{0\}$;
- (3) $\inf_{u \in \mathbb{R}^m} \{L_f V + L_g V \cdot u\} < 0 (\forall x \in \mathbb{R}^n \setminus \{0\})$, where $L_f V := \partial V / \partial x \cdot f(x)$ and $L_g V := \partial V / \partial x \cdot g(x)$.

Remark 4. Note that a C^1 proper positive-definite function $V : \mathbb{R}^n \rightarrow [0, +\infty)$ is a CLF if and only if the following implication holds:

$$\forall x \in \mathbb{R}^n \setminus \{0\}, L_g V = 0 \Rightarrow L_f V < 0.$$

Definition 5. (sector margin [15]). A continuous mapping $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is said to be a sector nonlinearity in $[\alpha, \beta]$ if $\phi(0) = 0$ and $\alpha u^2 \leq u\phi(u) < \beta u^2 (\forall u \neq 0)$. A state feedback controller $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ for (4) is said to have a sector margin $[\alpha, \beta]$ if the origin of the closed-loop system $\dot{x} = f(x) + g(x)\phi(u(x))$ is globally asymptotically stable, where $\phi(u) := (\phi_1(u_1), \dots, \phi_m(u_m))^T$ and each $\phi_j(u_j)$ is an arbitrary sector nonlinearity in $[\alpha, \beta]$.

3.3 Homogeneous system

Definition 6. (homogeneous system [14]). A mapping $\Delta_\epsilon^r x := (\epsilon^{r_1} x_1, \dots, \epsilon^{r_n} x_n)^T (\forall \epsilon > 0, \forall x \in \mathbb{R}^n \setminus \{0\})$ is said to be a dilation on \mathbb{R}^n , where $r = (r_1, \dots, r_n)$ is a constant

vector satisfying $0 < r_i < +\infty (i = 1, \dots, n)$. A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be homogeneous of degree $k \in \mathbb{R}$ with respect to a dilation $\Delta_\epsilon^r x$ if $V(\Delta_\epsilon^r x) = \epsilon^k V(x)$. System (4) is said to be homogeneous of degree $\tau \in \mathbb{R}$ with respect to dilations $\Delta_\epsilon^r x$ and $\Delta_\epsilon^s u$ if $f(\Delta_\epsilon^r x) + g(\Delta_\epsilon^r x)\Delta_\epsilon^s u = \epsilon^\tau \Delta_\epsilon^r \{f(x) + g(x)u\}$.

Definition 7. (homogeneous approximation [14]). A homogeneous function $V_h(x)$ of degree $k \in \mathbb{R}$ with respect to a dilation $\Delta_\epsilon^r x$ is said to be a homogeneous approximation of $V(x)$ if there exists $V_o(x)$ satisfying $V(x) = V_h(x) + V_o(x)$ and

$$\lim_{\epsilon \rightarrow 0} \frac{V_o(\Delta_\epsilon^r x)}{\epsilon^k} = 0,$$

uniformly on $S^{n-1} := \{x \in \mathbb{R}^n | \|x\| = 1\}$.

A homogeneous system $\dot{x} = f_h(x) + g_h(x)u$ of degree $\tau \in \mathbb{R}$ with respect to the dilations $\Delta_\epsilon^r x$ and $\Delta_\epsilon^s u$ is said to be a homogeneous approximation of (4) if there exists $f_o(x)$ and $g_o(x)$ satisfying $f(x) + g(x)u = f_h(x) + g_h(x)u + f_o(x) + g_o(x)u$ and

$$\lim_{\epsilon \rightarrow 0} \frac{f_{o,i}(\Delta_\epsilon^r x) + g_{o,i}(\Delta_\epsilon^r x)\Delta_\epsilon^s u}{\epsilon^{\tau+r_i}} = 0,$$

uniformly on $S^{n+m-1} := \{(x, u) \in \mathbb{R}^{n+m} | \|(x, u)\| = 1\}$.

3.4 Previous results

Theorem 8. (Nakamura's finite-time controller [13]). Suppose that system (4) is continuous, and satisfies a condition $k + \tau - \max_{1 \leq j \leq m} s_j > 0$ and the following hypothesis:

- H1) System Σ has a homogeneous approximation Σ_h of degree $\tau < 0$ with respect to $\Delta_\epsilon^r x$ and $\Delta_\epsilon^s u$;
- H2) System Σ has a CLF $V(\cdot)$ such that the homogeneous approximation $V_h(\cdot)$ of degree k with respect to $\Delta_\epsilon^r x$ is a CLF for system Σ_h .

Let $\gamma > 0$ and $0 < \alpha \leq 1$ be arbitrary constants. Then, the following input finite-time stabilizes the origin of (4):

$$u(x) = \begin{cases} -\frac{L_f V(x) + |L_f V(x)| + \gamma |L_g V(x)|^{\frac{\tau+k}{\tau+k-s}}}{2\alpha |L_g V(x)|^{\frac{\tau+k}{\tau+k-s}}} \cdot \text{sgn}(L_g V(x)) & (L_g V(x) \neq 0) \\ 0 & (L_g V(x) = 0) \end{cases} \quad (5)$$

Input (5) is continuous and achieves a sector margin $[\alpha, +\infty)$. \square

Remark 9. The ranges of uncertainties in modeling errors are previously known in many practical situations. Nakamura's method covers such ranges by sector margins.

Remark 10. The following statements are true for the closed-loop system (4) with (5):

- S1) If $\tau = 0$ in H1, the origin is exponentially stable.
- S2) If $\tau > 0$ in H1, the origin is rational stable.

4. VALIDITY OF NAKAMURA'S METHOD

In this section, we confirm that the controller (5) is valid to the discontinuous nonlinear dynamical system (2). In order to show the validity, we prepare the following lemma.

Lemma 11. Let $\tau \in \mathbb{R}$ and $q > |\tau|$ be constants. Then the following system is homogeneous of degree τ with respect to $\Delta_\varepsilon^{(q-\tau, q)}x$ and $\Delta_\varepsilon^{q+\tau}u$:

$$\Sigma_h : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{J}u \end{cases} . \quad (6)$$

□

Proof. $\varepsilon^q x_2 = \varepsilon^\tau (\varepsilon^{q-\tau} x_2) = \varepsilon^\tau (\varepsilon^{q-\tau} \dot{x}_1)$, and $(\varepsilon^{q+\tau} u)/J = \varepsilon^\tau (\varepsilon^q u/J) = \varepsilon^\tau (\varepsilon^q \dot{x}_2)$. ■

Remark 12. Note that homogeneous system (6) is not a homogeneous approximation of (2). The important fact is that (6) can be transformed from (2) by input transformation.

With the homogeneous system (6), we confirm that the controller (5) is valid to (2) as follows:

Lemma 13. Consider system (2), homogeneous system (6) of degree $\tau < 0$ with respect to $\Delta_\varepsilon^{(q-\tau, q)}x$ and $\Delta_\varepsilon^{q+\tau}u$, and a control Lyapunov function (7) for (2) such that the homogeneous approximation $V_h(\cdot)$ of degree k with respect to $\Delta_\varepsilon^{(q-\tau, q)}x$ is a CLF for system (6).

Let $\gamma > 0$ and $0 < \alpha \leq 1$ be constants. Then, input (5) finite-time stabilizes the origin of (2) and achieves a sector margin $[\alpha, +\infty)$. □

Proof. [Proof of Lemma 13] By Lemma 1 in [2], for any $x \in \mathbb{R}^2 \setminus \{0\}$, there exists a Carathéodory solution [5] of the directional continuous map (2). The rest of the proof follows Theorem 8 (Theorem 3 in [13]). ■

Remark 14. Note that the statements in Remark 10 are true. The fact also follows Theorem 3 in [13].

Thus, we confirmed that Nakamura's method is valid to (2).

5. CONTROLLER DESIGN

In this section, we design Nakamura's finite-time controller for system (2). To design the controller, we use a homogeneous system (6) and a CLF.

5.1 CLF

In this subsection, we design a CLF for (2).

Lemma 15. Let $\tau \in \mathbb{R}$, $q > |\tau|$ and $k \geq 2q - \tau$ be constants. Then the following continuous function $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ is C^1 -differentiable, homogeneous of degree k with respect to the dilation $\Delta_\varepsilon^{(q-\tau, q)}x$ and a global control Lyapunov function for (2):

$$V(x) = |x_1|^{\frac{k}{q-\tau}} + b|x_1|^{\frac{k-q}{q-\tau}} \text{sgn}(x_1)x_2 + c|x_2|^{\frac{k}{q}}, \quad (7)$$

where b and c be positive constants satisfying

$$c \left(\frac{k}{q}\right)^{\frac{k}{q}} > b^{\frac{k}{q}} \left(\frac{k-q}{q}\right)^{\frac{k-q}{q}} > 0. \quad (8)$$

Table 2. Parameter Values of PA-10

$J := I + ma^2$ [kg · m ²]	3. 2870
ma [kg · m]	2. 3126
D [N · m · sec]	18. 6916
F [N · m]	24. 2500
f [N · m]	18. 8785

□

We show the proof of Lemma 15 in the appendix.

Remark 16. If $k = 2q - \tau$, $b|x_1|^{(k-q)/(q-\tau)} \text{sgn}(x_1)x_2 = bx_1x_2$. Thus, the second term of the right-side of (7) becomes a real analytic function.

Remark 17. Note that (7) is also a CLF for (6).

5.2 Finite-time controller

Using the CLF (7) with $\tau < 0$, we can design a controller (5) as follows:

$$u(x) = \begin{cases} -\frac{L_f V + |L_f V| + \gamma |L_g V|^{\frac{k+\tau}{k-q}} |L_g V|^{\frac{\tau+q}{k-q}} \text{sgn}(L_g V)}{2\alpha |L_g V|^{\frac{k+\tau}{k-q}}} & (L_g V(x) \neq 0) \\ 0 & (L_g V(x) = 0) \end{cases}, \quad (9)$$

where

$$\begin{aligned} L_f V(x) &= \left(\frac{k}{q-\tau} |x_1|^{\frac{k}{q-\tau}-1} \text{sgn}(x_1) + \frac{k-q}{q-\tau} b |x_1|^{\frac{k-2q+\tau}{q-\tau}} x_2 \right) \\ &\cdot x_2 + \left(b |x_1|^{\frac{k-q}{q-\tau}} \text{sgn}(x_1) + \frac{k}{q} c |x_2|^{\frac{k-q}{q}} \text{sgn}(x_2) \right) \\ &\cdot \left(mag \sin(x_1) - Dx_2 - f \text{sgn}(x_2) - F\delta(x_2) \text{sgn}(u) \right), \\ L_g V(x) &= \left(b |x_1|^{\frac{k-q}{q-\tau}} \text{sgn}(x_1) + \frac{k}{q} c |x_2|^{\frac{k-q}{q}} \text{sgn}(x_2) \right) \frac{1}{J}. \end{aligned}$$

6. EXPERIMENTAL RESULTS

In this section, we confirm the effectiveness of the Nakamura's finite-time controller by three experiments using the robot manipulator PA-10 (see Fig. 1):

- In the first experiment, we confirm that the controller (5) is effective for the robot manipulator.
- In the second experiment, we compare the controller with other convergence rate controllers, and confirm the advantages of the controller.
- In the other experiment, we compare the controller with other finite-time control methods, and confirm the advantages of the Nakamura's method.

6.1 Experiment environment

Table 2 shows the mechanical parameters of PA-10 [12]. PA-10 is operated by PC (Windows XP, Pentium 4 3.0GHz) through ARCNET. The sampling period is 2 [msec].

6.2 Nakamura's finite-time controller

In this subsection, we show an experimental result that the controller (5) stabilizes the origin.

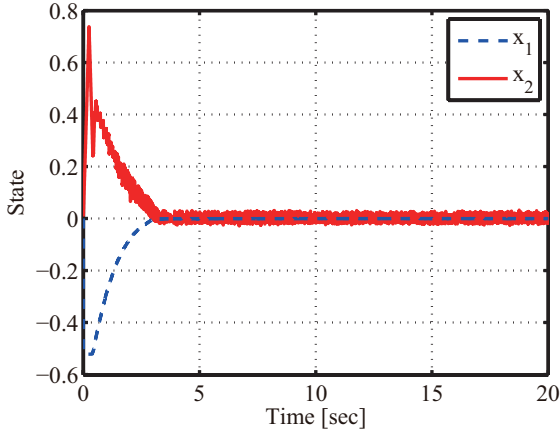


Fig. 3. state of system (2) with Nakamura's finite-time controller (10)

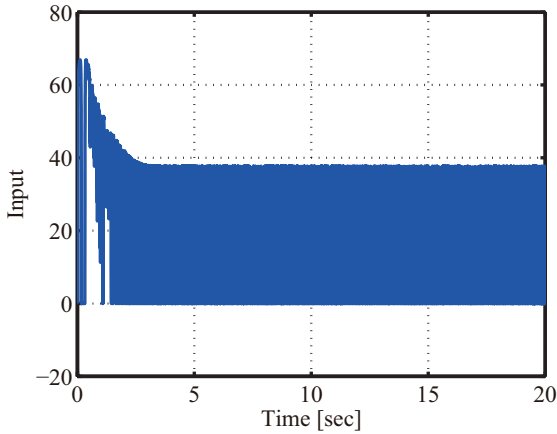


Fig. 4. input of system (2) with Nakamura's finite-time controller (10)

Consider the homogeneous system (6) of degree $\tau = -1/3$ with respect to the dilations $\Delta_\varepsilon^{(1,2/3)}x$ and $\Delta_\varepsilon^{1/3}u$, and the following CLF (7) for system (2):

$$V = x_1^2 + |x_1|^{4/3} \operatorname{sgn}(x_1)x_2 + \frac{1}{2}|x_2|^3.$$

By Theorem 8 and Lemma 13, we obtain the following finite-time controller (9) with the sector margin $[1/2, +\infty)$ for system (2):

$$u(x) = \begin{cases} -\frac{L_f V + |L_f V| + 0.001|L_g V|^{5/4}}{|L_g V|^{5/4}} |L_g V|^{1/4} \operatorname{sgn}(L_g V) & (L_g V(x) \neq 0) \\ 0 & (L_g V(x) = 0) \end{cases} \quad (10)$$

Figures 3 and 4 show the responses of the finite-time controller (10) for $x(0) = (-\pi/3, 0)^T$. By Fig. 3, we can observe that the controller successfully stabilizes the origin.

6.3 Comparison with other convergence rate controllers

In this subsection, we compare the finite-time controller with other convergence rate controllers. We can also design an exponential controller and an rational controller by Nakamura's method.

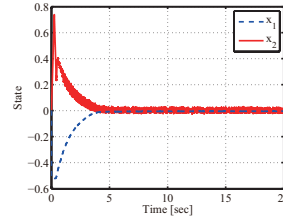


Fig. 5. states of system (2) with exponential controller

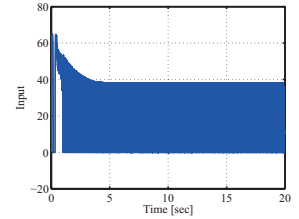


Fig. 6. input of system (2) with exponential controller

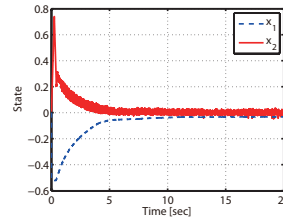


Fig. 7. states of system (2) with rational controller

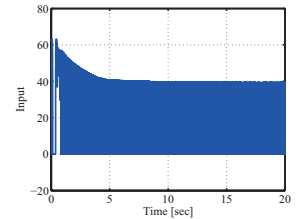


Fig. 8. input of system (2) with rational controller

Exponential controller Consider the homogeneous system (6) of degree $\tau = 0$ with respect to the dilations $\Delta_\varepsilon^{(1,1)}x$ and $\Delta_\varepsilon^1 u$, and the following CLF for system (2):

$$V = x_1^2 + x_1 x_2 + \frac{1}{2} x_2^2. \quad (11)$$

By Theorem 8 and Remark 14, we obtain the exponential controller for (2).

Remark 18. In this case, the CLF (11) coincides with the following CLF for obtained by Backstepping approach [16].

Rational controller Consider the homogeneous system (6) of degree $\tau = 1/3$ with respect to the dilations $\Delta_\varepsilon^{(2/3,1)}x$ and $\Delta_\varepsilon^{4/3}u$, and the following CLF for system (2):

$$V = |x_1|^3 + |x_1|^{3/2} \operatorname{sgn}(x_1)x_2 + \frac{1}{2}x_2^2.$$

By Theorem 8 and Remark 14, we obtain the rational controller for (2).

Experimental results Figures 5 and 6 show the responses of the exponential controller, and Figures 7 and 8 illustrate the responses of the rational controller. By Figs. 5 and 7, we can observe that the both controllers successfully stabilize the origin.

Figure 9 shows the responses of the state x_1 . We can confirm that the finite-time controller achieves much faster convergence than the other controllers.

Table 3 shows the mean errors and error variances of steady-state x from 15 [sec] to 20 [sec]. The mean error of x_1 with the finite-time controller is much smaller than the other controllers. Therefore, we can confirm that the finite-time controller is much higher convergence accuracy than the other convergence rate controllers.

6.4 Comparison with other finite-time control methods

In this subsection, we compare Nakamura's controller with other finite-time controllers proposed in [8, 17].

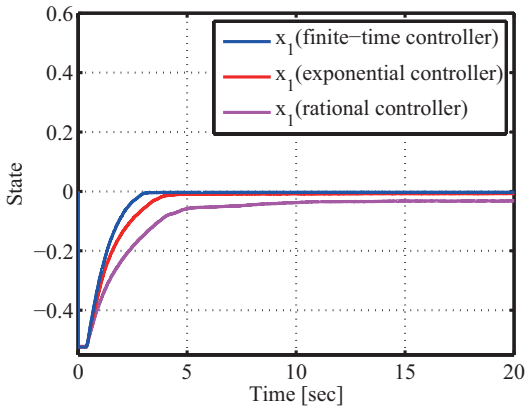


Fig. 9. state x_1 of system (2) with Nakamura's finite-time controller (10), exponential controller and rational controller

Table 3. Averages and variances of residuals (Nakamura's finite-time controller (10), exponential controller and rational controller)

	$E(x_1)$	$\sigma^2(x_1)$
Nakamura's finite-time controller	-3.4×10^{-4}	2.58×10^{-8}
exponential controller	-4.7×10^{-3}	3.87×10^{-8}
rational controller	-3.2×10^{-2}	1.51×10^{-7}
	$E(x_2)$	$\sigma^2(x_2)$
Nakamura's finite-time controller	8.6×10^{-4}	9.53×10^{-5}
exponential controller	1.2×10^{-3}	8.89×10^{-5}
rational controller	8.6×10^{-4}	1.22×10^{-4}

Hong's finite-time controller [8] Hong's controller is obtained as follows:

$$u = -mag \sin(x_1) + Dx_2 + f \operatorname{sgn}(x_2) - F\delta(x_2)\operatorname{sgn}(u) - J[k_1|x_1|^\zeta \operatorname{sgn}(x_1) + k_2|x_2|^{\frac{1}{1+\zeta}} \operatorname{sgn}(x_2)].$$

We choose $k_1 = 50$, $k_2 = 50$ and $\zeta = 0.5$ as control parameters. Note that Hong's method does not guarantee a sector margin.

Yu's finite-time controller [17] Yu's controller is obtained as follows:

$$\begin{cases} u = u_0 + u_1, \\ u_0 = -mag \sin(x_1) + Dx_2 + f \operatorname{sgn}(x_2) - F\delta(x_2)\operatorname{sgn}(u_0) - J\mu^{-1}\zeta^{-1}|x_2|^{2-\zeta} \operatorname{sgn}(x_2), \\ u_1 = -J[k_1s + k_2|s|^\rho \operatorname{sgn}(s)], \\ s = x_1 + \mu|x_2|^\zeta \operatorname{sgn}(x_2). \end{cases}$$

We choose $k_1 = 20$, $k_2 = 20$, $\rho = 1/3$, $\mu = 1$ and $\zeta = 1.5$ as control parameters.

Experimental results Figures 10 and 11 show the responses of the Hong's controller, and Figures 12 and 13 illustrate the responses of the Yu's controller. By Figs. 10 and 12, we can observe that both Hong's and Yu's controllers successfully stabilize the origin.

Table 4 shows the mean errors and the error variances of steady-state from 15 [sec] to 20 [sec]. The mean error and the error variance of x_1 with the Nakamura's controller are one digit smaller than the other controllers. Thus, we can confirm that Nakamura's method has much higher convergence accuracy than the other control methods.

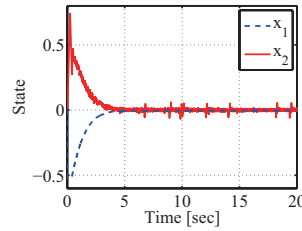


Fig. 10. states of system (2) with Hong's controller

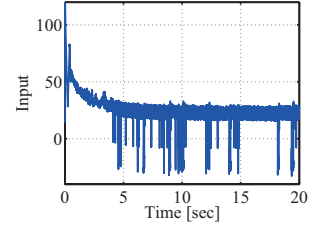


Fig. 11. input of system (2) with Hong's controller

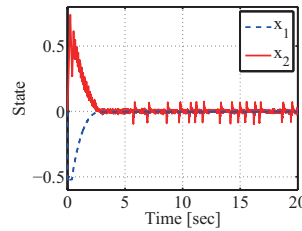


Fig. 12. states of system (2) with Yu's controller

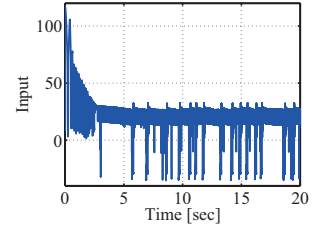


Fig. 13. input of system (2) with Yu's controller

Table 4. Averages and variances of residuals (Nakamura's finite-time controller (10), Hong's controller and Yu's controller)

	$E(x_1)$	$\sigma^2(x_1)$
Nakamura's finite-time controller	-3.4×10^{-4}	2.58×10^{-8}
Hong's controller	-3.9×10^{-3}	4.34×10^{-7}
Yu's controller	-2.9×10^{-3}	2.57×10^{-6}
	$E(x_2)$	$\sigma^2(x_2)$
Nakamura's finite-time controller	8.6×10^{-4}	9.53×10^{-5}
Hong's controller	1.5×10^{-4}	6.79×10^{-5}
Yu's controller	1.1×10^{-4}	3.52×10^{-4}

7. CONCLUSION

In this paper, we confirmed that Nakamura's method is valid to position control of the robot manipulator with discontinuous friction terms. By the experimental results, we show that Nakamura's finite-time controller has the higher convergence accuracy than the other convergence rate controllers and the other finite-time control methods.

ACKNOWLEDGEMENTS

This work was supported by Grant-in-Aid for Young Scientists (B) (21760333).

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Appendix A. PROOF OF LEMMA 15

For proving Lemma 15, we present the following definition.

Definition 19. (homogeneous norm [14]). The function $\|x\|_{\{r,p\}} := (\sum_{j=1}^n |x_j|^{p/r_j})^{1/p}$ for $x \in \mathbb{R}^n$ is said to be a homogeneous p -norm.

Proof. [Proof of Lemma 15] We prove the lemma by the following procedure: (i) we show that (7) is C^1 -differentiable; (ii) we show that (7) is homogeneous of degree k with respect to $\Delta_\varepsilon^{(q-\tau,q)}x$; (iii) we show that (7) is a global control Lyapunov function for (2).

- (i) By $q > |\tau|$ and $k \geq 2q - \tau$, (7) is C^1 -differentiable.
- (ii) By substituting $\Delta_\varepsilon^{(q-\tau,q)}x$ into (7),

$$\begin{aligned} V(\Delta_\varepsilon^r x) &= |\varepsilon^{q-\tau} x_1|^{\frac{k}{q-\tau}} + b|\varepsilon^{q-\tau} x_1|^{\frac{k-q}{q-\tau}} \\ &\quad \cdot \operatorname{sgn}(\varepsilon^{q-\tau} x_1) \varepsilon^q x_2 + c|\varepsilon^q x_2|^{\frac{k}{q}} \\ &= \varepsilon^k \left(|x_1|^{\frac{k}{q-\tau}} + b|x_1|^{\frac{k-q}{q-\tau}} \cdot \operatorname{sgn}(x_1) x_2 + c|x_2|^{\frac{k}{q}} \right). \end{aligned}$$

Hence, (7) is homogeneous of degree k with respect to $\Delta_\varepsilon^{(q-\tau,q)}x$.

(iii) We prove the fact as follows: (a) we prove that (7) is positive-definite; (b) we prove that (7) is also proper; (c) by Remark 4, we prove that if $x \neq 0$ and $L_g V = 0$, $L_f V < 0$.

- (a) According to Theorem 3 in [11], (7) is positive-definite.
- (b) Since $\{x \mid \|x\|_{\{r,p\}} = 1\}$ is compact, the following constant $V_{0 \min}$ is well-defined:

$$V_{0 \min} := \min_{x \in \{x \mid \|x\|_{\{r,p\}} = 1\}} V(x).$$

Note that for any $x \in \mathbb{R}^2$, there exist $\varepsilon > 0$ and $x_0 \in \{x \mid \|x\|_{\{r,p\}} = 1\}$ such that $x = \Delta_\varepsilon^r(x_0)$. Let $x := \Delta_\varepsilon^r(x_0)$ satisfy

$$\|x\|_{\{r,p\}} > \left(\frac{L}{V_{0 \min}} \right)^{\frac{1}{k}}, \quad \forall L > 0. \quad (\text{A.1})$$

Since $\|x\|_{\{r,p\}} = \varepsilon$,

$$V(x) = V(\Delta_\varepsilon^r(x_0)) = \varepsilon^k V(x_0) \geq \varepsilon^k V_{0 \min} > L.$$

Thus, for any $x \in \{V(x) \leq L\}$, $\|x\|_{\{r,p\}} \leq (L/V_{0 \min})^{1/k}$ is satisfied.

Note that (7) is positive-definite. If L is fixed, $\{x \mid \|x\|_{\{r,p\}} \leq (L/V_{0 \min})^{1/k}\}$ is bounded. Thus, for any $L > 0$, $\{x \in \mathbb{R}^2 \mid V(x) \leq L\}$ is bounded.

Note that (7) is continuous. By Theorem 1 in [3, I.2.I], the inverse mapping $V^{-1} : [0, L] \rightarrow \mathbb{R}^2$ is closed set.

Thus, for any $L > 0$, $\{x \in \mathbb{R}^2 \mid V(x) \leq L\}$ is bounded closed set. Therefore, (7) is proper.

- (c) Since $L_g V = 0$, the following equality is satisfied:

$$b|x_1|^{\frac{k-q}{q-\tau}} \operatorname{sgn}(x_1) = -\frac{k}{q} c|x_2|^{\frac{k}{q}-1} \operatorname{sgn}(x_2).$$

This is equivalent to the following equation:

$$x_2 = -\left(\frac{qb}{ck} \right)^{\frac{q}{k-q}} |x_1|^{\frac{q}{q-\tau}} \operatorname{sgn}(x_1). \quad (\text{A.2})$$

If $L_g V = 0$, $L_f V$ becomes as follows:

$$\begin{aligned} L_f V &= \left(\left(\frac{k}{q-\tau} |x_1|^{\frac{k}{q-\tau}-1} \operatorname{sgn}(x_1) \right) \right. \\ &\quad \left. + \frac{k-q}{q-\tau} b|x_1|^{\frac{k-q}{q-\tau}-1} x_2 \right) x_2. \end{aligned} \quad (\text{A.3})$$

By substituting (A.2) into (A.3),

$$\begin{aligned} L_f V &= -\frac{k}{q-\tau} \left[1 - \left(\frac{b^{\frac{k}{q}} \left(\frac{k-q}{q} \right)^{\frac{k-q}{q}}}{c \left(\frac{k}{q} \right)^{\frac{k}{q}}} \right)^{\frac{q}{k-q}} \right] \\ &\quad \cdot \left(\frac{qb}{ck} \right)^{\frac{q}{k-q}} |x_1|^{\frac{k+\tau}{q-\tau}}. \end{aligned}$$

Note that (7) satisfies the condition (8). Thus, if $x \neq 0$ and $L_g V = 0$, $L_f V < 0$. ■