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Optimal Trajectories for Vehicles with Energy Recovery Options ⋆

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Abstract: The fuel optimal control of modern vehicles involves the control of several components: the automated manual transmission, power split between the engine and secondary power converter, vehicle velocity, clutch position and motor start-stop. These controls are often optimized separate from each other, which leads to suboptimal results. In this paper we focus on the combined optimization of hybrid system use, gearbox and vehicle velocity.
A novel cost function description is used which describes the influence of the automated manual transmission, the potential of brake energy recovery, and the vehicle velocity with one control signal, and, therefore, reduces the computational complexity. The cost is modeled using a piecewise affine continuous function, which has the advantage of the control appearing affine in the Hamiltonian. Besides the standard optimal control solution for systems with an affine cost function, non-smooth optimal control theory is involved to obtain a sequence of subarcs that fulfills the necessary conditions of optimality.
Since the length and cost of each subarc, that fulfills the necessary conditions of optimality, in travel time and fuel consumption, can analytically be expressed in its initial and end velocity, the fuel optimal control of a vehicle with energy recovery options is rewritten as a nonlinear optimization problem.

Keywords: Hybrid vehicles, Vehicles, Optimal control, Maximum principle, Trajectory planning, Velocity control, Train control, Traffic control, Trajectories.

1. INTRODUCTION

Nowadays, control plays an increasing role in modern vehicle drive trains, for instance the control of automated manual transmissions (AMTs), of clutch and engine stop-start, of power split between different power converters in hybrid drive trains, often referred to as energy management strategy (EMS), and of vehicle velocity by the combined power output of engine and electric machine with an (adaptive) cruise control ((A)CC).

The different control systems have a common objective, to minimize the fuel consumption, while satisfying constraints on the driveability, comfort, components, and vehicle velocity. Optimization of individual systems generally leads to suboptimal results. From onboard navigation systems one can acquire the data, e.g., road curvature, road grade and velocity limitations, needed to set-up an optimization problem for the systems described above.

Several contributions regarding velocity trajectory optimization for vehicles (incl. trains) with an AMT and braking capabilities have been made (Monastyrsky and Golownykh, 1993; Ko et al., 2004; Hellström et al., 2008; Vašak et al., 2009). In Hellström et al. (2010) the velocity trajectory optimization is solved, accounting for the energy recovery potential of hybrid vehicles. In the previously mentioned contributions, the problem is attacked using Dynamic Programming techniques. These solutions are computationally heavy. Therefore, here a solution using the Maximum Principle (MP) is explored, since this is (possibly) computationally more efficient.

The fuel optimal velocity trajectory of a vehicle is related to the well known Optimal Control Problem (OCP) example of the fuel optimal flight of a rocket. When the cost function is affine on the control interval (zero trust to maximum thrust), the Pontryagin MP, extended with theory on singular extremals (Bell and Jacobson, 1975; Johnson and Gibson, 1963; Kelley, 1965; Kopp and Moyer, 1965), can be applied to solve the problem. The solution consists of the extremal controls and a singular control arc where the velocity is constant and the trust is in equilibrium with the aerodynamic losses, see, e.g., Geering (2007, p. 62) for details on the solution. In Schwartzkopf and Leipnik (1977) and Stoicescu (1995) an optimal solutions shape, for conventional vehicles, is derived using the Pontryagin MP, however, the non-smoothness, e.g., related to braking, is not fully addressed.

The Pontryagin MP does not apply to non-smooth systems as it requires the underlying data to be differentiable. Several extensions of the MP to the non-smooth case are known, see, e.g., Clarke (2005) and Vinter (2000) for an overview. The requirements to the underlying system can

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be relaxed, by considering generalizations of the derivative, that is, the adjoint multiplier functions are described in the form of a differential inclusion set rather then explicit differential equations.

In addition to the drive train of conventional vehicles, modern vehicles are often equipped with a secondary power converter and storage device. Since the primary power converter can in general only produce power, the secondary power converter enables energy recovery during braking or down hill driving. The EMS of the hybrid drive train components deals with the re-use of recovered energy and is often obtained using optimal control theory, see, e.g., Sciarretta and Guzzella (2007), for an overview. The optimal control solution is characterized by adjoint parameters of which the value depends on the future velocity and power trajectories. The real-time EMS is often designed and bench marked using predefined velocity and power trajectories.

In Van Keulen et al. (2010), it is shown that route information from a navigation system can be used to construct a velocity trajectory optimization problem using a novel non-smooth description of the hybrid drive train cost function which reduces the computational complexity since only one control signal is required instead of three (control of AMT, hybrid system and vehicle velocity). The solution in Van Keulen et al. (2010) was derived on physical insight alone.

The main contribution of this paper is the derivation of the optimal solution for the cost function proposed in Van Keulen et al. (2010). It is shown that describing the cost function with piecewise affine (PWA) relations, rather then a higher order continuous function, results in the analytical derivation of the optimal solution shape.

This paper is organized as follows. Section 2 discusses the cost function derivation. In Section 3 the derivation of a solution shape that fulfills the necessary conditions for optimality for the velocity trajectory is presented. Section 4 deals with the structure of the solution shape and sketches the possibilities of real-time implementation. Finally, in Section 5 we summarize with conclusions and give an outlook on future research.

2. COST FUNCTION DERIVATION

Finding a control that simultaneously optimizes the AMT, EMS and ACC is not trivial due to i) the nonlinear characteristics of the drive train components, and ii) the large number of control parameters that hamper the practical implementation of numerical solutions with, e.g., Dynamic Programming. To reduce the computational complexity, it is proposed to simplify the problem, by approximating the fuel cost of gearshifts, hybrid system use and vehicle velocity with a scalar function with a scalar argument.

Using a convex continuous PWA cost function has the advantage that the control appears linearly in the Hamiltonian, which enables the use of non-smooth optimal control theory developed in Clarke (1983, 2005) and Vinter (2000). In the remainder of this section, firstly, the approximation of the engine and AMT, and hereafter the hybrid system, is discussed and motivated, secondly, a formal system description is presented.

Similar to the cost function of the fuel optimal flight of a rocket, it is suggested to approximate the fuel cost of the engine of a vehicle, at rotational velocity ω, with an affine relation, sometimes referred to as a Willans approximation:

$$P_f = \gamma_{p,0}(\omega) + \gamma_{p,1}(\omega)P_p \tag{1}$$

with P_f the fuel power, $\gamma_{p,0} > 0$ and $\gamma_{p,1} > 1$, and P_p the demanded engine power. This relation will later be expanded to form the cost function for the different optimization problems.

It is also proposed to approximate the control of the AMT with (infinitely) many gear settings (a continuously variable transmission CVT), so the power converter rotational velocity can be chosen virtually independent of the vehicle speed, and to base the gear ratio selection on choosing the most efficient ω using a predefined level of power reserve P_v

$$P_p + P_v \leq \omega \max(T_p(\omega)) \tag{2}$$

with $T_p(\omega)$ the engine torque. The optimal gear setting is obtained from

$$\min_{\omega} \int P_f(\omega)dt. \tag{3}$$

Using $P_v = 0$, approximates *e-line* tracking: the line connecting the engine optimal operating points (rotational velocity and torque), for each power request. Fig. 1 depicts the equivalent fuel consumption P_{eq}, of a medium-duty truck, as a function of the tractive power P_r and different levels of power reserve P_v. Here, above $\frac{5}{8}\overline{P}_p$ the power reserve is linearly build off to become zero at the engine maximum output power $\overline{P}_p$.

Changing (1) to,

$$P_f = \gamma_{p,0}(P_v) + \gamma_{p,1}(P_v)P_p, \tag{4}$$

for $\underline{P}_p(P_v) < P_p < \overline{P}_p$ with

$$\underline{P}_p(P_v) = \frac{-\gamma_{p,0}(P_v)}{\gamma_{p,1}(P_v)}, \tag{5}$$

is a viable way to incorporate the gear selection strategy in the cost function.

This cost function reflects an approximation of the engine and AMT use. The simplification of discrete gear shifting with many gear settings has two main consequences i) the fuel to power conversion of the engine is too optimistic, due to the limited number of gears, engine power cannot always be delivered at the preferred rotational velocity of the engine, see Saerens et al. (2008), and ii) the time and cost required to change gear are not accounted for.

The power conversion characteristics of electric machine and battery combined, are approximated as

$$P_s = \max(\gamma^+_{m,1}P_m, P_m/\gamma^-_{m,1}) \tag{6}$$

or visa versa

$$P_m = \min(\gamma^-_{m,1}P_s, P_s/\gamma^+_{m,1}) \tag{7}$$

with $\gamma^+_{m,1} > 1$ and $\gamma^-_{m,1} > 1$. The power stored in the battery P_s is a PWA function of the mechanical power of the electric machine P_m. In this contribution, an electric machine is used as secondary power converter, a hydraulic,

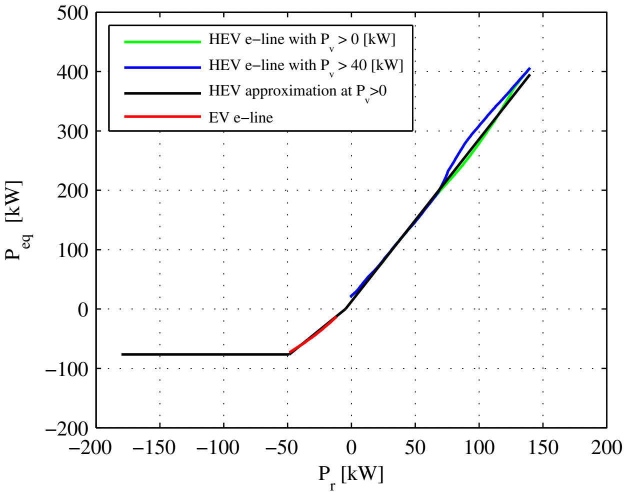

Fig. 1. The cost function description. Abbreviation (H)EV indicates (Hybrid) Electric Vehicle.

pneumatic or mechanical hybrid can be treated similarly, however.

The tractive power P_r is split over, or provided by, the two power converters as

$$P_r = P_p + P_m. \tag{8}$$

In case of hybrid vehicles, the power split is obtained from

$$\min_{P_m} \int P_f(P_m)dt \tag{9}$$

such that the initial and final battery charge is the same (or is a predefined difference). Note that $P_f \geq 0$, so the cost function is convex ($\gamma_{p,1} > 1$) and PWA. Eq. (9) can be formally solved, but it is also possible to obtain the solution based on physical insight alone.

The optimal strategy is to generate electricity and store it in the battery if P_r drops below the level where $P_f = 0$, i.e., $P_r < \underline{P}_p$, this is referred to as brake energy recovery. Given the losses associated with charging the battery, using the PWA model description, it is not profitable to charge the battery at any other occasion. The energy stored in the battery can then be re-used at any time where $P_r > \underline{P}_p$. Because the incremental cost $\gamma_{p,1}$ and discharge/motor factor $\gamma^+_{m,1}$ are constant, it does not matter *when* the battery is actually discharged, as long as the constraint on the battery end charge is met and the electric machine is used to provide tractive power.

The optimal power split can be incorporated in the cost function, to become a function of the tractive power P_r alone, using brake energy recovery as "negative" fuel consumption. This is described with an affine relation for P_r between the maximum regenerative power $\underline{P}_q$ and the drag power $\underline{P}_p$, $\underline{P}_q \leq P_r \leq \underline{P}_p$. The amount of fuel saved by storing energy in the battery and using it later is obtained from $\frac{\gamma_{p,1}}{\gamma^+_{m,1}\gamma^-_{m,1}} P_m$, with $P_m = P_r - \underline{P}_p \leq 0$ for charging/generating.

The EMS problem is simplified in several ways: the route dependent influences of load shifting, clutch opening, and engine stop-start are disregarded, the limited battery capacity or the temperature dependent electric machine overload capability is not accounted for, the maximum tractive power is limited to the maximum engine output power $P_r \leq \overline{P}_p$, and finally, the component description is simplified compared to Sciarretta and Guzzella (2007).

The application of the service brakes can be incorporated in the cost function as well. Clearly, applying the service brakes does not consume fuel or recover energy, and is, therefore, modeled with a horizontal line in the cost function for $\underline{P}_d < P_r < \underline{P}_q$, where $\underline{P}_d$ is the available brake power.

The above mentioned simplifications result in a non-smooth cost function, see Fig. 1, with a single control signal P_r, the mechanical power at the wheels, instead of three (the output power of engine and electric machine and gearbox use):

$$P_{eq}(t) = \max\left(\gamma_{p,1}(P_r(t) - \underline{P}_p), \frac{\gamma_{p,1}(P_r(t) - \underline{P}_p)}{\gamma^+_{m,1}\gamma^-_{m,1}}, \frac{\gamma_{p,1}(\underline{P}_q - \underline{P}_p)}{\gamma^+_{m,1}\gamma^-_{m,1}}\right) \tag{10}$$

here, $P_{eq}(t)$ is a PWA function representing the equivalent fuel consumption power.

The minimization of the functional

$$\min_{P_r} \int_{t_0}^{t_1} P_{eq}(P_r)dt, \tag{11}$$

subject to the nonlinear vehicle dynamics:

$$\dot{v}(t) = a_1 \frac{P_r(t)}{v(t)} - a_2(t,s) - a_3 v(t) - a_4 v^2(t), \tag{12}$$

$$\dot{s}(t) = v(t), \tag{13}$$

is considered. Here, $v > 0$ is the vehicle velocity, s the traveled distance, $a_1 > 0$ the reciprocal vehicle mass, a_2 a parameter related to rolling resistance and gravitational force, $a_3 > 0$ a loss parameter proportional to velocity, $a_4 > 0$ the parameter for (aerodynamic) losses quadratic to velocity.

The control P_r is bounded:

$$P_r \in [\underline{P}_d, \overline{P}_p], \tag{14}$$

and the following begin and end states are constrained:

$$v(t_0) = v_0,\ s(t_0) = s_0, \tag{15}$$

$$v(t_1) = v_e,\ s(t_1) = s_e, \tag{16}$$

where v_e is the desired final velocity and s_e is the distance to be reached. The velocity trajectory optimization problem can be defined as a *fixed-time*, *fixed-end-point*, non-smooth OCP, in which the fuel cost is described as a PWA continuous function. It is assumed that the final time t_1 is sufficiently large for the existence of a solution.

3. NECESSARY CONDITIONS FOR OPTIMALITY

In this section, the MP is applied to derive a set of controls that fulfils the necessary conditions of optimality. This comprises first the definition of the Hamiltonian, hereafter, the necessary conditions are stated, which includes higher order necessary conditions for the singular control subarcs in situations where the Hamiltonian does not explicitly depends on the control, finally, the optimal control subarcs are derived.

The system dynamics can be adjoined to the fuel cost function by a set of multiplier functions $p(t)$, leading to the Hamiltonian:

$$H = \max\left(\gamma_{p,1}(P_r(t) - \underline{P}_p), \frac{\gamma_{p,1}(P_r(t) - \underline{P}_p)}{\gamma^+_{m,1}\gamma^-_{m,1}}, \frac{\gamma_{p,1}(\underline{P}_q - \underline{P}_p)}{\gamma^+_{m,1}\gamma^-_{m,1}}\right) + p_1(t)\left(a_1\frac{P_r(t)}{v(t)} - a_2(t,s) - a_3v(t) - a_4v^2(t)\right) + p_2(t)v(t). \quad (17)$$

Applying the MP (Vinter, 2000, Theorem 6.2.1 on p. 203), it is stated that if the control is optimal, then there exist nontrivial continuous multiplier functions:

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \not\equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (18)$$

such that the following necessary conditions are satisfied:

- the adjoint inclusion $\dot{p} \in \text{co}\partial_{v,s}H$, in which $\partial_{v,s}H$ denotes the generalized subdifferential of H. Since the dynamics in (12) and (13) are smooth this reduces to differential equations on the multiplier functions:

$$\dot{p}_1(t) = -\frac{\partial H}{\partial v} = p_1(t)\left(a_1\frac{P_r(t)}{v^2(t)} + a_3 + 2a_4v(t)\right) - p_2(t), \quad (19)$$

$$\dot{p}_2(t) = -\frac{\partial H}{\partial s} = p_1(t)\frac{\partial a_2(t,s)}{\partial s}, \quad (20)$$

- the Hamiltonian has a global minimum with respect to P_r:

$$P_r^* = \arg\min_{P_r} H(v^*, s^*, P_r, p_1^*, p_2^*) \quad (21)$$

  where v^* is the optimal velocity state trajectory, s^* the optimal distance trajectory, P_r^* the optimal power input trajectory, p_1^*, and p_2^* the corresponding multiplier functions.

Note that, since the problem treated in Section 4 is a fixed-time fixed-end-point problem, the transversality conditions are omitted.

For convenience the Hamiltonian is written as an affine relation of the control parameter:

$$H = g(v, s, p_1, p_2) + h(v, p_1)P_r. \quad (22)$$

Minimizing the Hamiltonian yields:

$$h(t)P_r^* \leq h(t)P_r. \quad (23)$$

Switching function $h(t)$ is of first order with respect to the control P_r, and described with:

$$h(t) = \begin{cases} p_1(t)\dfrac{a_1}{v(t)} + \gamma_{p,1} & \text{in int } [\underline{P}_p, \overline{P}_p], \\ p_1(t)\dfrac{a_1}{v(t)} + \dfrac{\gamma_{p,1}}{\gamma^+_{m,1}\gamma^-_{m,1}} & \text{in int } [\underline{P}_q, \underline{P}_p], \\ p_1(t)\dfrac{a_1}{v(t)} & \text{in int } [\underline{P}_d, \underline{P}_q]. \end{cases} \quad (24)$$

Here "int" denotes the interior of the region.

A special situation occurs when h becomes identically zero, $h \equiv 0$. In that case H does not depend upon P_r explicitly. Although the control arc satisfies the MP, the optimal control cannot be found directly by minimizing H, it must satisfy additional higher order necessary conditions for optimality (Kelley, 1965; Kopp and Moyer, 1965; Johnson and Gibson, 1963; Bell and Jacobson, 1975), so singular control. The necessary conditions are that all of the derivatives of h, along the optimal trajectory, must vanish in this time interval as well, i.e., $\dot{h} \equiv 0$, $\ddot{h} \equiv 0$, $h^{(3)} \equiv 0$, and so on.

In case $P_r < \underline{P}_q$, $h \equiv 0$ as optimal singular control solution can not be obtained due to the non-triviality condition (18) on p_1 and p_2, so $P_r = \underline{P}_d$ or $P_r = \underline{P}_q$ depending on the sign of h. For $P_r > \underline{P}_q$, $h \equiv 0$ can occur and the higher order conditions are needed.

For the singular case, the following necessary condition for optimality is obtained:

$$h(t) = \frac{a_1p_1(t)}{v(t)} + \sigma \equiv 0, \quad (25)$$

which leads to:

$$p_1(t) \equiv \frac{-\sigma v(t)}{a_1}, \quad (26)$$

where σ is a piecewise constant:

$$\sigma = \begin{cases} \gamma_{p,1} & \text{in int } [\underline{P}_p, \overline{P}_p], \\ \dfrac{\gamma_{p,1}}{\gamma^+_{m,1}\gamma^-_{m,1}} & \text{in int } [\underline{P}_q, \underline{P}_p]. \end{cases} \quad (27)$$

The condition on the first derivative of h becomes:

$$\dot{h}(t) = \dot{p}_1(t)\frac{a_1}{v(t)} - p_1(t)\frac{a_1\dot{v}}{v^2(t)} \equiv 0. \quad (28)$$

Using (12) and (19), a condition on the multiplier p_2 can be derived:

$$p_2(t) = p_1(t)\left(\frac{a_2(t,s)}{v} + 2a_3 + 3a_4v(t)\right), \quad (29)$$

The condition on the second derivative of h becomes:

$$\ddot{h}(t) = \dot{p}_1(t)a_1\left(\frac{a_2(t,s)}{v^2(t)} + \frac{2a_3}{v(t)} + 3a_4\right) + p_1(t)a_1\dot{v}\left(\frac{-a_2(t,s)}{2v^3(t)} - \frac{2a_3}{v^2(t)}\right) + \frac{p_1(t)a_1\dot{a}_2(t,s)}{v^2(t)} - \frac{\dot{p}_2(t)a_1}{v(t)} + \frac{p_2(t)a_1\dot{v}(t)}{v^2(t)} \equiv 0. \quad (30)$$

Using (12), (19), (20), and (29) the condition above can be reduced to:

$$\ddot{h}(t) = p_1(t)a_1\dot{v}\left(\frac{2a_3}{v^2(t)} + \frac{6a_4}{v(t)}\right) \equiv 0.$$

Therefore, $\ddot{h}$ can only vanish for the singular control $\dot{v} \equiv 0$, so, v^* is constant and by (12) it then holds that:

$$P_r^*(t) \equiv \frac{a_2(t,s)v^* + a_3v^{*2} + a_4v^{*3}}{a_1}. \quad (31)$$

From the necessary conditions it can be seen that the singular control arcs have the following features:

- the velocity v^* is constant,
- the tractive power P_r^* is in equilibrium with the vehicle losses (31), the situation of negative velocity is neglected,
- the costate variable p_1^* is constant and attains the value $p_1^* = \frac{-\sigma v^*}{a_1}$,

- the costate variable p_2^* attains the value $p_2^*(t) = p_1^*\left(\frac{a_2(t,s)}{v^*} + 2a_3 + 3a_4 v^*\right)$,
- if $\frac{\partial a_2(s)}{\partial s} = 0$, p_2^* is constant, so the value for p_2^* obtained by condition (29) holds for all subarcs.

The singular control subarc on the control interval $\underline{P}_q < P_r < \underline{P}_p$, in practice, only occurs at a downhill where the electric machine force and aerodynamic drag force and rolling resistance are in equilibrium with the gravitational force.

The optimal control has the following subarcs:

$$P_r^*(t) = \begin{cases} \overline{P}_p & \text{for } p_1 < \dfrac{-\gamma_{p,1} v}{a_1}, \\ [\underline{P}_p,\ \overline{P}_p] & \text{for } p_1 \equiv \dfrac{-\gamma_{p,1} v}{a_1}, \\ \underline{P}_p & \text{for } \dfrac{-\gamma_{p,1} v}{a_1} < p_1 < \dfrac{-\gamma_{p,1} v}{\gamma_{m,1}^{+}\gamma_{m,1}^{-} a_1}, \\ [\underline{P}_q,\ \underline{P}_p] & \text{for } p_1 \equiv \dfrac{-\gamma_{p,1} v}{\gamma_{m,1}^{+}\gamma_{m,1}^{-} a_1}, \\ \underline{P}_q & \text{for } \dfrac{-\gamma_{p,1} v}{\gamma_{m,1}^{+}\gamma_{m,1}^{-} a_1} < p_1 < 0, \\ \underline{P}_d & \text{for } p_1 > 0. \end{cases} \tag{32}$$

4. NUMERICAL SOLUTION

To arrive at optimal trajectories $P_r^*(t)$ and $v^*(t)$, two problems remain to be solved: i) the structure of the solution (regularity), i.e., the sequence of nonsingular and singular subarcs composing the optimal trajectory, and ii) the junction points between nonsingular and singular subarcs describing the length of each subarc.

For simplicity the solution structure for constant road grade is derived. It is shown that the structure of the solution, in case of constant road grade, can be reduced to a set of possible solution shapes. The equivalent fuel cost and travel time can then be analytically expressed in the velocities at the junction points of the solution structure, which enables the construction of a nonlinear optimization problem.

4.1 Structure of the solution

Several observations limit the number of possible solution shapes. First note that by the continuity of $p_1(t)$, only a switch from one control extremal to the neighboring extremal, or sometimes a singular solution, is allowed. For instance during deceleration a switch from $\underline{P}_p$ followed by $\underline{P}_q$ to $\underline{P}_d$ is possible. However, the sequence $\underline{P}_p \rightarrow \underline{P}_d \rightarrow \underline{P}_q$ is not allowed.

In the remainder we assume a constant road grade and a singular solution $P_r \in [\underline{P}_p,\ \overline{P}_p]$. The following possible solution shapes can be expected:

- a maximum power subarc, a constant velocity, and another maximum power subarc: $\overline{P}_p \rightarrow P_r \in [\underline{P}_p,\ \overline{P}_p] \rightarrow \overline{P}_p$,
- a maximum power subarc, a constant velocity, and a sequence of coasting, energy recuperation and braking: $\overline{P}_p \rightarrow P_r \in [\underline{P}_p,\ \overline{P}_p] \rightarrow \underline{P}_p \rightarrow \underline{P}_q \rightarrow \underline{P}_d$, see Fig. 2,
- coasting and energy recuperation, a constant velocity, and a maximum power subarc: $\underline{P}_q \rightarrow \underline{P}_p \rightarrow P_r \in [\underline{P}_p,\ \overline{P}_p] \rightarrow \overline{P}_p$,
- coasting and energy recuperation, a constant velocity, and a sequence of coasting, energy recuperation and braking: $\underline{P}_q \rightarrow \underline{P}_p \rightarrow P_r \in [\underline{P}_p,\ \overline{P}_p] \rightarrow \underline{P}_p \rightarrow \underline{P}_q \rightarrow \underline{P}_d$,

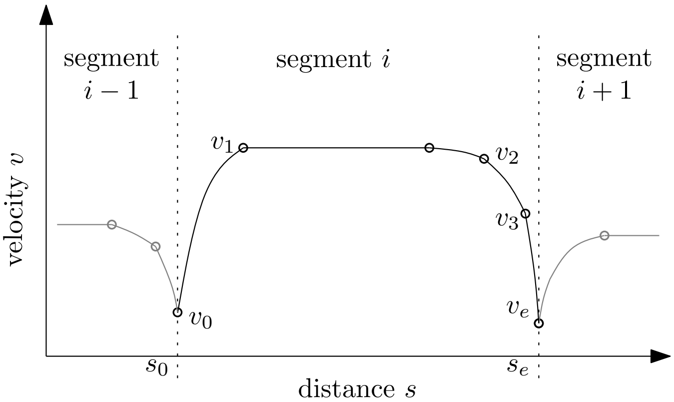

Fig. 2. Solution shape with a maximum power subarc, a singular solution in the interval $P_r \in [\underline{P}_p,\ \overline{P}_p]$, and a sequence of coasting, energy recuperation and braking.

In the next section it is shown that fuel consumption and travel time can analytically be expressed as a function of the velocities at the subarc junction points.

4.2 Junction points

In Van Keulen et al. (2010) it is shown that the length (time and distance), as well as the fuel cost, of each nonsingular subarc can be analytically expressed in the velocities at the junction points between the subarcs with:

$$\Delta t|_{t_a}^{t_b} = \frac{1}{a_1} \sum_{j=1}^{3} \frac{z_j \ln\left(\frac{v_b - z_j}{v_a - z_j}\right)}{a_2 + 2a_3 z_j + 3a_4 z_j^2}. \tag{33}$$

Here, v_a is the initial velocity of the subarc, v_b is the final velocity of the subarc, t_a is the initial time, t_b is the final time, and z_j is the jth root of the cubic equation:

$$-a_1 P_r + a_2 z + a_3 z^2 + a_4 z^3 = 0. \tag{34}$$

Two possible solutions can occur: three real roots, or one real root and two imaginary roots, of which one root provides the maximum velocity that can be attained with power P_r.

The covered distance $\Delta s|_{s_a}^{s_b}$ is calculated similarly by multiplying (33) with z_j, the jth root of the cubic equation, see Van Keulen et al. (2010). From the covered distance with the nonsingular subarcs, follows the distance to be covered with the singular subarc (constant velocity).

A positive argument is required in the "ln" function in (33). For the solution shape as presented in Fig. 2 this can be enforced by a constraint on the length of each subarc $t_{m,i} \geq 0$ with $m \in [0, 1, 2, 3, e]$.

The equivalent fuel consumption, and time to complete the segment becomes a function of the initial and final velocities of the subarcs: $E_{f_i}(\nu_i)$ and $t_i(\nu_i)$, in which E_{f_i}

is the equivalent fuel cost of segment i, t_i is the traveling time for segment i, and ν_i is a set of velocities describing the junction points of the solution structure. The size of ν_i varies between 3 and 5.

To obtain the fuel cost for all n segments, the segment fuel cost and traveling time can be simply summed up. So the minimization of (11) can be rewritten as a nonlinear optimization:

$$\min_{\nu} \sum_{i=1}^{n} E_{f_i}(\nu_i), \tag{35}$$

subject to a time constraint:

$$\sum_{i=1}^{n} t_i(\nu_i) \leq t_1 - t_0, \tag{36}$$

constraints on the length of each subarc:

$$-t_{m,i} \leq 0, \tag{37}$$

for all n segments and ν velocities at junction points.

Preliminary numerical and experimental results of this approach for a suboptimal subarc sequence can be found in Van Keulen et al. (2010).

5. CONCLUSIONS AND OUTLOOK

This research has been concerned with the fuel optimal control of vehicles with energy recovery options. The main contribution is the optimal control solution for a novel cost function description which approximates the control of gear shift, energy recovery, and vehicle velocity with an affine piecewise continuous function. This approach reduces the complexity of the problem considerably as only one control signal is used instead of three. Describing the cost function with piecewise affine functions enables an analytical derivation of optimal control subarcs that fulfil the necessary conditions of optimality.

It is shown that the control structure, under constant road grade, can be reduced to only a few possible sequences of subarcs. Each sequence can be analytically described with the velocities at the junction points of the subarcs. With this observation the initial control problem is rewritten as a nonlinear optimization. This reduces the computational complexity of the problem compared to direct methods, since these methods have difficulties to converge to singular solutions.

The drive train control involves several aspects, which are not yet included here. Future work will focus on including engine stop-start, discrete gearshift functionality, battery state of charge bounds and state constraints (velocity limitation), in the problem formulation. Numerical methods to solve the nonlinear optimization are to be studied. Further research is required to deal with the varying size of the set of velocities describing the solution structure.

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