A Dynamic Compensator for Parameter Varying Systems Subject to Actuator Limitations applied to a T-S Fuzzy System

Michael Klug, Eugênio B. Castelan* and Valter J.S. Leite**

* Grupo de Controle de Sistemas Mecatrônicos
Departamento de Automação e Sistemas- DAS / CTC / UFSC
88040-900, Florianópolis, Brazil. micklug, eugenio@das.ufsc.br

** CEFET-MG / Campus Divinópolis, Rua Álvares Azevedo 400,
35503-822, Divinópolis, MG, Brazil. valter@ieee.org

Abstract: We address the design of parameter-dependent dynamic output feedback controllers for discrete-time systems with time-varying parameters and subject to actuators saturations. The particularity of the proposed controller is that all its matrices are dependent of the varying parameters, differently from a previous proposed compensator that was partially dependent of the parameters. The proposed LMI stability and stabilization results are based on the use of a parameter dependent Lyapunov function and on the λ -contractivity of the associated level set to guarantee the local stability and some time performance. Two control strategies are addressed: avoiding saturation and allowing saturation, in both cases taking into account the realistic consideration of local validity of the plant model. The proposed numerical example shows the control design for a nonlinear system represented by a Takagi-Sugeno Fuzzy model.

Keywords: Dynamic Compensator, Bounded Control, LPV systems, Contractivity, T-S fuzzy model.

1. INTRODUCTION

In most practical control systems, it is required to consider the presence of certain dynamical nonlinear characteristics and practical constraints that are inherent to the plant, to the actuators and to the sensors. To cope with the presence of the saturation nonlinearity induced by the actuators limits, among the various existing nonlinear control systems approaches, absolute stability theory has been considered in the literature for analysis and synthesis of the so-called Lur'e system (Khalil (2003)). Furthermore, time-varying parameters techniques can be used to locally represent the behavior of nonlinear system, using, for instance, the Takagi-Sugeno fuzzy models, so that robust control techniques for uncertain systems can be used to deal with the plant nonlinearities in the design phase (see, for instance, (Geromel and Colaneri (2005); Daafouz and Bernussou (2001); Peres and Oliveira (2008))). In particular, if the time-varying parameters are real-time available, parameter-dependent control laws can be implemented.

Considering linear systems with saturating inputs, a large amount of works can be found in the literature as, for instance (see also references therein): (Hu et al. (2002); Zheng and Wu (2008); Gomes da Silva and Tarbouriech (2005)). In the context of nonlinear systems subject to control saturations, the synthesis of a dynamic output feedback controller has been considered in (Gomes da Silva et al. (2008)), where the proposed stabilization conditions are stated in LMI forms. In a more general context, in

(Castelan et al. (2010)) the authors present a methodology for the synthesis of output feedback control laws for Lur'e type discrete-time nonlinear systems with time-varying parameters and subject to actuators limitations. Thus, based on some tools from the absolute stability theory, on a modified sector condition, and using the concept of contractive sets applied to a level set obtained from a parameter dependent Lyapunov function, an LMI framework has been proposed to design a dynamic output feedback controller, called partially dependent of the time-varying parameters.

The results proposed in the present work extends the results in (Castelan et al. (2010)) by considering the design of a dynamic output feedback controller which is totally dependent of the time-varying parameters, in the sense that all the matrices associated to the controller state and output equation depend on the considered parameters. The system to be controlled is represented by a parameter-varying discrete-time system subject to actuator saturations but, differently from (Castelan et al. (2010)), there is no sector nonlinearity associated to the system dynamics, although the proposed results can be extended to this more general setting. To consider more realistic nonlinear applications, we present a discussion about the consideration of the region of validity of the parameter-varying model in the control design phase. The LMI based algorithms to synthesize the dynamic compensators consider two cases: with saturation avoidance and without saturation avoidance. In both cases, we guarantee that a large contractivity set is included in the region of validity. A numerical example based on the use of a Takagi-Sugeno model for some nonlinear system is used to show some features of the

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proposed compensator, including the consideration of the mentioned region of validity. The reader can also consult (Viana et al. (2010)) to see the application of a similar totally parameter-dependent compensator to discrete-time fuzzy systems with time-varying delay. See (Nachidi et al. (2008)) for an LMI-based approach using static output feedback.

Notations. For a vector $x \in \mathbb{R}^n$, $x \geq 0$ means that all components of x , denoted by $x_{(i)}$, are nonnegative. The elements of a matrix $A \in \mathbb{R}^{m \times n}$ are denoted by $A_{(i,j)}$, $i = 1, \dots, m, j = 1, \dots, n$. $A_{(i)}$ denotes the i^{th} row of the matrix A . For two symmetric matrices, A and B , $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of A . I_m denotes the identity matrix of order m and $diag(x)$ denotes the diagonal matrix obtained from vector x . The symbol $*$ represents symmetrical blocks and $\bullet$ represents an element that has no influence on the development.

2. PROBLEM FORMULATION

Consider a linear discrete-time system with time-varying parameters, subject to bounded control inputs, given by:

$$\begin{cases} x_{k+1} = A(h_k)x_k + B(h_k)sat(u_k) \\ y_k = Cx_k \end{cases} \quad (1)$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^p$ are, respectively, the state, the control and the output vectors. In addition, $h_k \in \mathbb{R}^r$ is a real-time available time-varying vector parameter belonging to the unit simplex: $\Xi = \{h_k \in \mathbb{R}^r; \sum_{i=1}^r h_{k(i)} = 1, h_{k(i)} \geq 0, i = 1, \dots, r\}$. The structure of system matrices is assumed as follows:

$$[A(h_k) \ B(h_k)] = \sum_{i=1}^r h_{k(i)} [A_i \ B_i] \quad (2)$$

and $C \in \mathbb{R}^{p \times n}$, where the matrices A_i and B_i have appropriate dimensions.

The control inputs are bounded in amplitude, so that the standard saturation function is verified:

$$sat(u_{(l)}(t)) = sign(u_{(l)}(t))min(\rho_{(l)}, |u_{(l)}(t)|)$$

$\forall l = 1, \dots, m$, where $\rho_{(l)} > 0$ denotes the symmetric amplitude bound relative to the l^{th} control.

We propose the following dynamic output feedback controller (DOFC), which is totally dependent of the parameters and denoted TDP-DOFC:

$$\begin{cases} x_{c,k+1} = A_c(h_k)x_{c,k} + B_c(h_k)u_{c,k} - E_c(h_k)\Psi(u_k) \\ y_{c,k} = C_c(h_k)x_{c,k} + D_c(h_k)u_{c,k} \end{cases} \quad (3)$$

with $x_{c,k} \in \mathbb{R}^{n \times n}$ and $\Psi(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the dead-zone nonlinearity, defined by: $\Psi(u_k) = u_k - sat(u_k)$. The matrices of the controller are structured as

$$\begin{aligned} [A_c(h_k) \ B_c(h_k)] &= \sum_{i=1}^r h_{k(i)}^2 [A_{ci} \ B_{ci}] + \\ &\sum_{i=1}^{r-1} \sum_{q=i+1}^r h_{k(i)} h_{k(q)} [A_{ciq} \ B_{ciq}] \\ [C_c(h_k) \ D_c(h_k) \ E_c(h_k)] &= \sum_{i=1}^r h_{k(i)} [C_{ci} \ D_{ci} \ E_{ci}] \end{aligned} \quad (4)$$

where $(A_{ci}, \ A_{ciq}) \in \mathbb{R}^{n \times n}$, $(B_{ci}, \ B_{ciq}) \in \mathbb{R}^{n \times m}$, $C_{ci} \in \mathbb{R}^{p \times n}$, $D_{ci} \in \mathbb{R}^{p \times m}$ and $E_{ci} \in \mathbb{R}^{n \times m}$. Observe that $E_c(h_k)$ is a parameter-varying anti-windup gain matrix that helps to mitigate the effects of saturation in the control inputs.

From the interconnection $u_{c,k} = y_k$ and $y_{c,k} = u_k$, and by defining the augmented state vector $\varsigma_k = [x_k' \ x_{c,k}']'$, the following closed-loop system is obtained:

$$\varsigma_{k+1} = (\mathbb{A}(h_k) + \mathbb{B}(h_k)\mathbb{K}(h_k))\varsigma_k - \mathbb{B}(h_k)\Psi(u_k) \quad (5)$$

with

$$\begin{aligned} \mathbb{A}(h_k) &= \begin{bmatrix} A(h_k) & 0 \\ B_c(h_k)C & A_c(h_k) \end{bmatrix}, \quad \mathbb{B}(h_k) = \begin{bmatrix} B(h_k) \\ 0 \end{bmatrix}, \\ \mathbb{B}(h_k) &= \begin{bmatrix} B(h_k) \\ E_c(h_k) \end{bmatrix}, \quad \text{and} \quad \mathbb{K}(h_k) = [D_c(h_k)C \ C_c(h_k)]. \end{aligned}$$

The matrix $\mathbb{K}(h_k)$ can be viewed as a parameter-dependent state feedback applied to an augmented system.

Problem 1. Determine A_{ci} , A_{ciq} , B_{ci} , B_{ciq} , C_{ci} , D_{ci} , E_{ci} and a region $S_0 \subseteq \mathbb{R}^{2n}$, as large as possible, such that the origin of the closed loop system (5) is robustly asymptotically stable for any initial condition $\varsigma_0 \in S_0$.

Two type of solutions for Problem 1 will be considered:

1st) to obtain robust asymptotic stability avoiding the saturation of actuators (i.e., to respect the control constraints, $|u_k| \leq \rho$, for any trajectory emanating from S_0); and

2nd) to obtain robust asymptotic stability allowing the saturation of actuators (i.e. it is assumed that $|u_k| > \rho$ may happen for trajectories emanating from S_0).

Remark 2. Similar problem and solutions have been investigated in (Castelan et al. (2010)) considering a DOFC that is only partially dependent of parameters (PDP-DOFC). For controlling the system (1) considered in the present work, the matrices of PDP-DOFC are given by

$$\begin{aligned} [A_c(h_k) \ B_c(h_k) \ E_c(h_k)] &= \sum_{i=1}^r h_{k(i)} [A_{ci} \ B_{ci} \ E_{ci}] \\ \text{and} \quad [C_c(h_k) \ D_c(h_k)] &= [C_c \ D_c]. \end{aligned}$$

3. PRELIMINARY RESULTS

We consider a parameter-dependent Lyapunov function (PDLF) such that $V(\varsigma_k, h_k) : \mathbb{R}^{2n} \times \Xi \rightarrow \mathbb{R}^+$. The associated level set is given by

$$\mathcal{L}_V \triangleq \{\varsigma_k \in \mathbb{R}^{2n}; V(\varsigma_k, h_k) \leq 1, \forall h_k \in \Xi\} \quad (6)$$

Definition 1. Consider a real scalar $\lambda \in (0, 1]$. The level set $\mathcal{L}_V$ is robustly λ -contractive (with respect to the trajectories of system (5)), if

$$\begin{aligned} \Delta V_\lambda(\varsigma_k, h_k) &\triangleq V(\varsigma_{k+1}, h_{k+1}) - \lambda V(\varsigma_k, h_k) < 0 \\ &\forall \varsigma_k \in \mathcal{L}_V \quad \text{and} \quad \forall h_k \in \Xi. \end{aligned} \quad (7)$$

In the sequel, we consider the PDLF of the form

$$V(\varsigma_k, h_k) = \varsigma_k' Q^{-1}(h_k) \varsigma_k \quad (8)$$

with $Q(h_k) = \sum_{i=1}^r h_{k(i)} Q_i$, $Q_i = Q_i' > 0$. Thus, the level set $\mathcal{L}_V$ obtained from (6) and (8) is an intersection of ellipsoidal sets (Hu et al. (2002); Castelan et al. (2010)):

$$\mathcal{L}_V \triangleq \mathcal{E}(\mathbb{Q}^{-1}(h_k)) = \bigcap_{i \in \{1, \dots, r\}} \mathcal{E}(\mathbb{Q}_i^{-1})$$

where, for $i = 1, \dots, r$: $\mathcal{E}(\mathbb{Q}_i^{-1}) = \{s_k \in \mathbb{R}^{2n}; s_k' \mathbb{Q}_i^{-1} s_k \leq 1\}$.

Let us also consider a (parameter-dependent) polyhedral set given by:

$$S(\rho, h_k) = \{s_k \in \mathbb{R}^{2n}; |\Gamma(h_k)s_k| \leq \rho, \forall h_k \in \Xi\}$$

where, by definition: $\Gamma(h_k) = \sum_{i=1}^r h_{k(i)} \Gamma_i$, $\Gamma_i \in \mathbb{R}^{m \times 2n}$.

Lemma 3. (Castelan et al. (2010)) Given the sets $S(\rho, h_k)$ and $\mathcal{E}(\mathbb{Q}^{-1}(h_k))$, the inclusion $\mathcal{E}(\mathbb{Q}^{-1}(h_k)) \subset S(\rho, h_k)$ is obtained if there exist $\mathbb{U} \in \mathbb{R}^{2n \times 2n}$ such that:

$$\begin{bmatrix} -\mathbb{Q}_i + \mathbb{U} + \mathbb{U}' & \mathbb{U}' \Gamma_i^{(l)} \\ * & \rho_{(l)}^2 \end{bmatrix} \geq 0 \quad (9)$$

$\forall i = 1, \dots, r \quad \forall l = 1, \dots, m$

Also, by setting $\mathbb{Y}(h_k) = \mathbb{K}(h_k) - \Gamma(h_k)$, (9) imply

$$\Psi'(u_k) S^{-1} (\Psi(u_k) - \mathbb{Y}(h_k)s_k) \leq 0 \quad (10)$$

for all $s_k \in \mathcal{E}(\mathbb{Q}^{-1}(h_k))$ and for any diagonal positive definite matrix $S \in \mathbb{R}^{m \times m}$.

Let us suppose that (9) is verified and consider a $\lambda \in (0, 1]$. By considering (10) and Definition 1, the robust λ -contractivity of $\mathcal{E}(\mathbb{Q}^{-1}(h_k))$ is guaranteed if, for some diagonal positive matrix S , the following inequality holds true

$$V(s_{k+1}, h_{k+1}) - \lambda V(s_k, h_k) - 2\Psi'(u_k) S^{-1} (\Psi(u_k) - \mathbb{Y}(h_k)s_k) < 0 \quad (11)$$

In the saturation avoidance case, $|u_k| \leq \rho$, we have $\Psi(u_k) = u_k - \text{sat}(u_k) = 0$, which is obtained by setting $\Gamma(h_k) = \mathbb{K}(h_k) \Leftrightarrow \mathbb{Y}(h_k) = 0$, and $E_{ci} = 0$. Then, (11) reduces to $\Delta V_\lambda(s_k, h_k) < 0$, which is guaranteed if $\exists \mathbb{U} \in \mathbb{R}^{2n \times 2n}$ such that (see Daafouz and Bernussou (2001)):

$$\mathbb{M}_{av}(h_k) = \begin{bmatrix} -\mathbb{Q}(h_{k+1}) & \mathbb{A}_{MF}(h_k)\mathbb{U} \\ * & \lambda(\mathbb{Q}(h_k) - \mathbb{U} - \mathbb{U}') \end{bmatrix} < 0 \quad (12)$$

$\forall h_k \in \Xi$

Lemma 4. (LA-Stability with Saturation Avoidance)

Let A_{ci} , A_{ciq} , B_{ci} , B_{ciq} , C_{ci} and D_{ci} be known matrices that form controller (3). For a given real scalar $\lambda \in (0, 1]$, consider the existence of symmetric positive definite matrices $\mathbb{Q}_i \in \mathbb{R}^{2n \times 2n}$ and of a matrix $\mathbb{U} \in \mathbb{R}^{2n \times 2n}$, satisfying:

$$\mathbb{M}_{ji}^+ = \begin{bmatrix} -\mathbb{Q}_j & (\mathbb{A}_i + \mathbb{B}_i \mathbb{K}_i) \mathbb{U} \\ * & \lambda(\mathbb{Q}_i - \mathbb{U} - \mathbb{U}') \end{bmatrix} < 0 \quad (13)$$

$\forall i, j = 1, \dots, r$

$$\mathbb{M}_{jiq}^+ = \begin{bmatrix} -2\mathbb{Q}_j & (\mathbb{A}_{iq} + \mathbb{B}_i \mathbb{K}_q + \mathbb{B}_q \mathbb{K}_i) \mathbb{U} \\ * & \lambda(\mathbb{Q}_i + \mathbb{Q}_q - 2\mathbb{U} - 2\mathbb{U}') \end{bmatrix} < 0 \quad (14)$$

$\forall j = 1, \dots, r, \quad \forall i = 1, \dots, r-1 \quad \text{and} \quad \forall q = i+1, \dots, r$

$$\mathbb{N}_i = \begin{bmatrix} -\mathbb{Q}_i + \mathbb{U} + \mathbb{U}' & \mathbb{U}' \mathbb{K}_{i(l)}' \\ * & \rho_{(l)}^2 \end{bmatrix} \geq 0 \quad (15)$$

$\forall i = 1, \dots, r \quad \text{and} \quad \forall l = 1, \dots, m$

Then, the set $S_0 \triangleq \bigcap_i \{\mathcal{E}(\mathbb{Q}_i^{-1})\} = \mathcal{E}(\mathbb{Q}^{-1}(h_k))$ is robustly λ -contractive and is included in $S(\rho, h_k) = \{s_k \in \mathbb{R}^{2n}; |\mathbb{K}(h_k)s_k| \leq \rho, \forall h_k \in \Xi\}$.

Proof. We first recall that (15) is used to guarantee the saturation avoidance. From condition (12) and the structure of the matrices defined in (2) and (4) we rewrite

$$\mathbb{A}_{MF}(h_k) = \sum_{i=1}^r h_{k(i)}^2 \underbrace{\begin{bmatrix} A_i + B_i D_{ci} C & B_i C_{ci} \\ B_{ci} C & A_{ci} \end{bmatrix}}_{\mathbb{A}_i + \mathbb{B}_i \mathbb{K}_i} + \sum_{i=1}^{r-1} \sum_{q=i+1}^r h_{k(i)} h_{k(q)} \underbrace{\begin{bmatrix} \mathbb{M}_{a1iq} & \mathbb{M}_{a2iq} \\ B_{ciq} C & A_{ciq} \end{bmatrix}}_{\mathbb{A}_{iq} + \mathbb{B}_i \mathbb{K}_q + \mathbb{B}_q \mathbb{K}_i}$$

where $\mathbb{M}_{a1iq} = (A_i + A_q) + (B_i D_{cq} + B_q D_{ci})C$, $\mathbb{M}_{a2iq} = B_i C_{cq} + B_q C_{ci}$ and $\mathbb{A}_{iq} = \begin{bmatrix} A_i + A_q & 0 \\ B_{ciq} C & A_{ciq} \end{bmatrix}$.

Similarly, we can rewrite $\mathbb{M}_{av}(h_k)$, for instant k , as

$$\mathbb{M}_{av}(h_k) = \sum_{j=1}^r \sum_{i=1}^r h_{k(j)} h_{k(i)}^2 \mathbb{M}_{ji}^+ + \sum_{j=1}^r \sum_{i=1}^{r-1} \sum_{q=i+1}^r h_{k(j)} h_{k(i)} h_{k(q)} \mathbb{M}_{jiq}^+ < 0$$

with $\mathbb{Q}(h_{k+1}) = \sum_{i=1}^r h_{k+1(i)} \mathbb{Q}_i = \sum_{j=1}^r h_{k(j)} \mathbb{Q}_j$. Hence, by considering the admissible parameters values ($\sum_{i=1}^r h_{k(i)} = 1, h_{k(i)} \geq 0$), conditions (13) and (14) guarantee that (12) is verified and we conclude that $\mathcal{E}(\mathbb{Q}^{-1}(h_k)) \subset S(\rho, h_k)$ is robustly λ -contractive. $\square$

In the case where saturations are allowed, $|u_k| > \rho \Leftrightarrow \Psi(u_k) \neq 0$ may happen, and we set $\Gamma(h_k) = \mathbb{K}(h_k) - \mathbb{Y}(h_k)$. Then, (11) is guaranteed if $\exists \mathbb{U} \in \mathbb{R}^{2n \times 2n}$, where $\mathbb{H}(h_k) = \mathbb{Y}(h_k)\mathbb{U}$, such that

$$\mathbb{M}_{al}(h_k) = \left[\begin{array}{c|c} \mathbb{M}_{av}(h_k) & -\bar{\mathbb{B}}(h_k)S \\ * & \mathbb{H}'(h_k) \\ \hline * & * & -2S \end{array} \right] < 0 \quad \forall h_k \in \Xi \quad (16)$$

Lemma 5. (LA-Stability Allowing Saturation)

Let A_{ci} , A_{ciq} , B_{ci} , B_{ciq} , C_{ci} and D_{ci} be known matrices that form controller (3). For a given real scalar $\lambda \in (0, 1]$, consider the existence of symmetric positive definite matrices $\mathbb{Q}_i \in \mathbb{R}^{2n \times 2n}$, of a positive diagonal matrix $S \in \mathbb{R}^{m \times m}$ and of matrices $\mathbb{U} \in \mathbb{R}^{2n \times 2n}$ and $\mathbb{H}_i \in \mathbb{R}^{m \times 2n}$, satisfying:

$$\left[\begin{array}{c|c} \mathbb{M}_{ji}^+ & -\bar{\mathbb{B}}_i S \\ * & \mathbb{H}_i' \\ \hline * & * & -2S \end{array} \right] < 0 \quad (17)$$

$\forall i, j = 1, \dots, r$

$$\left[\begin{array}{c|c} \mathbb{M}_{jiq}^+ & -(\bar{\mathbb{B}}_i + \bar{\mathbb{B}}_q)S \\ * & \mathbb{H}_i' + \mathbb{H}_q' \\ \hline * & * & -4S \end{array} \right] < 0 \quad (18)$$

$\forall j = 1, \dots, r, \quad \forall i = 1, \dots, r-1 \quad \text{and} \quad \forall q = i+1, \dots, r$

$$\left[\begin{array}{c|c} -\mathbb{Q}_i + \mathbb{U} + \mathbb{U}' & \mathbb{U}' \mathbb{K}_{i(l)}' - \mathbb{H}_{i(l)}' \\ * & \rho_{(l)}^2 \end{array} \right] \geq 0 \quad (19)$$

$\forall i = 1, \dots, r \quad \text{and} \quad \forall l = 1, \dots, m$

Then, the set $S_0 \triangleq \bigcap_i \{\mathcal{E}(\mathbb{Q}_i^{-1})\} = \mathcal{E}(\mathbb{Q}^{-1}(h_k))$ is robustly absolutely λ -contractive.

4. MAIN RESULTS

Let us consider the following matrices, adapted from Scherer et al. (1997) and used in Castelan et al. (2010):

$$\mathbb{U} = \begin{bmatrix} X & N \\ Z & \bullet \end{bmatrix}, \quad \mathbb{U}^{-1} = \begin{bmatrix} Y & M \\ W & \bullet \end{bmatrix} \quad \text{and} \quad \Theta = \begin{bmatrix} Y & I \\ W & 0 \end{bmatrix}.$$

For these matrices, we have $\mathbb{U}\Theta = \begin{bmatrix} I & X \\ 0 & Z \end{bmatrix} \Rightarrow \Theta' \mathbb{U} \Theta = \begin{bmatrix} Y' & T' \\ I & X \end{bmatrix}$, where, by construction, $T' = Y'X + W'Z$.

From $\mathbb{U}\mathbb{U}^{-1} = I$ and $\mathbb{U}^{-1}\mathbb{U} = I$, we also have $XY + NW = YX + MZ = I$. Furthermore, by partitioning $\mathbb{Q}_i = \begin{bmatrix} Q_{11i} & Q_{12i} \\ * & Q_{22i} \end{bmatrix}$, we can define: $\Theta' \mathbb{Q}_i \Theta = \begin{bmatrix} \bar{Q}_{11i} & \bar{Q}_{12i} \\ * & \bar{Q}_{22i} \end{bmatrix}$.

Proposition 6. (LA-Stabilization with Saturation Avoidance) Given a real scalar $\lambda \in (0, 1]$, suppose that there exist symmetric positive definite matrices $(\bar{Q}_{11i}, \bar{Q}_{22i}) \in \mathbb{R}^{n \times n}$, and matrices $(\bar{Q}_{12i}, \hat{A}_i, \hat{A}_{iq}, X, Y, T) \in \mathbb{R}^{n \times n}$, $(\hat{B}_i, \hat{B}_{iq}) \in \mathbb{R}^{n \times p}$, $\hat{C}_i \in \mathbb{R}^{m \times n}$ and $\hat{D}_i \in \mathbb{R}^{m \times p}$, satisfying conditions (20), (21) and (22).

Let $W \in \mathbb{R}^{n \times n}$ be any nonsingular matrix, determine:

$$Z = (W')^{-1}(T' - Y'X) \quad (26)$$

Then, the matrices of TDP-DOFC (3) obtained by

$$\begin{aligned} D_{ci} &= \hat{D}_i, \quad C_{ci} = (\hat{C}_i - D_{ci}CX)Z^{-1} \\ B_{ci} &= (W')^{-1}(\hat{B}_i - Y'B_iD_{ci}) \\ A_{ci} &= (W')^{-1}(\hat{A}_i - Y'A_iX - Y'B_iD_{ci}CX - \\ &\quad Y'B_iC_{ci}Z - W'B_{ci}CX)Z^{-1} \\ B_{ciq} &= (W')^{-1}[\hat{B}_{iq} - Y'(B_iD_{cq} + B_qD_{ci})] \\ A_{ciq} &= (W')^{-1}[\hat{A}_{iq} - Y'(A_i + A_q)X - Y'(B_iC_{cq} + B_qC_{ci})Z - \\ &\quad Y'(B_iD_{cq} + B_qD_{ci})CX - W'B_{ciq}CX]Z^{-1} \end{aligned} \quad (27)$$

and the set $S_0 \triangleq \bigcap_i \{\mathcal{E}(\mathbb{Q}_i^{-1})\} = \mathcal{E}(\mathbb{Q}^{-1}(h_k))$, are solutions to Problem 1.

Sketch of the Proof. Pre and post multiplying (13) and (14) by $\text{diag}[\Theta' \quad \Theta']$ and its transpose, and defining the change of variables $\hat{A}_i, \hat{A}_{iq}, \hat{B}_i, \hat{B}_{iq}, \hat{C}_i, \hat{D}_i$ in accordance to (27), we have that (13) is equivalent to (20) and (14) is equivalent to (21). Also, pre and post-multiplication of (15) by $\text{diag}[\Theta' \quad I]$ and its transpose, lead to (22). $\square$

Proposition 7. (LA-Stabilization Allowing Saturation)

Given a real scalar $\lambda \in (0, 1]$, suppose that there exist symmetric positive definite matrices $(\bar{Q}_{11i}, \bar{Q}_{22i}) \in \mathbb{R}^{n \times n}$, a positive diagonal matrix $S \in \mathbb{R}^{m \times m}$ and matrices $(\bar{Q}_{12i}, \hat{A}_i, \hat{A}_{iq}, X, Y, T) \in \mathbb{R}^{n \times n}$, $(\hat{B}_i, \hat{B}_{iq}) \in \mathbb{R}^{n \times p}$, $\hat{C}_i \in \mathbb{R}^{m \times n}$, $\hat{D}_i \in \mathbb{R}^{m \times p}$ and $(\hat{E}_i, \hat{H}_{1i}, \hat{H}_{2i}) \in \mathbb{R}^{n \times m}$, satisfying conditions (23), (24) and (25).

Let W be nonsingular, determine Z by (26). Then, the matrices of TDP-DOFC (3) described in (27) with, in addition,

$$E_{ci} = (W')^{-1}(\hat{E}_iS^{-1} - Y'B_i) \quad (28)$$

and the set $S_0 \triangleq \bigcap_i \{\mathcal{E}(\mathbb{Q}_i^{-1})\} = \mathcal{E}(\mathbb{Q}^{-1}(h_k))$, are solutions to Problem 1.

Remark 8. Propositions 6 and 7 can be adapted to consider the PDP-DOFC commented in Remark 2, by eliminating conditions (21) and (24) and by considering $\hat{C}_i = \hat{C}$ and $\hat{D}_i = \hat{D}$ for the remaining conditions in both propositions.

In several realistic problems, the model (1) is valid only locally (regionally), *i.e.* it describes the system behavior only in a subset of the state space, $\mathcal{V}_0 \subset \mathbb{R}^n$, called domain of validity of the model. In such problems, we must ensure that the states of the controlled system evolve inside the domain of validity $\mathcal{V}_0$, to guarantee the pre-specified stability and degree of performance. Otherwise, if the trajectories of the closed-loop system evolve outside of $\mathcal{V}_0$, where the model (1) does not describe the plant model, or does not sufficiently approximate it, stability and performance cannot be guaranteed. Thus, another condition can be added to the conditions given in propositions 6 and 7, with the aim of ensuring the inclusion of the λ -contractive domain in the region of validity of the closed-loop system, *i.e.*, $\mathcal{L}_V \subseteq \chi_0$, where $\chi_0 \triangleq \left\{ \varsigma_k = \begin{bmatrix} x_k \\ x_{c,k} \end{bmatrix} \in \mathbb{R}^{2n}; \quad x \in \mathcal{V}_0, \quad \forall x_c \in \mathbb{R}^n \right\}$. This inclusion will guarantee that for any initial condition $\varsigma_0 \in \mathcal{L}_V$, the corresponding trajectory will evolve within $\mathcal{L}_V$ and tend asymptotically to the origin.

Consider, for instance, the domain of validity defined by $\mathcal{V}_0 = \text{Co}\{v_\tau \in \mathbb{R}^n, \tau = 1, \dots, n_\tau\} = \{x_k \in \mathbb{R}^n; |Lx_k| \leq \omega\}$. Its extension to the augmented state space of the closed-loop system is given by:

$$\begin{aligned} \chi_0 &= \text{Co}\{\bar{v}_\tau \in \mathbb{R}^{2n}; \quad \bar{v}_\tau = \begin{bmatrix} v'_\tau & 0 \end{bmatrix}', \tau = 1, \dots, n_\tau\} \\ &= \{\varsigma_k \in \mathbb{R}^{2n}; \quad |L_a \varsigma_k| \leq \omega\}, \quad \text{with } L_a = \begin{bmatrix} L & 0 \end{bmatrix}. \end{aligned}$$

The inclusion $\mathcal{L}_V \subseteq \chi_0$ can be described by the following convex constraint, similar to (9):

$$\begin{bmatrix} -\mathbb{Q}_i + \mathbb{U}' + \mathbb{U} \mathbb{U}' L_{a(\ell)}' \\ * \quad \omega_{(\ell)}^2 \end{bmatrix} \geq 0 \quad \forall i = 1, \dots, r \quad \text{and} \quad \forall \ell = 1, \dots, 2n$$

equivalent, for the computational purpose, to:

$$\begin{bmatrix} -\bar{Q}_{11i} + Y' + Y & -\bar{Q}_{12i} + T' + I & L_{a(\ell)}' \\ -\bar{Q}_{12i}' + T + I & -\bar{Q}_{22i} + X' + X & X_{(\ell)}' \\ * & * & \omega_{(\ell)}^2 \end{bmatrix} \geq 0. \quad (29)$$

$\forall i = 1, \dots, r \quad \text{and} \quad \forall \ell = 1, \dots, n$

For the optimization of the size of the contractive domain $\mathcal{L}_V \subset \chi_0$, we can adopt a shape set model for synthesis. This set can be chosen, for instance, with the same shape as the domain of validity χ_0 . The objective is to maximize $\beta > 0$ such that the inclusion $\beta\chi_0 \subseteq \mathcal{L}_V$ is also verified. Considering $\mu = 1/\beta^2$, this inclusion reads

$$\begin{bmatrix} \mu & v'_\sigma \\ * & \mathbb{Q}_i \end{bmatrix} \geq 0 \Leftrightarrow \begin{bmatrix} \mu & v'_\tau Y & v'_\tau \\ * & \bar{Q}_{11i} & \bar{Q}_{12i} \\ * & * & \bar{Q}_{22i} \end{bmatrix} \geq 0. \quad (30)$$

$\forall i = 1, \dots, r \quad \text{and} \quad \forall \tau = 1, \dots, n_\tau$

$$\begin{bmatrix} -\bar{Q}_{11j} & -\bar{Q}_{12j} & Y' A_i + \hat{B}_i C & \hat{A}_i \\ * & -\bar{Q}_{22j} & A_i + B_i \hat{D}_i C & A_i X + B_i \hat{C}_i \\ * & * & \lambda(\bar{Q}_{11i} - Y' - Y) & \lambda(\bar{Q}_{12i} - T' - I) \\ * & * & * & \lambda(\bar{Q}_{22i} - X' - X) \end{bmatrix} < 0 \quad (20)$$

$\forall i, j = 1, \dots, r$

$$\begin{bmatrix} -2\bar{Q}_{11j} & -2\bar{Q}_{12j} & Y' (A_i + A_q) + \hat{B}_{iq} C & \hat{A}_{iq} \\ * & -2\bar{Q}_{22j} & A_i + A_q + (B_i \hat{D}_q + B_q \hat{D}_i) C & (A_i + A_q) X + B_i \hat{C}_q + B_q \hat{C}_i \\ * & * & \lambda(\bar{Q}_{11i} + \bar{Q}_{11q} - 2Y' - 2Y) & \lambda(\bar{Q}_{12i} + \bar{Q}_{12q} - 2T' - 2I) \\ * & * & * & \lambda(\bar{Q}_{22i} + \bar{Q}_{22q} - 2X' - 2X) \end{bmatrix} < 0 \quad (21)$$

$\forall j = 1, \dots, r, \quad \forall i = 1, \dots, r-1 \quad \text{and} \quad \forall q = i+1, \dots, r$

$$\begin{bmatrix} -\bar{Q}_{11i} + Y + Y' & -\bar{Q}_{12i} + T' + I & (\hat{D}_{i(l)} C)' \\ * & -\bar{Q}_{22i} + X' + X & \hat{C}_{i(l)} \\ * & * & \rho_{(l)}^2 \end{bmatrix} \geq 0 \quad (22)$$

$\forall i = 1, \dots, r \quad \text{and} \quad \forall l = 1, \dots, m$

$$\begin{bmatrix} -\bar{Q}_{11j} & -\bar{Q}_{12j} & Y' A_i + \hat{B}_i C & \hat{A}_i & \hat{E}_i \\ * & -\bar{Q}_{22j} & A_i + B_i \hat{D}_i C & A_i X + B_i \hat{C}_i & -B_i S \\ * & * & \lambda(\bar{Q}_{11i} - Y' - Y) & \lambda(\bar{Q}_{12i} - T' - I) & \hat{H}_{1i}' \\ * & * & * & \lambda(\bar{Q}_{22i} - X' - X) & \hat{H}_{2i}' \\ * & * & * & * & -2S \end{bmatrix} < 0 \quad (23)$$

$\forall i, j = 1, \dots, r$

$$\begin{bmatrix} -2\bar{Q}_{11j} & -2\bar{Q}_{12j} & Y' (A_i + A_q) + \hat{B}_{iq} C & \hat{A}_{iq} & \hat{E}_i + \hat{E}_q \\ * & -2\bar{Q}_{22j} & A_i + A_q + (B_i \hat{D}_q + B_q \hat{D}_i) C & (A_i + A_q) X + B_i \hat{C}_q + B_q \hat{C}_i & -(B_i + B_q) S \\ * & * & \lambda(\bar{Q}_{11i} + \bar{Q}_{11q} - 2Y' - 2Y) & \lambda(\bar{Q}_{12i} + \bar{Q}_{12q} - 2T' - 2I) & \hat{H}_{1i}' + \hat{H}_{1q}' \\ * & * & * & \lambda(\bar{Q}_{22i} + \bar{Q}_{22q} - 2X' - 2X) & \hat{H}_{2i}' + \hat{H}_{2q}' \\ * & * & * & * & -4S \end{bmatrix} < 0 \quad (24)$$

$\forall j = 1, \dots, r, \quad \forall i = 1, \dots, r-1 \quad \text{and} \quad \forall q = i+1, \dots, r$

$$\begin{bmatrix} -\bar{Q}_{11i} + Y + Y' & -\bar{Q}_{12i} + T' + I & (\hat{D}_{i(l)} C)' - \hat{H}_{1i(l)}' \\ * & -\bar{Q}_{22i} + X' + X & \hat{C}_{i(l)} - \hat{H}_{2i(l)}' \\ * & * & \rho_{(l)}^2 \end{bmatrix} \geq 0 \quad (25)$$

$\forall i = 1, \dots, r \quad \text{and} \quad \forall l = 1, \dots, m$

Notice that the addition of conditions (29) and (30) to propositions 6 and 7 only changes the numerical complexity of the proposed stability conditions. For the synthesis of a TDP-DOFC with saturation avoidance (SatAv), we propose the following convex programming problem (CPP):

$$\begin{aligned} & \min_{\substack{\bar{Q}_{11i}, \bar{Q}_{12i}, \bar{Q}_{22i}, \hat{A}_i, \hat{A}_{ciq}, \\ X, Y, T, \hat{B}_i, \hat{B}_{ciq}, \hat{C}_i, \hat{D}_i}} \mu \\ & \text{subject to} \\ & \text{LMIs (20), (21), (22) and (30).} \end{aligned} \quad (31)$$

And, for the TDP-DOFC allowing saturation (SatAl)

$$\begin{aligned} & \min_{\substack{\bar{Q}_{11i}, \bar{Q}_{12i}, \bar{Q}_{22i}, \hat{G}_i, \hat{A}_i, \hat{A}_{ciq}, S \\ X, Y, T, \hat{B}_i, \hat{B}_{ciq}, \hat{C}_i, \hat{D}_i, \hat{Z}_{1i}, \hat{Z}_{2i}}} \mu \\ & \text{subject to} \\ & \text{LMIs (23), (24), (25) and (30).} \end{aligned} \quad (32)$$

Remark 9. Similar CPPs can be used for the synthesis of PDP-DOFC considered in previous remarks 2 and 8.

5. APPLICATION TO A T-S FUZZY SYSTEM

Consider the equations representing the problem of balancing an inverted pendulum on a car:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g \sin(x_1) - a m l x_2^2 \sin(2x_1) / 2 - a \cos(x_1) u}{4l/3 - a m l \cos^2(x_1)} \end{aligned} \quad (33)$$

with: x_1 , the angle (in radians) of the pendulum about the vertical; x_2 , the angular velocity; $g = 9.8m/s^2$, the acceleration of gravity; m , the mass of the pendulum; M , the mass of the car; $2l$ the length of the pendulum; and u , the force applied to the car. By definition: $a = 1/(m + M)$.

The system in (33) can be described by a linear continuous-time model with varying parameters, called Takagi-Sugeno fuzzy model. The region of validity of the model, $\mathcal{V}_0$ is defined by the intervals $x_1 \in [-x_{1max}, x_{1max}]$ and $x_2 \in [-x_{2max}, x_{2max}]$, with $x_{1max} = \pi/3$ e $x_{2max} = 5$. Thus, we have (Tanaka and Wang (2001)):

$$\dot{x} = \sum_{j=1}^2 \sum_{k=1}^2 \sum_{l=1}^2 \sum_{m=1}^2 E_j M_k N_l S_m \{ \hat{A}_{jklm} x + \hat{B}_{jklm} u \}$$

$$\text{with } \hat{A}_{jklm} = \begin{bmatrix} 0 & 1 \\ g q_j b_k - \frac{a m l}{2} q_j c_l \end{bmatrix} \text{ and } \hat{B}_{jklm} = \begin{bmatrix} 0 \\ -a q_j d_m \end{bmatrix}$$

which, aggregating the sums, can be written:

$$\dot{x} = \sum_{i=1}^{16} h_i \{ \hat{A}_i x + \hat{B}_i u \} \quad (34)$$

where $i = m + 2(l-1) + 4(k-1) + 8(j-1)$ and $h_i = E_j M_k N_l S_m$ with $E_1 = \frac{z_1 - q_2}{q_1 - q_2}$, $E_2 = \frac{q_1 - z_1}{q_1 - q_2}$, $M_1 = \frac{\sin(x_1) - (\frac{2}{\pi}) z_2}{(1 - \frac{2}{\pi}) z_2}$, $M_2 = \frac{x_1 - z_2}{(1 - \frac{2}{\pi}) z_2}$, $N_1 = \frac{z_3 - c_2}{c_1 - c_2}$, $N_2 = \frac{c_1 - z_3}{c_1 - c_2}$, $S_1 = \frac{z_4 - d_2}{d_1 - d_2}$, $S_2 = \frac{d_1 - z_4}{d_1 - d_2}$, $z_1 = \frac{4l}{3} - a m l \cos^2(x_1)$, $z_2 = \sin(x_1)$, $z_3 = x_2 \sin(2x_1)$, $z_4 = \cos(x_1)$

and $q_1 = \max(z_1)$, $q_2 = \min(z_1)$, $b_1 = 1$, $b_2 = \frac{2}{\pi}$, $c_1 = \max(z_3)$, $c_2 = \min(z_3)$, $d_1 = \max(z_4)$, $d_2 = \min(z_4)$.

The maximum and minimum are calculated for $\mathcal{V}_0$. We choose: $m = 2.0kg$, $M = 8.00kg$, $2l = 1.0m$ and $\rho = 300N$. For each rule in (34) and using the Matlab command *c2d* with a fixed sampling period, $T_s = 0.1s$, we obtain the matrices that constitute (1): A_i, B_i and $C = [1 \ 0]$. To consider the region of validity $\mathcal{V}_0$, we define: $\omega = [x_{1max} \ x_{2max}]'$, $L = I_2$ and $v_\tau = \left\{ \begin{bmatrix} x_{1max} \\ x_{2max} \end{bmatrix}, \begin{bmatrix} x_{1max} \\ -x_{2max} \end{bmatrix}, \begin{bmatrix} -x_{1max} \\ x_{2max} \end{bmatrix}, \begin{bmatrix} -x_{1max} \\ -x_{2max} \end{bmatrix} \right\}$.

In figures 1 ($\lambda = 1$) and 2 ($\lambda = 0.7$) we show the regions of stability (RS) and confinement (RC) of the trajectories in the state space of the plant, represented by “•”, for SatAv, and “+”, for SatAl, respectively. Each RS is given by

$$S_{0r} \triangleq \bigcap_i \{ \mathcal{E}_r(\mathcal{Q}_i^{-1}) \}, \mathcal{E}_r(\mathcal{Q}_i^{-1}) = \left\{ x \in \mathbb{R}^2; \begin{bmatrix} x \\ 0 \end{bmatrix} \mathcal{Q}_i^{-1} \begin{bmatrix} x \\ 0 \end{bmatrix} \leq 1 \right\},$$

and gives the set of initial conditions, $s_0 = [x'_0 \ 0]'$, for which the trajectories of the system is contained in RC. The RC region is the orthogonal projection of $\mathcal{L}_V = \mathcal{E}(\mathcal{Q}^{-1}(h_k))$ on the plane formed by the states of the plant

$$S_{0p} = \{ s_{kproj} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} s_k \mid s_k \in \mathcal{L}_V \triangleq \bigcap_i \mathcal{E}(\mathcal{Q}_i^{-1}) \}.$$

We see that the size of S_{0r} in the SatAv case is more

Fig. 1. Stabilization and confinement regions for $\lambda = 1$

conservative than the size obtained in the SatAl case. Also, we can note that the requirement for a higher performance (lower contractive factor) causes a decrease in RS and RC regions, i.e. we verify the trade-off between performance and size of the regions of stability and confinement.

6. CONCLUSION

We have extended the results presented in (Castelan et al. (2010)) to design a dynamic output feedback controller that is totally dependent of the parameters. A main practical aspect discussed in the present paper has been about the domain of validity for the considered parameter-varying system, which can be addressed in the design phase. A numerical example issued from the Takagi-Sugeno model of a nonlinear system has been considered.

Fig. 2. Stabilization and confinement regions for $\lambda = 0.7$

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