$\mathcal{L}_1$ Adaptive Controller for Uncertain Nonlinear Multi-Input Multi-Output Systems with Input Quantization

Hui Sun * Naira Hovakimyan * Tamer Başar *

* Coordinated Science Laboratory, University of Illinois at
Urbana-Champaign, Urbana, IL 61801 USA (e-mail: huisun1@illinois.edu,
nhovakim@illinois.edu, basar1@illinois.edu).

Abstract: We address the problem of tracking for a general class of uncertain nonlinear MIMO systems with input quantization, without requiring any matching conditions. We consider the $\mathcal{L}_1$ adaptive controller and analyze its performance bounds in the presence of input quantization. We study two common types of quantization, the uniform quantization and the logarithmic quantization. In both cases we provide the transient performance bounds, which are decoupled into two positive terms. One of these terms can be made arbitrarily small by increasing the rate of adaptation, while the other term can be made small by increasing the quantization density. The performance bounds imply that with sufficiently dense quantization and fast adaptation, the output of an uncertain MIMO nonlinear system can follow the desired reference input sufficiently closely. We notice that with $\mathcal{L}_1$ adaptive control architecture fast adaptation does not lead to high-gain control and retains guaranteed time-delay margin, which is bounded away from zero. Simulations included in the paper illustrate the results.

Keywords: adaptive control, quantized systems, quantization errors

1. INTRODUCTION

The field of adaptive control emerged to provide solutions for control of systems in the presence of uncertainty and component failure. Despite the significant advances over the years, the transient performance analysis of adaptive controllers was reduced to persistency of excitation type assumption, or alternatively it was achieved via gain-scheduling, and with that defeating the point of adaptation. The recently developed $\mathcal{L}_1$ adaptive controller has guaranteed transient and steady-state performance bounds without introducing persistency of excitation or gain-scheduling in the controller parameters. $\mathcal{L}_1$ adaptive controller introduces a separation between adaptation and robustness, and for the first time allows for arbitrary increase of the rate of adaptation with guaranteed time-delay margin, bounded away from zero (Cao and Hovakimyan (2008, 2010); Hovakimyan and Cao (2010)). Similar to linear systems, with $\mathcal{L}_1$ adaptive control architecture the performance of uncertain systems scales uniformly with all the variations in initial conditions, unknown parameters and reference inputs. In this paper we obtain performance bounds for the $\mathcal{L}_1$ adaptive controller in the presence of input quantization.

As a brief review of the quantization literature (without adaptation), we note that the effect of quantization on stability was first investigated in Delchamps (1990), relaxation of the definition of stability was undertaken in Wong and Brockett (1999); Ishii et al. (2004), the design of coarse quantization was investigated in Elia and Mitter (2001), and stabilization of quantized systems was analyzed in Ishii and Başar (2005); Brockett and Liberzon (2000). In Sun et al. (2010), we obtained performance bounds for the $\mathcal{L}_1$ adaptive controller for a SISO LTI system

* Research presented here is supported in part by AFOSR under Grant Nos FA9550-09-1-0157 and FA9550-09-1-0249.

in the presence of input quantization. Here we extend that result to general nonlinear MIMO systems, without requiring any matching conditions. The issues addressed are not only the extension from SISO to MIMO systems, but also from linear to nonlinear, and from systems with matched uncertainties to those with both matched and unmatched uncertainties. For that, we refer to Xargay et al. (2010) for the general architecture of $\mathcal{L}_1$ adaptive controller and analyze the performance bounds of this architecture in the presence of logarithmic as well as uniform quantization.

The paper is organized as follows. We formulate the problem in Section 2 and introduce the $\mathcal{L}_1$ adaptive controller in Sections 3 and 4. Performances in the two cases of uniform and logarithmic quantization are studied in Sections 6 and 5, followed by simulations in Section 7. Section 8 concludes the paper.

Throughout the paper, we use $\|\cdot\|_\infty$ to denote the infinity norm of a vector, $\|\cdot\|_{\mathcal{L}_\infty}$ for the $\mathcal{L}_\infty$ norm of a signal, and $\|\cdot\|_{\mathcal{L}_1}$ for the $\mathcal{L}_1$ norm of a transfer function.

2. PROBLEM FORMULATION

We consider a general adaptive system with input quantization. As depicted in Figure 1, the state-feedback control designed (that is the output of the controller), $u_{\text{qin}}(t)$, is quantized before it is input to the plant, with that input denoted by $u_q(t)$. The controller itself consists of two tandem blocks: $\mathcal{L}_1$ adaptive controller followed by a preprocessor which generates the input, $u_{\text{qin}}(t)$, to the quantizer.

The quantization block can be characterized by a map $Q : U_{\text{qin}} \rightarrow U_q$, $U_{\text{qin}} \subset \mathbb{R}$, $U_q \subset \mathbb{R}$, which maps a continuous range of values into a set of countably many elements. In

Fig. 1. System with Quantized Input

general, a quantizer induces a partition of U_{qin} . For each subset of U_{qin} , the quantizer outputs a single element in U_q .

Let the system dynamics be described by the following set of equations:

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + B_m \omega u_q(t) + f(x(t), t), \quad x(0) = x_0, \\ y(t) &= C x(t), \\ u_q(t) &= Q(u_{\text{qin}}(t)), \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the system state vector (measured), $\|x_0\|_\infty \leq \rho_0$; $u_q(t) \in \mathbb{R}^m$ ($n \geq m$) is the quantized control; $y(t) \in \mathbb{R}^m$ is the regulated output; A_m is a known Hurwitz $n \times n$ matrix; $B_m \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{m \times n}$ are known constant matrices; $\omega \in \mathbb{R}^{m \times m}$ is the unknown high-frequency gain matrix; $f(\cdot)$ is an unknown nonlinear function; $u_{\text{qin}}(t)$ is the designed control signal, and $Q(\cdot)$ is the quantization function.

We can rewrite the system in (1) as

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + B_m (\omega u_q(t) + f_1(x(t), t)) + B_{um} f_2(x(t), t), \\ y(t) &= C x(t), \quad x(0) = x_0, \\ u_q(t) &= Q(u_{\text{qin}}(t)), \end{aligned} \quad (2)$$

where $B_{um} \in \mathbb{R}^{n \times (n-m)}$ is a constant matrix such that $B_m^\top B_{um} = 0$ and $\text{rank}([B_m \ B_{um}]) = n$; and $f_1(\cdot)$, $f_2(\cdot)$, are unknown nonlinear functions, such that

$$\begin{bmatrix} f_1(x(t), t) \\ f_2(x(t), t) \end{bmatrix} = [B_m \ B_{um}]^{-1} f(x(t), t).$$

Here $B_m f_1(\cdot)$ represents the matched part of the uncertainties, whereas $B_{um} f_2(\cdot)$ represents the unmatched part (cross-coupling dynamics).

As in the case of the system without quantization Xargay et al. (2010), we introduce the following assumptions.

Assumption 1. [Conservative knowledge of the high-frequency gain matrix] The high-frequency gain matrix ω is assumed to be an unknown (nonsingular) strictly row-diagonally dominant matrix with $\text{sgn}(\omega_{ii})$ known. Also, we assume that there exists a known compact set Ω , such that $\omega \in \Omega \subset \mathbb{R}^{m \times m}$.

Assumption 2. [Uniform boundedness of $f_i(0, t)$] There exists $B_{10} > 0$ and $B_{20} > 0$, such that $\|f_i(0, t)\| \leq B_{i0}$, $i = 1, 2$, uniformly in t .

Assumption 3. [Semiglobal uniform boundedness of partial derivatives] For $i = 1, 2$, and any $\nu > 0$, there exist $d_{f_{xi}}(\nu)$ and $d_{f_{ti}}(\nu) > 0$ such that, for any $\|x(t)\|_\infty < \nu$, the partial derivatives of $f_i(x, t)$ are piecewise continuous and bounded:

$$\|\partial f_i(x, t)/\partial x\| \leq d_{f_{xi}}(\nu), \quad \|\partial f_i(x, t)/\partial t\| \leq d_{f_{ti}}(\nu).$$

Assumption 4. [Stability of matched transmission zeros] The transmission zeros of the transfer matrix $H_m(s) = C(s\mathbb{I} - A_m)^{-1}B_m$ from the quantized input $u_q(t)$ to the output of the system $y(t)$ lie in the open left-half plane.

Now, given a piecewise continuous, bounded reference signal $r(t)$, the control objective is to design an adaptive state feedback controller to ensure that $y(t)$ tracks the output $y_m(t)$ of the system $y_m(s) = M(s)r(s) \triangleq C(s\mathbb{I} - A_m)^{-1}B_m K_g(s)r(s)$, where $K_g(s)$ is a feedforward prefilter.

3. $\mathcal{L}_1$ ADAPTIVE CONTROLLER

3.1 Notations

In the following analysis, we use the notations listed below. Let $\mathcal{R}_p^{k \times l}(s)$ and $\mathcal{RH}_\infty^{k \times l}(s)$ denote the set of real rational proper transfer matrices of dimensions $k \times l$ and the set of real rational proper and Hurwitz-stable transfer matrices of dimensions $k \times l$, respectively.

For any $\nu > 0$, let

$$L_{i\nu} \triangleq \frac{\bar{\nu}}{\nu} d_{f_{xi}}(\bar{\nu}), \quad (3)$$

where $\bar{\nu} \triangleq \nu + \bar{\gamma}_x$, and $\bar{\gamma}_x > 0$ is an arbitrary, small constant. Let $H_{xm}(s) \triangleq (s\mathbb{I}_n - A_m)^{-1}B_m$, $H_{xum}(s) \triangleq (s\mathbb{I}_n - A_m)^{-1}B_{um}$, $H_m(s) \triangleq C H_{xm}(s) = C(s\mathbb{I}_n - A_m)^{-1}B_m$, $H_{um}(s) \triangleq C H_{xum}(s) = C(s\mathbb{I}_n - A_m)^{-1}B_{um}$, and $r_0(t)$ be the signal with Laplace transform $(s\mathbb{I}_n - A_m)^{-1}x_0$. Since A_m is Hurwitz and x_0 is finite, $\|r_0\|_{\mathcal{L}_\infty}$ is bounded.

The design of $\mathcal{L}_1$ adaptive controller involves a strictly proper transfer matrix $D(s) \in \mathcal{R}_p^{m \times m}(s)$ and a matrix gain $K \in \mathbb{R}^{m \times m}$, which lead to a strictly proper stable transfer matrix

$$C_u(s) \triangleq \omega K (\mathbb{I}_m + D(s)\omega K)^{-1} D(s) \quad (4)$$

with DC gain $C(0) = \mathbb{I}_m$. The choice of $D(s)$ needs to ensure also that $C_u(s)H_m^{-1}(s)$ is a proper stable transfer matrix, i.e. $C_u(s)H_m^{-1}(s) \in \mathcal{RH}_\infty^{m \times m}(s)$. For a particular class of systems, a possible choice for $D(s)$ might be $D(s) = \frac{1}{s}\mathbb{I}_m$, which yields a strictly proper $C_u(s)$ of the form $C_u(s) = \omega K (s\mathbb{I}_m + \omega K)^{-1}$, with the condition that the choice of K must ensure that $-\omega K$ is Hurwitz.

For proofs of stability and performance bounds, the choices of $D(s)$ and K need to ensure that there exists $\rho_{x_r} > \rho_0$, such that

$$\begin{aligned} \|G_m(s)\|_{\mathcal{L}_1} + \|G_{um}(s)\|_{\mathcal{L}_1} \ell_0 &\leq \\ \frac{\rho_{x_r} - \|H_{xm}(s)C_u(s)K_g(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} - \|s(s\mathbb{I} - A_m)^{-1}\|_{\mathcal{L}_1} \rho_0}{L_1 \rho_{x_r} \rho_{x_r} + B_0}, \end{aligned} \quad (5)$$

$$G_m(s) \triangleq H_{xm}(s) (\mathbb{I}_m - C_u(s)), \quad (6)$$

$$G_{um}(s) \triangleq (\mathbb{I}_n - H_{xm}(s)C_u(s)H_m^{-1}(s)C) H_{xum}(s), \quad (7)$$

$\ell_0 \triangleq L_2 \rho_{x_r} / L_1 \rho_{x_r}$, $B_0 \triangleq \max\{B_{10}, \frac{B_{20}}{\ell_0}\}$, and $K_g(s) \in \mathcal{RH}_\infty^{m \times m}(s)$ is the feedforward prefilter.

For the analysis results in the following sections to hold, the inequality in (5) is only a sufficient condition and is required only for the real ω . However, since ω is unknown, (5) can not be used to guide the filter design. More conservatively, we can make the choices of $D(s)$ and K to ensure that for all $\omega \in \Omega$ there exists a constant $\rho_{x_r} > \rho_0$, such that (5) holds.

Define the constants

$$\begin{aligned} \rho_{u_r} &\triangleq \|\omega^{-1}C_u(s)\|_{\mathcal{L}_1} (L_1 \rho_{x_r} \rho_{x_r} + B_{10}) \\ &\quad + \|\omega^{-1}C_u(s)H_m^{-1}(s)H_{um}(s)\|_{\mathcal{L}_1} (L_2 \rho_{x_r} \rho_{x_r} + B_{20}) \\ &\quad + \|\omega^{-1}C_u(s)K_g(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty}, \end{aligned} \quad (8)$$

$$\begin{aligned} c_{ue} &= \|H_{xm}(s)\omega\|_{\mathcal{L}_1} + \|H_{xm}(s)C_u(s)\|_{\mathcal{L}_1} 2\omega_1 \max, \\ c_{xe} &= \|H_{xm}(s)C_u(s)H_m^{-1}(s)C\|_{\mathcal{L}_1}, \end{aligned} \quad (9)$$

where $\omega_{1\max} = \max_{\omega \in \Omega} \max_{i=1,\dots,m} \|\omega_i\|_1$, and ω_i is the i^{th} row of ω .

Define the functions

$$c_{de}(a, b) = \|G_m(s)\|_{\mathcal{L}_1} a + \|G_{um}(s)\|_{\mathcal{L}_1} b, \quad (10)$$

$$c_{eu}(a, b) = \|\omega^{-1} C_u(s)\|_{\mathcal{L}_1} a + \|\omega^{-1} C_u(s) H_m^{-1}(s) H_{um}(s)\|_{\mathcal{L}_1} b.$$

3.2 $\mathcal{L}_1$ Adaptive Controller

The $\mathcal{L}_1$ adaptive controller consists of three elements: the state predictor, the adaptive law and the control law. We now introduce these three elements, separately.

State predictor: The state predictor is generated by

$$\begin{aligned} \hat{x}(t) = & A_m \hat{x}(t) + B_m (\hat{\omega}(t) u_q(t) + \hat{\theta}_1(t) \|x_t\|_{\mathcal{L}_\infty} + \hat{\sigma}_1(t)) \\ & + B_{um} (\hat{\theta}_2(t) \|x_t\|_{\mathcal{L}_\infty} + \hat{\sigma}_2(t)), \hat{x}(0) = x_0, \end{aligned} \quad (11)$$

where $\hat{\omega}(t) \in \mathbb{R}^{m \times m}$, $\hat{\theta}_1(t) \in \mathbb{R}^m$, $\hat{\sigma}_1(t) \in \mathbb{R}^m$, $\hat{\theta}_2(t) \in \mathbb{R}^{n-m}$, and $\hat{\sigma}_2(t) \in \mathbb{R}^{n-m}$ are the adaptive estimates.

Adaptive laws: The adaptation laws for $\hat{\omega}(t)$, $\hat{\theta}_1(t)$, $\hat{\sigma}_1(t)$, $\hat{\theta}_2(t)$, and $\hat{\sigma}_2(t)$ are defined as

$$\begin{aligned} \dot{\hat{\omega}}(t) = & \Gamma \text{Proj}(\hat{\omega}(t), -(\tilde{x}^\top(t) P B_m)^\top u_q^\top(t)), \\ \dot{\hat{\theta}}_1(t) = & \Gamma \text{Proj}(\hat{\theta}_1(t), -(\tilde{x}^\top(t) P B_m)^\top \|x_t\|_{\mathcal{L}_\infty}), \\ \dot{\hat{\sigma}}_1(t) = & \Gamma \text{Proj}(\hat{\sigma}_1(t), -(\tilde{x}^\top(t) P B_m)^\top), \\ \dot{\hat{\theta}}_2(t) = & \Gamma \text{Proj}(\hat{\theta}_2(t), -(\tilde{x}^\top(t) P B_{um})^\top \|x_t\|_{\mathcal{L}_\infty}), \\ \dot{\hat{\sigma}}_2(t) = & \Gamma \text{Proj}(\hat{\sigma}_2(t), -(\tilde{x}^\top(t) P B_{um})^\top), \end{aligned} \quad (12)$$

where $\hat{\omega}(0) = \hat{\omega}_0$, $\hat{\theta}_i(0) = \hat{\theta}_{i0}$, $\hat{\sigma}_i(0) = \hat{\sigma}_{i0}$, $i = 1, 2$; $\tilde{x}(t) = \hat{x}(t) - x(t)$; $\Gamma \in \mathbb{R}^+$ is the adaptation gain; $P = P^\top > 0$ is the solution of the algebraic Lyapunov equation $A_m^\top P + P A_m = -Q$, $Q = Q^\top > 0$; and $\text{Proj}(\cdot, \cdot)$ denotes the projection operator (Pomet and Praly (1992)).

Control law: We first design the control signal $u(t)$, and then the input for the quantization $u_{\text{qin}}(t)$ is designed according to different types of quantizers. The control law is generated by

$$u(t) = -K\chi(t), \quad \chi(s) = D(s)\hat{\eta}(s), \quad (13)$$

where $\hat{\eta}(s)$ is the Laplace transform of the signal

$$\hat{\eta}(t) = \hat{\omega}(t) u_q(t) + \hat{\eta}_1(t) + \hat{\eta}_{2m}(t) - r_g(t), \quad (14)$$

with $r_g(s) = K_g(s)r(s)$, $\hat{\eta}_{2m}(s) = H_m^{-1}(s)H_{um}(s)\hat{\eta}_2(s)$, and $\hat{\eta}_1(t)$ and $\hat{\eta}_2(t)$ are defined by

$$\hat{\eta}_i(t) = \hat{\theta}_i(t) \|x_t\|_{\mathcal{L}_\infty} + \hat{\sigma}_i(t), \quad i = 1, 2. \quad (15)$$

Conventional design methods from multivariable control theory can be used to design the prefilter $K_g(s)$ to achieve desired decoupling properties. As an example, if one chooses $K_g(s)$ as the constant matrix $K_g = -(C A_m^{-1} B_m)^{-1}$, then the diagonal elements of the desired transfer matrix $M(s) = C(sI_n - A_m)^{-1} B_m K_g$ have DC gain equal to one, while the off-diagonal elements have zero DC gain.

4. ANALYSIS OF $\mathcal{L}_1$ ADAPTIVE CONTROLLER

4.1 Closed-Loop Reference System

Let the closed-loop reference system be given as:

$$\begin{aligned} \dot{x}_{\text{ref}}(t) = & A_m x_{\text{ref}}(t) + B_m (\omega u_{\text{ref}}(t) + f_1(x_{\text{ref}}(t), t)) \\ & + B_{um} f_2(x_{\text{ref}}(t), t), x_{\text{ref}}(0) = x_0 \\ u_{\text{ref}}(s) = & -\omega^{-1} C_u(s) (\eta_{1\text{ref}}(s) + H_m^{-1}(s) H_{um}(s) \eta_{2\text{ref}}(s) \\ & - K_g(s)r(s)) \\ y_{\text{ref}}(t) = & C x_{\text{ref}}(t), \end{aligned} \quad (16)$$

where $\eta_{1\text{ref}}(s)$ and $\eta_{2\text{ref}}(s)$ are the Laplace transforms of the signals $\eta_{i\text{ref}}(t) = f_i(x_{\text{ref}}(t), t)$, $i = 1, 2$.

Lemma 1. (Xargay et al. (2010)). For the closed-loop reference system in (16), subject to the $\mathcal{L}_1$ -norm condition (5), if $\|x_0\|_\infty < \rho_{x_r}$, then $\|x_{\text{ref}t}\|_{\mathcal{L}_\infty} < \rho_{x_r}$, $\|u_{\text{ref}t}\|_{\mathcal{L}_\infty} < \rho_{u_r}$, where ρ_{x_r} and ρ_{u_r} are defined in (5) and (8), respectively.

4.2 Equivalent Linear Time-Varying System

In this subsection we show that the nonlinear system with unmodeled dynamics as in (2) can be transformed into a linear system with unknown time-varying parameters and time-varying disturbances.

Lemma 2. (Xargay et al. (2010)). For the system in (2), if there exist ρ_x and ρ_u , such that

$$\|x_t\|_{\mathcal{L}_\infty} \leq \rho_x, \quad \|u_t\|_{\mathcal{L}_\infty} \leq \rho_u, \quad (17)$$

then for all $\tau \in [0, t]$, there exist differentiable $\theta_1(\tau) \in \mathbb{R}^m$, $\sigma_1(\tau) \in \mathbb{R}^m$, $\theta_2(\tau) \in \mathbb{R}^{n-m}$, and $\sigma_2(\tau) \in \mathbb{R}^{n-m}$ such that

$$\|\theta_i(\tau)\|_\infty < \theta_{b_i}(\rho_x), \quad \|\dot{\theta}_i(\tau)\|_\infty < d_{\theta_i}(\rho_x), \quad (18)$$

$$\|\sigma_i(\tau)\|_\infty < \sigma_{b_i}(\rho_x), \quad \|\dot{\sigma}_i(\tau)\|_\infty < d_{\sigma_i}(\rho_x), \quad (19)$$

$$f_i(x(\tau), \tau) = \theta_i(\tau) \|x_\tau\|_{\mathcal{L}_\infty} + \sigma_i(\tau), \quad i = 1, 2, \quad (20)$$

where θ_{b_i} and σ_{b_i} are given by $\theta_{b_i}(\rho_x) \triangleq L_{i\rho_x}$, $\sigma_{b_i}(\rho_x) \triangleq B_{i0}$.

If the bounds in (17) hold, then Lemma 2 implies that the system in (2) can be rewritten over $\tau \in [0, t]$ as

$$\begin{aligned} \dot{x}(\tau) = & A_m x(\tau) + B_m (\omega u_q(\tau) + \theta_1(\tau) \|x_\tau\|_{\mathcal{L}_\infty} + \sigma_1(\tau)) \\ & + B_{um} (\theta_2(\tau) \|x_\tau\|_{\mathcal{L}_\infty} + \sigma_2(\tau)), \quad x(0) = x_0, \\ y(\tau) = & C x(\tau), \end{aligned} \quad (21)$$

where $\theta_1(\tau)$, $\sigma_1(\tau)$, $\theta_2(\tau)$, and $\sigma_2(\tau)$ are unknown bounded time-varying signals with bounded derivatives.

4.3 Prediction Error

We define the following quantities

$$\begin{aligned} \theta_m(\rho_x) \triangleq & 4(\max_{\omega \in \Omega} \text{tr}(\omega^\top \omega) + (\theta_{b_1}^2 + \sigma_{b_1}^2)m + (\theta_{b_2}^2 + \sigma_{b_2}^2)(n-m)) \\ & + 4 \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)} ((\theta_{b_1} d_{\theta_1} + \sigma_{b_1} d_{\sigma_1})m + (\theta_{b_2} d_{\theta_2} + \sigma_{b_2} d_{\sigma_2})(n-m)). \end{aligned} \quad (22)$$

$$\tilde{\omega}(t) = \hat{\omega}(t) - \omega, \quad \tilde{\theta}_i(t) = \hat{\theta}_i(t) - \theta_i(t), \quad \tilde{\sigma}_i(t) = \hat{\sigma}_i(t) - \sigma_i(t), \quad (23)$$

$$\tilde{\eta}_i(t) = \hat{\eta}_i(t) - \eta_i(t), \quad \tilde{\eta}_i(t) = f_i(x(t), t), \quad i = 1, 2. \quad (24)$$

Using the notations above, the following error dynamics can be derived from (21) and (11):

$$\dot{\tilde{x}}(t) = A_m \tilde{x}(t) + B_m (\tilde{\omega}(t) u_q(t) + \tilde{\eta}_1(t)) + B_{um} \tilde{\eta}_2(t), \quad (25)$$

where $\tilde{x}(0) = 0$. Next we show that if the adaptation gain Γ is lower bounded by

$$\Gamma > \frac{\theta_m(\rho_x)}{\lambda_{\min}(P)\gamma_0^2}, \quad (26)$$

where $\gamma_0 > 0$ can be an arbitrarily small number, the projection is confined to the following bounds

$$\hat{\omega}(t) \in \Omega, \quad \|\hat{\theta}_i(t)\|_\infty \leq \theta_{b_i}, \quad \|\hat{\sigma}_i(t)\|_\infty \leq \sigma_{b_i}, \quad i = 1, 2, \quad (27)$$

then the truncated $\mathcal{L}_\infty$ -norm of the prediction error $\tilde{x}(t)$ can be arbitrarily reduced both in transient and steady-state by increasing the adaptation gain. The uniform $\mathcal{L}_\infty$ norm bound for $x(t)$ is proved later in Theorem 4. The following lemma summarizes this result.

Lemma 3. (Xargay et al. (2010)). Let the adaptation gain be lower bounded as in (26), and the projection be confined to the bounds in (27). Given the system in (2) and the $\mathcal{L}_1$ adaptive controller defined by (11)-(13) subject to (5), if there exist ρ_x and ρ_u , such that $\|x_t\|_{\mathcal{L}_\infty} \leq \rho_x$, $\|u_t\|_{\mathcal{L}_\infty} \leq \rho_u$, then we have $\|\tilde{x}_t\|_{\mathcal{L}_\infty} < \gamma_0$, where γ_0 was introduced in (26).

The proof is similar to Lemma 3 in Xargay et al. (2010) and is omitted.

5. LOGARITHMIC QUANTIZATION

Let $Q_{\log}$ be the quantization function of the logarithmic quantizer. If the input signal is $u_{\text{qin}}(t) \geq 0$, the quantization function $u_q(t) = Q_{\log}(u_{\text{qin}}(t))$ is defined by

$$u_q(t) = u_i, \text{ if } u_i \leq u_{\text{qin}}(t) < \frac{1}{\rho}u_i, \quad u_{i+1} = \rho u_i, \quad (28)$$

where the constant $u_0 > 0$ and the density constant $0 < \rho < 1$ are the parameters of the quantizer (Hayakawa et al. (2009)). For $u_{\text{qin}}(t) < 0$, $u_q(t) = -Q_{\log}(-u_{\text{qin}}(t))$. The logarithmic quantization function $Q_{\log}$ is shown in Figure 2.

Fig. 2. Logarithmic Quantization Function, $\Delta = \frac{1}{3}$, $\rho = \frac{1}{2}$.

Let $M_1 = 1$ and $M_2 = \rho$. We know from (28) that $M_2|u_{\text{qin}}| \leq |u_q| \leq M_1|u_{\text{qin}}|$. Thus, $|u_q| - \frac{M_1+M_2}{2}|u_{\text{qin}}| \leq \frac{M_1-M_2}{2}|u_{\text{qin}}|$. Since u_q and u_{qin} always have the same sign, we have

$$|u_q - \frac{M_1+M_2}{2}u_{\text{qin}}| \leq \frac{M_1-M_2}{2}|u_{\text{qin}}|. \quad (29)$$

Following Hayakawa et al. (2009), we choose

$$u_{\text{qin}} = \frac{2}{M_1+M_2}u. \quad (30)$$

Let

$$u_{\text{qe}} = u_q - u. \quad (31)$$

Since $u_q = u + u_{\text{qe}} = \frac{M_1+M_2}{2}u_{\text{qin}} + u_{\text{qe}}$, (29) implies

$$|u_{\text{qe}}(t)| \leq \frac{M_1-M_2}{2}|u_{\text{qin}}(t)| = \Delta|u(t)|,$$

where $\Delta = \frac{M_1-M_2}{M_1+M_2}$. Since the inequality above holds for all $t \in (0, \infty)$, we have

$$\|u_{\text{qe}}\|_{\mathcal{L}_\infty} \leq \Delta\|u\|_{\mathcal{L}_\infty}. \quad (32)$$

Note that $\Delta = \frac{M_1-M_2}{M_1+M_2} = \frac{1-\rho}{1+\rho}$ is also a constant representing the coarseness of the quantizer as ρ in (28). When the quantizer is finer, ρ increases and Δ decreases, and when the quantizer is coarser, ρ decreases and Δ increases, with $0 < \rho < 1$, $0 < \Delta < 1$.

Given the quantization density Δ , we introduce some more notation. Let $\rho_{x_{\log}}$ be defined as

$$\rho_{x_{\log}} \triangleq \rho_{x_r} + \bar{\gamma}_{x_{\log}}, \quad (33)$$

where $\bar{\gamma}_{x_{\log}} > 0$ is a small constant. Also define

$$\gamma_{x_{\log}} \triangleq \gamma_{x_{o_{\log}}} + \gamma_{x_{q_{\log}}} + \epsilon, \quad (34)$$

$$\begin{aligned} \gamma_{x_{o_{\log}}} &\triangleq \frac{c_{xe} + \frac{c_{ue}\Delta\|\omega^{-1}C_u(s)H_m^{-1}(s)C\|_{\mathcal{L}_1}}{1-2\omega_1\max\Delta\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1}}}{1 - c_{de}(L_1\rho_{x_r}, L_2\rho_{x_r}) - \frac{c_{ue}\Delta c_{eu}(L_1\rho_{x_r}, L_2\rho_{x_r})}{1-2\omega_1\max\Delta\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1}}} \gamma_0, \\ \gamma_{x_{q_{\log}}} &\triangleq \frac{\frac{c_{ue}}{1-2\omega_1\max\Delta\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1}}}{1 - c_{de}(L_1\rho_{x_r}, L_2\rho_{x_r}) - \frac{c_{ue}\Delta c_{eu}(L_1\rho_{x_r}, L_2\rho_{x_r})}{1-2\omega_1\max\Delta\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1}}} \Delta\rho_{u_r}, \end{aligned}$$

where $\epsilon < \bar{\gamma}_{x_{\log}}$ is a small positive constant, c_{xe} and c_{ue} are defined in (9), $c_{de}(a, b)$ and $c_{eu}(a, b)$ are defined in (10), and γ_0

and Δ are sufficiently small so that $\gamma_{x_{\log}} = \gamma_{x_{o_{\log}}} + \gamma_{x_{q_{\log}}} + \epsilon \leq \bar{\gamma}_{x_{\log}}$. Similarly, let

$$\rho_{u_{\log}} \triangleq \rho_{u_r} + \gamma_{u_{\log}}, \quad (35)$$

$$\begin{aligned} \gamma_{u_{\log}} &\triangleq \frac{(1+\Delta)c_{eu}(L_1\rho_{x_r}, L_2\rho_{x_r})}{1-2\omega_1\max\Delta\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1}} \gamma_{x_{\log}} \\ &+ \frac{(1+\Delta)\|\omega^{-1}C_u(s)H_m^{-1}(s)C\|_{\mathcal{L}_1}}{1-2\omega_1\max\Delta\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1}} \gamma_0 + \frac{1+2\omega_1\max\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1}}{1-2\omega_1\max\Delta\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1}} \Delta\rho_{u_r}. \end{aligned} \quad (36)$$

Theorem 4. Let the adaptive gain be lower bounded as in (26) and the projection be confined to the bounds in (27). Given the closed-loop system with the $\mathcal{L}_1$ adaptive controller defined via (11)–(13), subject to the $\mathcal{L}_1$ -norm condition in (5), and the closed-loop reference system in (16), if $\|x_0\|_\infty \leq \rho_0 < \rho_{x_r}$, then we have

$$\|x_{\text{ref}}\|_{\mathcal{L}_\infty} \leq \rho_{x_r}, \quad \|u_{\text{ref}}\|_{\mathcal{L}_\infty} \leq \rho_{u_r}, \quad \|\tilde{x}\|_{\mathcal{L}_\infty} < \gamma_0, \quad (37)$$

$$\|x\|_{\mathcal{L}_\infty} \leq \rho_{x_{\log}}, \quad \|u_q\|_{\mathcal{L}_\infty} \leq \rho_{u_{\log}}, \quad \|x - x_{\text{ref}}\|_{\mathcal{L}_\infty} < \gamma_{x_{\log}},$$

$$\|u_q - u_{\text{ref}}\|_{\mathcal{L}_\infty} < \gamma_{u_{\log}}, \quad \|y - y_{\text{ref}}\|_{\mathcal{L}_\infty} < \|C\|_\infty \gamma_{x_{\log}},$$

where $\gamma_{x_{\log}}$ and $\gamma_{u_{\log}}$ are defined in (34) and (36).

Proof. See Appendix.

Remark 5. The performance bounds for the state and the control signal in (37) contain a term linear in the quantization parameter Δ . When Δ goes to zero and there is no quantization, this term vanishes and the bounds reduce to the ones in Theorem 1 of Xargay et al. (2010).

Remark 6. To obtain the result in Theorem 4, the selection of $C_u(s)$ should satisfy the condition in (5). Without the unmatched uncertainty, $G_{um}(s) = 0$, this condition can always be satisfied by increasing the bandwidth of $C_u(s)$. However, in the presence of unmatched uncertainty, the existence of ρ_{x_r} is not obvious. When the unmatched uncertainty is small, the left-hand-side is small and a ρ_{x_r} can be determined. If the unmatched uncertainty increases, the positive invariant set shrinks, which agrees with intuition.

6. UNIFORM QUANTIZER

For a uniform quantizer, every quantization interval has a fixed length. Let Q_{unif} be the quantization function for the uniform quantizer. If the control signal is $u_{\text{qin}}(t) \geq 0$, the quantization function $u_q(t) = Q_{\text{unif}}(u_{\text{qin}}(t))$ is defined by

$$u_q(t) = u_i, \text{ if } u_i - \frac{1}{2}l \leq u_{\text{qin}}(t) < u_i + \frac{1}{2}l, \quad (38)$$

where $u_{i+1} = u_i + l$, $u_0 > 0$ is a constant, and l is the length of the quantization interval. For $u_{\text{qin}}(t) < 0$, $u_q(t) = -Q_{\text{unif}}(-u_{\text{qin}}(t))$. This uniform quantization function Q_{unif} is shown in Figure 3.

Fig. 3. Uniform Quantization Function, $l = 2$

Recall that $u_{qe}(t) = u_q(t) - u(t)$ as in (31). If we pick

$$u_{qin}(t) = u(t), \quad (39)$$

then from (38) we have the quantization error bound

$$\|u_{qe}\|_{\mathcal{L}_\infty} \leq \frac{1}{2}l, \quad (40)$$

which shows that the error introduced by a uniform quantizer is bounded by a constant.

Given the quantization interval length l , we introduce some more notations. Let $\rho_{x_{unif}}$ be defined as

$$\rho_{x_{unif}} \triangleq \rho_{x_r} + \bar{\gamma}_{x_{unif}}, \quad (41)$$

$$\gamma_{x_{unif}} \triangleq \gamma_{x_{o_{unif}}} + \gamma_{x_{q_{unif}}} + \epsilon, \quad (42)$$

$$\gamma_{x_{o_{unif}}} \triangleq \frac{c_{xe}}{1 - c_{de}(L_1\rho_{x_r}, L_2\rho_{x_r})}\gamma_0, \quad \gamma_{x_{q_{unif}}} \triangleq \frac{\frac{1}{2}c_{ue}}{1 - c_{de}(L_1\rho_{x_r}, L_2\rho_{x_r})}l,$$

where $\epsilon < \bar{\gamma}_{x_{unif}}$ is a small positive constant, c_{xe} , c_{ue} , and $c_{de}(a, b)$ are defined in (9) and (10), and γ_0 and Δ are sufficiently small, so that $\gamma_{x_{unif}} = \gamma_{x_{o_{unif}}} + \gamma_{x_{q_{unif}}} + \epsilon \leq \bar{\gamma}_{x_{unif}}$.

Similarly, let

$$\rho_{u_{unif}} \triangleq \rho_{u_r} + \gamma_{u_{unif}}, \quad (43)$$

$$\gamma_{u_{unif}} \triangleq c_{eu}(L_1\rho_{x_r}, L_2\rho_{u_r})\gamma_{x_{unif}} + \|\omega^{-1}C_u(s)H_m^{-1}(s)C\|_{\mathcal{L}_1}\gamma_0 + (1 + 2\omega_{1\max}\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1})\frac{1}{2}l. \quad (44)$$

We have the following theorem.

Theorem 7. Let the adaptation gain be lower bounded as in (26) and the projection be confined to the bounds in (27). Given the closed-loop system with the $\mathcal{L}_1$ adaptive controller defined by (11)-(13), subject to the $\mathcal{L}_1$ -norm condition in (5), and the closed-loop reference system in (16), if $\|x_0\|_\infty \leq \rho_0 < \rho_{x_r}$, then we have

$$\begin{aligned} \|x_{ref}\|_{\mathcal{L}_\infty} &\leq \rho_{x_r}, \quad \|u_{ref}\|_{\mathcal{L}_\infty} \leq \rho_{u_r}, \quad \|\tilde{x}\|_{\mathcal{L}_\infty} < \gamma_0, \quad (45) \\ \|x\|_{\mathcal{L}_\infty} &\leq \rho_{x_{unif}}, \quad \|u_q\|_{\mathcal{L}_\infty} \leq \rho_{u_{unif}}, \quad \|x - x_{ref}\|_{\mathcal{L}_\infty} < \gamma_{x_{unif}}, \\ \|u_q - u_{ref}\|_{\mathcal{L}_\infty} &< \gamma_{u_{unif}}, \quad \|y - y_{ref}\|_{\mathcal{L}_\infty} < \|C\|_\infty \gamma_{x_{unif}}. \end{aligned}$$

The proof is similar to that of Theorem 4 and is thus omitted.

Remark 8. The performance bounds in (45) are decoupled into two terms, one of which depends linearly on l , representing the length of the quantization interval for the uniform quantizer, and the other one is independent of the quantizer's parameters. When l decreases to zero and there is no quantization, the corresponding term vanishes, and the performance bounds reduce to the ones in Theorem 1 in Xargay et al. (2010).

7. SIMULATIONS

Consider the system

$$\begin{aligned} \dot{x}(t) &= (A_m + A_\Delta)x(t) + B_m\omega u_q(t) + f_\Delta(x(t), t), \\ y(t) &= Cx(t), \quad x(0) = x_0, \\ u_q(t) &= Q(u_{qin}(t)), \end{aligned}$$

where

$$A_m = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1.8 \end{bmatrix}, B_m = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

while $A_\Delta \in \mathbb{R}^{3 \times 3}$ and $\omega \in \mathbb{R}^{2 \times 2}$ are unknown constant matrices satisfying

$$\|A_\Delta\|_\infty \leq 1, \quad \omega \in \begin{bmatrix} [0.6, 1.2] & [-0.2, 0.2] \\ [-0.2, 0.2] & [0.6, 1.2] \end{bmatrix} = \Omega.$$

For example, we pick

$$A_\Delta = \begin{bmatrix} 0.2 & -0.2 & -0.3 \\ 0.1 & -0.4 & 0.3 \\ -0.1 & 0 & -0.9 \end{bmatrix}, \quad \omega = \begin{bmatrix} 0.9 & -0.2 \\ 0.1 & 1.1 \end{bmatrix},$$

and let f_Δ be the following (unknown) nonlinear function

$$f_\Delta(x, t) = \begin{bmatrix} \frac{1}{3}x^\top x + \tanh(\frac{1}{2}x_1)x_1 \\ \frac{1}{2}\text{sech}(x_2)x_2 - \frac{1}{5}x_3^2 - \frac{1}{2}(1 - e^{-0.3t}) \\ -x_3 \cos(t) \end{bmatrix}.$$

Fig. 4. $y(t)$, $y_m(t)$ and $r(t)$ for logarithmic quantization with $\Delta = 0.99$, $\Delta = 0.95$ and step references $r(t) = [u(t) - 2u(t-7) + u(t-14)u(t-10) - 2u(t-17) + u(t-24)]^\top$.

Fig. 5. $y(t)$, $y_m(t)$ and $r(t)$ for logarithmic quantizer with $\Delta = 0.99$ and sinusoidal references $r(t) = [0.5 \sin(\frac{\pi}{3}t) \ 0.2 + 0.8 \cos(\frac{\pi}{6}t)]^\top$.

In the implementation of the $\mathcal{L}_1$ controller, we set $Q = \mathbb{I}_3$,

$$K = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}, D(s) = \frac{1}{s(\frac{s}{25}+1)(\frac{s}{70}+1)(\frac{s^2}{40^2}+\frac{1.8s}{40}+1)}\mathbb{I}_2,$$

$$K_g(s) \equiv K_g = -(CA_m^{-1}B_m)^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \Gamma = 80000.$$

We choose conservatively $L_1\rho_x = L_2\rho_x = 40$ and $B_{10} = B_{20} = 5$, and thus the projection bounds can be chosen as

$$\begin{aligned} \hat{\theta}_1(t) &\in [-40, 40]\mathbf{1}_m, \quad \hat{\sigma}_1(t) \in [-5, 5]\mathbf{1}_m, \\ \hat{\theta}_2(t) &\in [-40, 40]\mathbf{1}_{(n-m)}, \quad \hat{\sigma}_2(t) \in [-5, 5]\mathbf{1}_{(n-m)}, \\ \hat{\omega}_{11}(t), \hat{\omega}_{22}(t) &\in [0.25, 3], \quad \hat{\omega}_{12}(t), \hat{\omega}_{21}(t) \in [-0.2, 0.2], \end{aligned}$$

where $\mathbf{1}_r \in \mathbb{R}^r$ is the vector with all elements being 1.

Figure 4 shows the cases where the reference signal is the sum of steps. In both cases, the reference signal has different amplitudes in different channels, and the system output tracks the desired signal. When the quantization is coarse, the tracking error is larger. When the quantization is sufficiently dense, the tracking error is reduced and is close to the case without quantization.

These figures also show the tracking of different step signals. For different amplitudes of steps, the controller provides scaled control signals and scaled system outputs, per the properties guaranteed by $\mathcal{L}_1$ adaptive control theory Hovakimyan and Cao (2010).

Figure 5 shows the system response to sinusoidal reference signals. The uniform quantization gives similar results and the figures are omitted here. The $\mathcal{L}_1$ adaptive controller ensures that the system output tracks the desired signal closely and smoothly. We notice that in all these cases we did not redesign or retune the $\mathcal{L}_1$ adaptive controller.

8. CONCLUSION

This paper has extended the analysis of $\mathcal{L}_1$ adaptive controller to a class of uncertain MIMO nonlinear systems with input quantization. We can foresee extensions to systems with both input and state quantization, as well as to underactuated systems, as directions for future work.

REFERENCES

- Brockett, R.W. and Liberzon, D. (2000). Quantized feedback stabilization of linear systems. *IEEE Transactions on Automatic Control*, 45(7), 1279–1280.
- Cao, C. and Hovakimyan, N. (2008). Design and analysis of a novel $\mathcal{L}_1$ adaptive control architecture with guaranteed transient performance. *IEEE Transactions on Automatic Control*, 53(2), 586–591.
- Cao, C. and Hovakimyan, N. (2010). Stability margins of $\mathcal{L}_1$ adaptive control architecture. *IEEE Transactions on Automatic Control*, 55(2).
- Delchamps, D. (1990). Stabilizing a linear system with quantized state feedback. *IEEE Transactions on Automatic Control*, 35(8), 916–924.
- Elia, N. and Mitter, S. (2001). Stabilization of linear systems with limited information. *IEEE Transactions on Automatic Control*, 46(9), 1384–1400.
- Hayakawa, T., Ishii, H., and Tsumura, K. (2009). Adaptive quantized control for linear uncertain discrete-time systems. *Automatica*, 45(3), 692–700.
- Hovakimyan, N. and Cao, C. (2010). *L1 Adaptive Control Theory: Guaranteed Robustness with Fast Adaptation*. SIAM, Philadelphia, PA.
- Ishii, H. and Başar, T. (2005). Remote control of LTI systems over networks with state quantization. *Systems & Control Letters*, 54(1), 15–31.
- Ishii, H., Başar, T., and Tempo, R. (2004). Randomized algorithms for quadratic stability of quantized sampled-data systems. *Automatica*, 40(5), 839–846.
- Pomet, J.B. and Praly, L. (1992). Adaptive nonlinear regulation: estimation from the Lyapunov equation. *IEEE Transactions on Automatic Control*, 37(6), 729–740.
- Sun, H., Hovakimyan, N., and Başar, T. (2010). $\mathcal{L}_1$ adaptive controller for systems with input quantization. In *American Control Conference*, 253–258. Baltimore, MD.
- Wong, W.S. and Brockett, R.W. (1999). Systems with finite communication bandwidth constraints II: Stabilization with limited information feedback. *IEEE Transactions on Automatic Control*, 44(5), 1049–1053.
- Xargay, E., Hovakimyan, N., and Cao, C. (2010). $\mathcal{L}_1$ adaptive controller for Multi-Input Multi-Output systems in the presence of uncertain nonlinear cross-coupling. In *American Control Conference*, 875–879. Baltimore, MD.

APPENDIX

Proof of Theorem 4. (By contradiction) Assume that the bounds in (37) do not hold. Then since $\|x(0) - x_{\text{ref}}(0)\|_\infty = 0 < \gamma_{x_{\log}}$, $\|u(0) - u_{\text{ref}}(0)\|_\infty = 0 < \gamma_{u_{\log}}$, and $x(t)$, $x_{\text{ref}}(t)$, $u(t)$, and $u_{\text{ref}}(t)$ are continuous, there exists t' such that

$$\|x(t') - x_{\text{ref}}(t')\|_\infty = \gamma_{x_{\log}}, \text{ or } \|u(t') - u_{\text{ref}}(t')\|_\infty = \gamma_{u_{\log}}, \quad (46)$$

while $\|x(\tau) - x_{\text{ref}}(\tau)\|_\infty < \gamma_{x_{\log}}$, $\|u(\tau) - u_{\text{ref}}(\tau)\|_\infty < \gamma_{u_{\log}}$, $\forall \tau \in [0, t']$, which implies that

$$\|(x - x_{\text{ref}})_{t'}\|_{\mathcal{L}_\infty} \leq \gamma_{x_{\log}}, \|(u - u_{\text{ref}})_{t'}\|_{\mathcal{L}_\infty} \leq \gamma_{u_{\log}}. \quad (47)$$

Then, Lemma 1 implies that

$$\|x_{\text{ref}t'}\|_{\mathcal{L}_\infty} \leq \rho_{x_r}, \|u_{\text{ref}t'}\|_{\mathcal{L}_\infty} \leq \rho_{u_r}. \quad (48)$$

Using the definitions of $\rho_{x_{\log}}$ and $\rho_{u_{\log}}$ in (33) and (35), together with the bounds in (47) and (48), we have

$$\|x_{t'}\|_{\mathcal{L}_\infty} \leq \rho_{x_r} + \gamma_{x_{\log}} \leq \rho_x, \|u_{t'}\|_{\mathcal{L}_\infty} \leq \rho_{u_r} + \gamma_{u_{\log}} \leq \rho_u.$$

Hence, if one chooses the adaptive gain according to (26) and the projection is confined to the bounds in (27), Lemma 3 implies that $\|\tilde{x}_{t'}\|_{\mathcal{L}_\infty} < \gamma_0$.

Next, let $\tilde{\eta}(t) = \tilde{\omega}(t)u(t) + \tilde{\eta}_1(t) + \tilde{\eta}_{2m}(t)$, where $\tilde{\eta}_{2m}(t)$ is the signal with Laplace transform $\tilde{\eta}_{2m}(s) = H_m^{-1}(s)H_{um}(s)\tilde{\eta}_2(s)$, and $\tilde{\eta}_1(t)$ and $\tilde{\eta}_2(t)$ are defined in (24). It follows from (13) that

$$\chi(s) = D(s)(\omega u(s) + \eta_1(s) + H_m^{-1}(s)H_{um}(s)\eta_2(s) - K_g(s)r(s) + \tilde{\eta}(s)),$$

where $\eta_1(s)$, $\eta_2(s)$, and $\tilde{\eta}(s)$ are the Laplace transforms of the signals $\eta_1(t)$, $\eta_2(t)$ (defined in (24)), and $\tilde{\eta}(t)$. Consequently

$$\chi(s) = (\mathbb{I}_m + D(s)\omega K)^{-1}D(s)(\eta_1(s) + H_m^{-1}(s)H_{um}(s)\eta_2(s) - K_g(s)r(s) + \tilde{\eta}(s)),$$

which together with the definition of $C_u(s)$ in (4) leads to

$$\omega u(s) = -C_u(s)(\eta_1(s) + H_m^{-1}(s)H_{um}(s)\eta_2(s) - K_g(s)r(s) + \tilde{\eta}(s)). \quad (49)$$

Recall from (31) that $u_{\text{qe}}(t) = u_q(t) - u(t)$, and the system in (1) consequently takes the form

$$\begin{aligned} x(s) &= G_m(s)\eta_1(s) + G_{um}(s)\eta_2(s) - H_{xm}(s)C_u(s)\tilde{\eta}(s) \\ &\quad + H_{xm}(s)\omega u_{\text{qe}}(s) + H_{xm}(s)C_u(s)K_g(s)r(s) + r_0(s). \end{aligned} \quad (50)$$

Next, let $e(t) = x(t) - x_{\text{ref}}(t)$. From (16) and (50) we have

$$\begin{aligned} e(s) &= G_m(s)(\eta_1(s) - \eta_{1\text{ref}}(s)) + G_{um}(s)(\eta_2(s) - \eta_{2\text{ref}}(s)) \\ &\quad + H_{xm}(s)\omega u_{\text{qe}}(s) - H_{xm}(s)C_u(s)\tilde{\eta}(s), \quad e(0) = 0. \end{aligned}$$

Moreover, it follows from the error dynamics in (25) that

$$\begin{aligned} H_m^{-1}(s)C\tilde{x}(s) &= \tilde{\eta}_u(s) + \tilde{\eta}_1(s) + H_m^{-1}(s)H_{um}(s)\tilde{\eta}_2(s) + \tilde{\eta}_{uq}(s) \\ &= \tilde{\eta}(s) + \tilde{\eta}_{uq}(s), \end{aligned} \quad (51)$$

where $\tilde{\eta}_u(s)$ is the Laplace transform of $\tilde{\eta}_u(t) = \tilde{\omega}(t)u(t)$, and $\tilde{\eta}_{uq}(s)$ is the Laplace transform of $\tilde{\eta}_{uq}(t) = \tilde{\omega}(t)u_{\text{qe}}(t)$. The two equations above lead to

$$\begin{aligned} e(s) &= G_m(s)(\eta_1(s) - \eta_{1\text{ref}}(s)) + G_{um}(s)(\eta_2(s) - \eta_{2\text{ref}}(s)) \\ &\quad - H_{xm}(s)C_u(s)H_m^{-1}(s)C\tilde{x}(s) \\ &\quad + H_{xm}(s)\omega u_{\text{qe}}(s) + H_{xm}(s)C_u(s)\tilde{\eta}_{uq}(s). \end{aligned} \quad (52)$$

For the first two terms, Assumption 3 implies that, for $i = 1, 2$,

$$\|(\eta_i - \eta_{i\text{ref}})_{t'}\|_{\mathcal{L}_\infty} \leq d_{f_{x_i}}(\bar{\rho}_{x_r})\|(x - x_{\text{ref}})_{t'}\|_{\mathcal{L}_\infty}. \quad (53)$$

For the last term, we have $\|\tilde{\eta}_{uq_{t'}}\|_{\mathcal{L}_\infty} \leq 2\omega_1 \max\|u_{\text{qe}_{t'}}\|_{\mathcal{L}_\infty}$, where $\omega_1 \max$ is defined after (9). Then, from (52) we have

$$\begin{aligned} \|e_{t'}\|_{\mathcal{L}_\infty} &\leq \|G_m(s)\|_{\mathcal{L}_1} d_{f_{x_1}}(\bar{\rho}_{x_r})\|e_{t'}\|_{\mathcal{L}_\infty} \\ &\quad + \|G_{um}(s)\|_{\mathcal{L}_1} d_{f_{x_2}}(\bar{\rho}_{x_r})\|e_{t'}\|_{\mathcal{L}_\infty} \\ &\quad + \|H_{xm}(s)C_u(s)H_m^{-1}(s)C\|_{\mathcal{L}_1}\|\tilde{x}_{t'}\|_{\mathcal{L}_\infty} \\ &\quad + \|H_{xm}(s)\omega\|_{\mathcal{L}_1}\|u_{\text{qe}_{t'}}\|_{\mathcal{L}_\infty} + \|H_{xm}(s)C_u(s)\|_{\mathcal{L}_1} 2\omega_1 \max\|u_{\text{qe}_{t'}}\|_{\mathcal{L}_\infty} \end{aligned} \quad (54)$$

For logarithmic quantizer, the quantization error is bounded by

$$\|u_{\text{qe}_{t'}}\|_{\mathcal{L}_\infty} \leq \Delta\|u_{t'}\|_{\mathcal{L}_\infty}, \quad (55)$$

To bound $\|u_{t'}\|_{\mathcal{L}_\infty}$ in the above inequality, note that

$$\|u_{t'}\|_{\mathcal{L}_\infty} \leq \|u_{\text{ref}_{t'}}\|_{\mathcal{L}_\infty} + \|(u - u_{\text{ref}})_{t'}\|_{\mathcal{L}_\infty}, \quad (56)$$

where $\|u_{\text{ref}_{t'}}\|_{\mathcal{L}_\infty} \leq \rho_{u_r}$ by Lemma 1. From (49), (16) and (53) $u(t) - u_{\text{ref}}(t)$ can be bound as follows

$$\begin{aligned} \|(u - u_{\text{ref}})_{t'}\|_{\mathcal{L}_\infty} &\leq \|\omega^{-1}C_u(s)\|_{\mathcal{L}_1} d_{f_{x_1}}(\bar{\rho}_{x_r})\|e_{t'}\|_{\mathcal{L}_\infty} \\ &\quad + \|\omega^{-1}C_u(s)H_m^{-1}(s)H_{um}(s)\|_{\mathcal{L}_1} d_{f_{x_2}}(\bar{\rho}_{x_r})\|e_{t'}\|_{\mathcal{L}_\infty} \\ &\quad + \|\omega^{-1}C_u(s)H_m^{-1}(s)C\|_{\mathcal{L}_1}\|\tilde{x}_{t'}\|_{\mathcal{L}_\infty} + 2\omega_1 \max\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1}\|u_{\text{qe}}\|_{\mathcal{L}_\infty}. \end{aligned} \quad (57)$$

Substitute (56) and (57) into (55) to get

$$\begin{aligned} \|u_{\text{qe}_{t'}}\|_{\mathcal{L}_\infty} &\leq \frac{\Delta\rho_{u_r}}{1 - 2\omega_1 \max\Delta\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1}} \\ &\quad + \frac{\Delta\|\omega^{-1}C_u(s)H_m^{-1}(s)C\|_{\mathcal{L}_1}}{1 - 2\omega_1 \max\Delta\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1}}\|\tilde{x}_{t'}\|_{\mathcal{L}_\infty} \\ &\quad + \frac{\Delta(\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1} d_{f_{x_1}}(\bar{\rho}_{x_r}) + \|\omega^{-1}C_u(s)H_m^{-1}(s)H_{um}(s)\|_{\mathcal{L}_1} d_{f_{x_2}}(\bar{\rho}_{x_r}))}{1 - 2\omega_1 \max\Delta\|\omega^{-1}C_u(s)\|_{\mathcal{L}_1}}\|e_{t'}\|_{\mathcal{L}_\infty}, \end{aligned} \quad (58)$$

which together with (54) lead to the upper bound

$$\|e_{t'}\|_{\mathcal{L}_\infty} \leq \gamma_{x_{\log}} + \gamma_{x_{\log}} < \gamma_{x_{\log}}, \quad (59)$$

where $\gamma_{x_{\log}}$, $\gamma_{x_{\log}}$, and $\gamma_{x_{\log}}$ are defined in (34).

For the tracking error of the control signal, we have

$$\begin{aligned} \|(u_q - u_{\text{ref}})_{t'}\|_{\mathcal{L}_\infty} &= \|(u_q - u + u - u_{\text{ref}})_{t'}\|_{\mathcal{L}_\infty} \\ &\leq \|u_{\text{qe}_{t'}}\|_{\mathcal{L}_\infty} + \|(u - u_{\text{ref}})_{t'}\|_{\mathcal{L}_\infty}. \end{aligned}$$

From (57) and (58), we have $\|(u_q - u_{\text{ref}})_{t'}\|_{\mathcal{L}_\infty} < \gamma_{u_{\log}}$.

Thus, the inequality above and (59) contradict the assumption in (46) that the tracking errors hit $\gamma_{x_{\log}}$ or $\gamma_{u_{\log}}$. Hence the error is upper bounded and (37) holds. $\square$