Preprints of the 18th IFAC World Congress
Milano (Italy) August 28 - September 2, 2011

Distributed Localization Via Barycentric Coordinates: Finite-Time Convergence

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Abstract: We consider distributed localization in a sensor network in $\mathbb{R}^2$ from inter-agent distances. Sensors and anchors exchange data with their neighbors. No centralized data processing is required. We establish a differential equation for the unknown sensor positions, and show that the estimated positions of sensors converge to their actual values in finite time (assuming noise-free measurements). The key assumption is that all sensors are in the convex hull of three or more anchors. The proposed localization method uses the barycentric coordinates of each sensor with respect to some of its neighbors (which may not include those anchors), assuming the sensor falls in the convex hull of these neighbors.

Keywords: Localization; Multi-agent systems; Sensor networks; Networks of robots and intelligent sensors; Multi-vehicle systems; Finite-time convergence

1. INTRODUCTION

In many multi-agent applications, it is desired to know the positions of the agents. For example, in bushfire surveillance or search and rescue operations, to accomplish the task at hand, it is necessary to know the positions of the agents sensing the data. A trivial solution to this problem is to install global positioning systems (GPS) on each of the agents. However, in large scale sensor networks, it is too expensive to either equip sensors with GPS or manually determine the position of sensors. Furthermore, use of GPS for localization purposes may be infeasible or limited due to possible loss or corruption of GPS signals, or when the agents are operating indoors. Thus, it is important to estimate the positions of the sensing agents, i.e., localize the agents. Localization is accomplished by combining inter-agent measurements and knowledge about the positions of some of the agents in the environment. Those agents with known positions are generally referred to as anchors. Many studies in localization assume the availability of inter-agent distances between a sensor and its neighbors. We recall the following four key questions posed in Anderson et al. [2009]. Similar questions can be posed for other types of inter-agent measurements.

⋆ This work is supported by the Swedish Research Council and Knut and Alice Wallenberg Foundation, NSF grants CCF:101-1903 and CCF:101-8509 and AFOSR grant FA 9550 101 0291, and NICTA, which is funded by the Australian Government as represented by the Department of Broadband, Communications and the Digital Economy and the Australian Research Council through the ICT Centre of Excellence program.

(i) What is the minimal amount of distance data needed to be collected to localize a network, at least if the data is free of noise?
(ii) What is the computational complexity of localization?
(iii) Can localization be carried out sequentially, sensor by sensor, in a distributed fashion, or are central calculations required?
(iv) What is the effect of noise (i.e., errors in the distance measurements) in a localization algorithm?

Questions (i) and (ii) have been addressed in the sensor network localization literature, where it is typically assumed that a small fraction of sensors, called *anchors*, have *a priori* information about their global coordinates. Exploiting the fact that the positions of these anchor nodes are known in a global coordinate system and that for each other sensor node a number of inter-node distances are known, all the other nodes in the network can be localized under a condition that will be discussed later in this section, i.e., global rigidity of the underlying graph of the network, see Aspnes et al. [2006], Eren et al. [2004]. Further, the network localization problem using inter-agent distances is, in general, NP-hard, see Aspnes et al. [2006]. The third question has been discussed in Aspnes et al. [2006], Fang et al. [2009], Anderson et al. [2009] where some special network topologies that permit solving the localization problem in a computationally efficient way were identified and some sequential algorithms with polynomial complexity were introduced. Such networks can be termed easily localizable networks. Moreover, in Khan et al. [2009] a different style of method is proposed to solve the localization problem by a distributed iterative linear algorithm, based on a different kind of structural assumption for the network and using a provably exponentially convergent iteration for the sensor positions. Regarding question (iv), in almost all engineering applications, having noiseless measurements is not a realistic assumption. Many computational algorithms have been proposed to solve the noisy localization problem (assuming graphical conditions for solvability of the noiseless problem are fulfilled), e.g., convex optimization based algorithms (Beck et al. [2008], Carter et al. [2006], Biswas et al. [2006], Ding et al. [2010]), algorithms using sum of squares relaxation (Nie [2009], Shames et al. [2009b,a]), graph connectivity based algorithms (Shang et al. [2003], Lederer et al. [2009]), methods that use multidimensional scaling (Costa et al. [2006]), or other approaches described in Moore et al. [2004], Bruck et al. [2009], Bachrach and Taylor [2005]. Furthermore, the formal analysis of the noisy distance-based localization problem is studied recently in Anderson et al. [2010].

The main contribution of this paper is that it proposes a continuous time algorithm along the lines of the one in Khan et al. [2009] that solves the localization problem in finite time. Moreover, the algorithm allows the nodes to use more distance measurements to their neighbors (previously, only three distance measurements were used) and that relaxes the assumption on the number of the anchors in the network.

The outline of this paper is as follows. In the next section, we introduce some background that we use in this paper. In Section 3, we propose a discrete-time algorithm to address the localization problem. Most of the results in this section are borrowed from Khan et al. [2009]. In Section 4, we provide a modified continuous-time version of the algorithm proposed in Section 3 that solves the localization problem in finite time. We show the applicability of the solution in Section 5 via simulation results. Concluding remarks come in the end.

2. BACKGROUND AND PRELIMINARIES

Consider a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ with vertex set $\mathcal{V} = \{i\}_{i=1}^n$ and edge set $\mathcal{E}$. Furthermore, consider a map $\mathbb{P} : \mathcal{V} \mapsto \mathbb{R}^2$, where $\mathbb{P}(i) = P_i$ is the row vector of the coordinates of the i-th vertex in $\mathbb{R}^2$, and each undirected edge $\{i, j\} \in \mathcal{E}$ corresponds to the distance $d_{ij} = \|P_i - P_j\|$.

The following definitions are standard in graph theory and have been used in the study of network localization literature (see Aspnes et al. [2006]).

Definition 1 (Sensor Network). *A sensor network $\mathcal{F}_\mathbf{P}$ is defined by a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ and a map $\Lambda : \mathcal{V} \to \mathbb{R}^2$ which takes sensor i in $\mathcal{V}$ to its respective position ${P_i}^\top \in \mathbb{R}^2$.*

Definition 2 (Congruent Networks). *A network $\mathcal{F}_\mathbf{P}$ and a network $\mathcal{F}_\mathbf{Q}$ are said to be congruent if there is an isometry $\mathcal{I} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathcal{I}(P_i) = Q_i$.*

Definition 3 (Equivalent Networks). *$\mathcal{F}_\mathbf{Q}$ and $\mathcal{F}_\mathbf{P}$ are said to be equivalent if (i) their underlying graphs are identical, i.e., $\mathcal{G}_\mathbf{Q} = \mathcal{G}_\mathbf{P} = \mathcal{G}$, and (ii) $\|P_i - P_j\| = \|Q_i - Q_j\|$ for all $\{i, j\} \in \mathcal{E}$.*

Definition 4 (Rigidity). *A network $\mathcal{F}_\mathbf{P}$ is rigid if there exists a sufficiently small positive ϵ such that if $\mathcal{F}_\mathbf{Q}$ is equivalent to $\mathcal{F}_\mathbf{P}$ and $\|P_i - Q_i\| \le \epsilon$ for all $i \in \mathcal{V}$ then $\mathcal{F}_\mathbf{Q}$ is congruent to $\mathcal{F}_\mathbf{P}$.*

Intuitively, a rigid network is one that cannot flex. Note that there exist rigid networks $\mathcal{F}_\mathbf{P}$ and $\mathcal{F}_\mathbf{Q}$ which are equivalent but not congruent, see Hendrickson [1992].

Definition 5 (Global Rigidity). *A network $\mathcal{F}_\mathbf{P}$ at $\mathbf{P} = [P_1^\top \ \dots P_n^\top]^\top$ is globally rigid if every network $\mathcal{F}_\mathbf{Q}$ that is equivalent to $\mathcal{F}_\mathbf{P}$ is also congruent to $\mathcal{F}_\mathbf{P}$.*

Generally, rigidity and global rigidity are generic properties of networks. This means that the rigidity (global rigidity) of a generic realization of a graph $\mathcal{G}$ depends only on the graph $\mathcal{G}$ and not on the particular realization. The following result from Eren et al. [2004] relates global rigidity to unique localizability of a network with underlying graph $\mathcal{G}$.

Theorem 1 (Uniquely Localizable Network). *A network in $\mathbb{R}^2$ is uniquely localizable if and only if (i) its underlying graph is globally rigid, and (ii) there are at least three noncollinear nodes with known coordinates (anchors).*

According to Theorem 1, any localizable network in $\mathbb{R}^2$ needs at least three anchors. If the measurements in a network are noisy, then using more than three anchors might be useful to reduce the effects of noise. In the algorithm proposed in this paper, it is assumed that there are *at least* three anchors in the network; however, three anchors are enough to perform the localization task in a noiseless situation.

Let $\mathcal{V}_A$ be the set of anchors whose coordinates are known with respect to a global coordinate system and $\mathcal{V}_S$ be the set of nonanchor sensors whose locations are to be determined. So, we can decompose $\mathcal{V}$ as $\mathcal{V} = \mathcal{V}_A \cup \mathcal{V}_S$ where $\mathcal{V}_A \cap \mathcal{V}_S = \phi$. We assume that all sensors are stationary, and there are exactly u anchors and $m = n-u$ nonanchors ; $|\mathcal{V}_A| = u$, $|\mathcal{V}_S| = m$ and $|\mathcal{V}| = n$. The set of n_ℓ neighbors of $\ell \in \mathcal{V}_S$ with sensing radius r is thus denoted by $\mathcal{N}_{\ell,r}$, which is a subset of $\mathcal{V}$. We formally define the set of neighbors of ℓ as $\mathcal{N}_{\ell,r} = \{i \in \mathcal{V} \mid \|P_i - P_\ell\| \le r\}$, and $\mathcal{V}_\ell$ as a subset of $\mathcal{N}_{\ell,r}$ that is the set of neighbors of ℓ such that these neighbors are corner points of the maximal convex polygon with at least three vertices that covers all the nodes in $\mathcal{N}_{\ell,r}$ and ℓ lies strictly inside it. Furthermore, each node ℓ knows the graph $\mathcal{G}_{\ell,r}(\mathcal{V}_\ell \cup \{\ell\}, \mathcal{E}_\ell)$ induced by itself and $\mathcal{V}_\ell$, together with the distances. We introduce the following assumption and trivial result on $\mathcal{G}_{\ell,r}$ and the network corresponding to it.

Assumption 1. *The neighborhood graph of sensor ℓ, $\mathcal{G}_{\ell,r}$, is complete and ℓ lies in the convex hull of the neighbor nodes.*

As argued in Khan et al. [2009], with a dense enough random network, or perhaps by design for a network in which sensor positions are roughly but not exactly known, or by

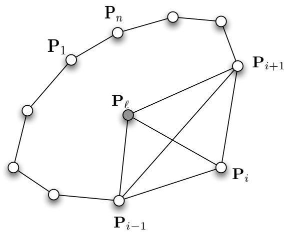


Fig. 1. The convex polygon Ξ with CCW ordering.

increasing the sensing radius/transmit power sufficiently, Assumption 1 can be fulfilled. With Assumption 1, we have

Proposition 1. *A network satisfying Assumption 1 with underlying graph $\mathcal{G}$ is globally rigid.*

Knowing the global coordinates of anchors, Proposition 1 establishes that the class of the networks considered in this paper can be localized uniquely.

Now we introduce the concept of *barycentric coordinates*. Let Ξ be a convex polygon in the plane with vertices at positions $P_1, \cdots, P_n$, $n \geq 3$, in a counter-clockwise (CCW) ordering ($P_i = [x_i, y_i]$). An example of such a configuration is depicted in Fig. 1. We call any set of functions $\pi_{\ell i} : \Xi \mapsto \mathbb{R}$, $\ell \in \mathcal{V}_S$, $i \in \mathcal{V}_\ell$ barycentric coordinates if they satisfy, for all $P_\ell \in \Xi$, the three properties (Floater et al. [2006]):

$$\pi_{\ell i} \geq 0 \quad i = 1, \cdots, n \tag{1a}$$

$$\sum_{i=1}^{n} \pi_{\ell i} = 1 \tag{1b}$$

$$P_\ell = \sum_{i=1}^{n} \pi_{\ell i} P_i \tag{1c}$$

Well known barycentric coordinates for the points inside a convex polygon are the *Wachspress coordinates*, where

$$\pi_{\ell i} = \frac{A(P_{i-1}, P_i, P_{i+1}) \prod\limits_{j \neq i, i-1} A(P_\ell, P_j, P_{j+1})}{\sum\limits_{k=1}^{n} A(P_{k-1}, P_k, P_{k+1}) \prod\limits_{j \neq k, k-1} A(P_\ell, P_j, P_{j+1})} \tag{2}$$

in which $A(P_i, P_j, P_k)$ is the signed area of the triangle $\triangle P_i P_j P_k$, i.e.,

$$A(P_i, P_j, P_k) = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{vmatrix} \tag{3}$$

In this paper, we use an alternative method (see Sippl and Scheraga [1986]) to calculate $A(P_i, P_j, P_k)$ using only the distances between the points (which is what is readily available in the scenarios considered in this paper).

$$A^2(P_i, P_j, P_k) = \frac{-1}{16} \det \begin{bmatrix} 0 & d_{ij}^2 & d_{ik}^2 & 1 \\ d_{ji}^2 & 0 & d_{jk}^2 & 1 \\ d_{ki}^2 & d_{kj}^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \tag{4}$$

Note that distance measurements alone are sufficient for the computation of Wachspress barycentric coordinates; the Euclidean positions of the various points are not required. For the rest of this paper, for brevity, we use the barycentric coordinates to mean Wachspress barycentric coordinates.

3. DISCRETE-TIME LOCALIZATION ALGORITHM

In this section, we propose a discrete-time distributed localization algorithm. First, we calculate the barycentric coordinates of each sensor $\ell \in \mathcal{V}_S$ in terms of the coordinates of the sensors in $\mathcal{V}_\ell$.

It should be noted that the determination of $\mathcal{V}_\ell$ and the calculation of the barycentric coordinates of sensor ℓ is done once, and does not use the position estimates, just the inter-agent distances. After determining the barycentric coordinates of nonanchor sensors, each sensor $\ell \in \mathcal{V}_S$ iteratively updates its current location estimate at the $k+1$-th iteration, $\widehat{P}_\ell(k+1)$, using a convex combination of the estimated positions of the nodes in $\mathcal{V}_\ell$ at the k-th iteration. The anchors do not update their estimated positions, since their locations are known *a priori*. The iterations are given by (Khan et al. [2009])

$$\widehat{P}_\ell(k+1) = \begin{cases} \widehat{P}_\ell(k) & \ell \in \mathcal{V}_A \\ \sum\limits_{i \in \mathcal{V}_\ell} \pi_{\ell i} \widehat{P}_i(k) & \ell \in \mathcal{V}_S \end{cases} \tag{5}$$

Let $\widehat{\mathbf{P}}(k)$= $[\widehat{\mathbf{P}}_A^\top(k) \;\; \widehat{\mathbf{P}}_S^\top(k)]^\top$ be the matrix obtained from stacking all the row vectors $\widehat{P}_i(k)$, $i \in \mathcal{V}$ such that the first u rows of $\widehat{\mathbf{P}}(k)$ are the positions of the anchors ($\widehat{\mathbf{P}}_A(k)$) and the rest are the estimated positions of the sensors ($\widehat{\mathbf{P}}_S(k)$). Then (5) can be written in matrix form

$$\widehat{\mathbf{P}}(k+1) = \Upsilon \, \widehat{\mathbf{P}}(k) = \begin{bmatrix} \mathbf{I}_u & \mathbf{0}_{u \times m} \\ \mathbf{B}_{m \times u} & \mathbf{A}_{m \times m} \end{bmatrix} \widehat{\mathbf{P}}(k) \tag{6}$$

where $\mathbf{I}_u \in \mathbb{R}^{u \times u}$ denotes the identity matrix. Note that (5) does not update the anchors' positions because their positions are known, and Υ in (6) is a stochastic matrix. Further, despite the origins of $\Upsilon = [\Upsilon_{ij}]$, $i, j \in \mathcal{V}$ in terms of barycentric coordinates computed using distance data, it has the structure of the probability transition matrix of a discrete-time Markov chain (DTMC). The matrix Υ has associated absorbing states (anchors) and transient states (nonanchor sensors). It can be shown that if $\mathbf{A}$ is the submatrix associated with the transient states of an absorbing Markov chain, then

$$\rho(\mathbf{A}) < 1 \tag{7}$$

where $\rho(.)$ is the spectral radius of a matrix (see Khan et al. [2009]). The subblock $\mathbf{A} = [a_{ij}]$, $i, j \in \mathcal{V}_S$ is the one step transition matrix of the transient state and therefore

$$\sum_{n=0}^{\infty} \mathbf{A}^n = (\mathbf{I}_m - \mathbf{A})^{-1}. \tag{8}$$

Before showing that the position estimates of each sensor converge to the actual positions under (6), we present the following lemmas.

Lemma 1. *The underlying Markov chain with the transition probability matrix given by the iteration matrix Υ is absorbing and*

$$\lim_{k \to \infty} \widehat{\mathbf{P}}(k) = \begin{bmatrix} \mathbf{I}_u & \mathbf{0} \\ (\mathbf{I}_m - \mathbf{A})^{-1}\mathbf{B} & \mathbf{0} \end{bmatrix} \widehat{\mathbf{P}}(0). \tag{9}$$

Proof. See section III of Khan et al. [2009] for the proof. $\square$

Lemma 2. *Let P_i be the exact coordinates of node $i \in \mathcal{V}_S$ and let the matrix $\mathbf{W} = [w_{ij}]$, $i \in \mathcal{V}_S$, $j \in \mathcal{V}_A$ be the $m \times u$ matrix of the barycentric coordinates of the m sensors in terms of the u anchors in $\mathcal{V}_A$, i.e., $\mathbf{P}_S = \mathbf{W}\mathbf{P}_A$ where $\mathbf{P}_A$ and $\mathbf{P}_S$ are the coordinates of anchors and sensors, respectively. Then*

$$\mathbf{W} = (\mathbf{I}_m - \mathbf{A})^{-1}\mathbf{B} \tag{10}$$

Proof. The proof is similar to that of Khan et al. [2009]. We need to show that

$$\mathbf{W} = \mathbf{A}\mathbf{W} + \mathbf{B} \tag{11}$$

Hence, for the il-th element of $\mathbf{W}$, it is sufficient to show that

$$w_{il} = \sum_{j \in \mathcal{V}_S} a_{ij} w_{jl} + a_{il} \tag{12}$$

For any arbitrary sensor $i \in \mathcal{V}_S$ we can write its coordinates in terms for the coordinates of $j \in \mathcal{V}_i$:

$$P_i = \sum_{j \in \mathcal{V}_S} a_{ij} P_j + \sum_{l \in \mathcal{V}_A} a_{il} P_l$$

Then we have

$$\begin{aligned} P_i &= \sum_{j \in \mathcal{V}_S} a_{ij} \sum_{l \in \mathcal{V}_A} w_{jl} P_l + \sum_{l \in \mathcal{V}_A} a_{il} P_l \\ &= \sum_{l \in \mathcal{V}_A} \left(\sum_{j \in \mathcal{V}_S} a_{ij} w_{jl} + a_{il} \right) P_l \end{aligned}$$

□

The main convergence result follows.

Theorem 2 (Khan et al. [2009])**.** *Under the iteration law (6) the position estimate of each sensor ℓ, $\widehat{P}_\ell(k)$, converges to the actual value P_ℓ as $k \to \infty$.*

Proof. The proof is the direct consequence of Lemmas 1 and 2. □

4. A CONTINUOUS-TIME VERSION

In this section, we aim to write, with some modifications, the updating equation (6) in a continuous-time framework and prove that the estimated locations of nonanchor sensors go to their actual values exponentially fast. Then we modify the proposed continuous-time equation such that the estimated locations go to their actual values in finite time. The proposed finite-time algorithm is inspired from the formation tracking algorithm proposed by Cao et al. [2010].

We now consider the continuous-time version of (6). We need to embed the DTMC in a continuous time Markov chain (CTMC). What we want is to find $\Upsilon_c = [\Upsilon_{c_{ij}}]$, $i, j \in \mathcal{V}$ such that

$$\dot{\widehat{\mathbf{P}}}(t) = \Upsilon_c \, \widehat{\mathbf{P}}(t). \tag{13}$$

and $\widehat{\mathbf{P}}(t)$ converges to $[\mathbf{P}_A^\top \;\; \mathbf{P}_B^\top]^\top$. For the transient states we get

$$\Upsilon_{c_{ij}} = \Upsilon_{c_i} \Upsilon_{ij} \quad \forall i, j, \; i \neq j, \quad \Upsilon_{c_{ii}} = -\Upsilon_{c_i} \tag{14}$$

where $\Upsilon_{c_i} = \sum_{i \neq j} \Upsilon_{c_{ij}}$, and for the absorbing states (anchors) we get

$$\Upsilon_{c_{ij}} = 0 \quad \forall i, j \tag{15}$$

Therefore, a continuous form of (6) is

$$\dot{\widehat{\mathbf{P}}}(t) = \Upsilon_c \, \widehat{\mathbf{P}}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \widehat{\mathbf{A}} \end{bmatrix} \widehat{\mathbf{P}}(t) \tag{16}$$

where

$$\widehat{\mathbf{A}} = \mathbf{A} - \mathbf{I}_m \tag{17}$$

Lemma 3. *Using* (16)*, $\widehat{\mathbf{P}}(t)$ goes to $\mathbf{P}$ exponentially fast.*

Proof. Clearly, $\widehat{\mathbf{P}}_A(t) = \mathbf{P}_A$. So, we only need to prove that $\widehat{\mathbf{P}}_S(t)$ goes to $\mathbf{P}_S$ exponentially fast.

Let $\widetilde{\mathbf{P}}_S(t) = \widehat{\mathbf{P}}_S(t) - \mathbf{P}_S$. Then, by considering (10), (16), (17) and the fact that all sensors and anchors are stationary, we have

$$\dot{\widetilde{\mathbf{P}}}_S(t) = \widehat{\mathbf{A}} \, \widetilde{\mathbf{P}}_S(t) \tag{18}$$

Since every eigenvalue of $\widehat{\mathbf{A}}$ has negative real part (see (7) and (17)), $\widetilde{\mathbf{P}}_S(t)$ goes to zero exponentially fast. □

Now, we modify the update equation (16) to yield a form that makes $\widehat{\mathbf{P}}(t)$ go to $\mathbf{P}$ in *finite time*. For the sake of simplicity, we assume that all sensors are in one-dimensional space, however, the results are valid for higher dimensions.

Lemma 4. *Suppose $\widehat{\mathbf{P}}(t)$ is updated by*

$$\dot{\widehat{\mathbf{P}}}(t) = \alpha \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \widehat{\mathbf{A}} \end{bmatrix} \widehat{\mathbf{P}}(t) + \beta \;\mathrm{sgn}\left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \widehat{\mathbf{A}} \end{bmatrix} \widehat{\mathbf{P}}(t) \right) \tag{19}$$

where α and β are positive constant scalars and sgn *is the sign function (letting the* sgn *of a vector be the vector of signs) and let $\widetilde{P}^+(t) = \max_{i \in \mathcal{V}_S} \widetilde{P}_i(t)$ and $\widetilde{P}^-(t) = \min_{i \in \mathcal{V}_S} \widetilde{P}_i(t)$. If $\widetilde{P}^+(t) > 0$, then $\dot{\widetilde{P}}^+(t) \leq 0$, and $\dot{\widetilde{P}}^+(t) = 0$ only occurs at isolated points. Also if $\widetilde{P}^-(t) < 0$ then $\dot{\widetilde{P}}^-(t) \geq 0$, and $\dot{\widetilde{P}}^-(t) = 0$ only occurs at isolated points.*

Proof. The proof is inspired form Cao et al. [2010]. Similarly to (18), the update equation (19) can be written as

$$\dot{\widetilde{\mathbf{P}}}_S(t) = \alpha \widehat{\mathbf{A}} \, \widetilde{\mathbf{P}}_S(t) + \beta \;\mathrm{sgn}\left(\widehat{\mathbf{A}} \, \widetilde{\mathbf{P}}_S(t) \right) \tag{20}$$

and for $i \in \mathcal{V}_S$, the update equation would be

$$\dot{\widetilde{P}}_i(t) = \alpha \sum_{j \in \mathcal{V}_S} \widehat{a}_{ij} \widetilde{P}_j(t) + \beta \;\mathrm{sgn}\left(\sum_{j \in \mathcal{V}_S} \widehat{a}_{ij} \widetilde{P}_j(t) \right) \tag{21}$$

where $\widehat{a}_{ij}$ is the element of the ith row and the jth column of $\widehat{\mathbf{A}}$. Suppose at time t we have $\widetilde{P}_k(t) = \widetilde{P}^+(t)$ for sensor k. Then for sensor k, (21) can be written as

$$\begin{aligned} \dot{\widetilde{P}}^+(t) = \alpha &\left(-\widetilde{P}^+(t) + \sum_{j \in \mathcal{V}_S, j \neq k} \widehat{a}_{kj} \widetilde{P}_j(t) \right) \\ &+ \beta \;\mathrm{sgn}\left(-\widetilde{P}^+(t) + \sum_{j \in \mathcal{V}_S, j \neq k} \widehat{a}_{kj} \widetilde{P}_j(t) \right) \end{aligned} \tag{22}$$

If $\widetilde{P}^+(t) > 0$ then $\dot{\widetilde{P}}^+(t) \leq 0$ because $0 \leq \left(\sum_{j \in \mathcal{V}_S, j \neq k} \widehat{a}_{kj} \right) \leq 1$. In particular, if sensor k is in the convex-hull of nonanchor sensors, then $\sum_{j \in \mathcal{V}_S, j \neq k} \widehat{a}_{kj} = 1$. In this case

$\dot{\widetilde{P}}^+(t) = 0$ if and only if $\widetilde{P}^+(t) = \widetilde{P}_j(t)\ \forall j \in \mathcal{V}_k$. On the other hand, if k is in the convex-hull of at least one anchor and two or more nonanchor sensors, then $\sum_{k\in\mathcal{V}_S, j\neq k} \widehat{a}_{kj} < 1$ and $\dot{\widetilde{P}}^+(t) = 0$ if and only if $\widetilde{P}^+(t) = \widetilde{P}_j(t) = 0\ \forall j \in \mathcal{V}_k$.

We next show that when $\widetilde{P}^+(\tau) > 0$ for some $\tau > 0$, $\dot{\widetilde{P}}^+(t) = 0$ only occurs at isolated time instants when $t \leq \tau$. In order to obtain a contradiction assume $\dot{\widetilde{P}}^+(t) = 0$ for $t \in [t_1, t_2]$ where $0 < t_1 < t_2 \leq \tau$. Then there is an index k such that $P_k(t) = P^+(t)$ for $t \in [t_1, t_3]$ where $t_1 < t_3 \leq t_2$. This includes the possibility that $P_k(t) = P_j(t) = P^+(t)$ for some $j \in \mathcal{V}_S$ and switching is occurring in $t \in [t_1, t_3]$. If k is in the convex-hull of at least one anchor and two or more nonanchor sensors, then $\widetilde{P}^+(t) = \widetilde{P}_j(t) = 0\ \forall j \in \mathcal{V}_k,\ t \in [t_1, t_3]$ which is a contradiction because we assumed that $\widetilde{P}^+(t) > 0$. If k is in the convex-hull of three nonanchor sensors, then $\widetilde{P}^+(t) = \widetilde{P}_j(t)\ \forall j \in \mathcal{V}_k,\ t \in [t_1, t_3]$ and therefore $\dot{\widetilde{P}}^+(t) = \dot{\widetilde{P}}_j(t) = 0$. By following a similar analysis for the neighbors of j which are the two-hop neighbors of k, we obtain that if at least one of the two-hop neighbors is a neighbor of an anchor, then $\widetilde{P}^+(t) = 0$, otherwise we check three-hop neighbors, four-hop neighbors, etc. Since all of the sensors are in the convex-hull of the anchors and since the underlying graph of the network is connected, it follows that $\widetilde{P}_i(t) = \widetilde{P}^+(t) = 0\ \forall i \in \mathcal{V}_S$ which results in a contradiction.

Similarly, it can be shown that if $\widetilde{P}^-(t) < 0$ then $\dot{\widetilde{P}}^-(t) \geq 0$, and $\dot{\widetilde{P}}^-(t) = 0$ only occurs at isolated points. □

Theorem 3. *Let $\widehat{\mathbf{P}}(t)$ be updated by (19), then $\widehat{\mathbf{P}}(t) \to \mathbf{P}$ in finite time. In particular, $\widehat{\mathbf{P}}(t) = \mathbf{P}$ for any $t \geq T$ where*

$$T = \max_{i\in\mathcal{V}_A} \left\{ \frac{|\widehat{P}_i(0) - P_i|}{\beta} \right\} \tag{23}$$

Proof. By considering (21), it is clear that if $\widetilde{P}^+(0) = \widetilde{P}^-(0) = 0$ then $\widetilde{P}^+(t) = \widetilde{P}^-(t) = 0\ \forall t > 0$.
If $\widetilde{P}^+(0) > 0$ and $\widetilde{P}^-(0) \geq 0$, then according to Lemma 4, $\widetilde{P}^-(t) \geq 0\ \forall t > 0$ because if for some $t > 0$, $\widetilde{P}^-(t) < 0$ then $\dot{\widetilde{P}}^-(t) > 0$, except at some isolated points where $\dot{\widetilde{P}}^-(t) = 0$, and therefore $\widetilde{P}^-(t)$ is always nonnegative. Since $\widetilde{P}^+(t) \geq \widetilde{P}^-(t)$, we have $\widetilde{P}^+(t) \geq 0$. If $\widetilde{P}^+(t) = 0$ then, $\widetilde{P}^-(t) = 0$. Otherwise, if $\widetilde{P}^+(t) > 0$, then $\dot{\widetilde{P}}^+(t) < 0$ for all $t > 0$, except some isolated points. Hence, in the light of (21), $\widetilde{P}^+(t)$ decreases with rate at least equal to β until it reaches zero. Therefore, $\widetilde{P}^+(T) = \widetilde{P}^-(T) = 0$ after $T = \frac{\widetilde{P}^+(0)}{\beta}$. Similarly, if $\widetilde{P}^-(0) < 0$ and $\widetilde{P}^+(0) \leq 0$, then, after $T = \frac{\widetilde{P}^-(0)}{\beta}$, $\widetilde{P}^+(T) = \widetilde{P}^-(T) = 0$. By combining these two cases, it can be concluded that $T = \max\{\frac{\widetilde{P}^-(0), \widetilde{P}^+(0)}{\beta}\}$ if $\widetilde{P}^+(0) \geq 0$ and $\widetilde{P}^-(0) \leq 0$ □

It should be noted that for $t > T$, $\widetilde{P}^+(t) = \widetilde{P}^-(t) = \widetilde{P}_i(t) = 0\ \forall i \in \mathcal{V}_S$ and chattering does not occur. The reason for that is $\widetilde{P}_i(t)$ can not increase or decrease after $t = T$ because if for example it increases, then there exists some $t' > T$ such that $\widetilde{P}^+(t') > 0$ and $\dot{\widetilde{P}}^+(t') > 0$. But this is a contradiction because, according to Lemma 4, if $\widetilde{P}^+(t) > 0\ \forall t > 0$ then $\dot{\widetilde{P}}^+(t)$ is always non-positive. For further information about chattering, see Anderson and Moore [1971].

5. SIMULATION RESULTS

Suppose there are 4 anchors and 5 nonanchor sensors in 2-dimensional space as shown in Fig. 2 and suppose that α and β in (19) are 1 and 0.2, respectively. The estimation error of the positions of nonanchor sensors using the updating equations of (16) and (19) are shown in Fig. 3 and Fig. 4(a), respectively. The estimation errors using (19) in the case that $\alpha = .001$ and $\beta = 0.2$ are depicted in Fig. 4(b). It can be seen that larger α results in better transient response. Furthermore, if α is small, then the second term of (19) is dominant and the time at which the estimation errors go to zero can be calculated using (23). Fig. 4(c) shows the estimation errors when $\alpha = .001$ and $\beta = 0.5$. It can be seen by comparing Fig. 4(b) and Fig. 4(c) that larger β results in faster convergence time. But if α is very small and β is very large, then $\widetilde{P}_i,\ i \in \mathcal{V}_S$ may change from a small negative number to a large positive number, or vice versa, when the sign of $\sum_{j\in\mathcal{V}_S} \widehat{a}_{ij}\widetilde{P}_j(t)$ in (21) is changed.

The estimation error in the case that noisy inter-agent distances are used is depicted in Fig. 4(d). The noise is zero mean Gaussian with a variance equal to .001 unit of distance squared.

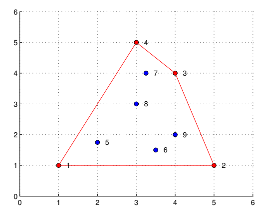


Fig. 2. Location of sensors in 2-D coordinates. The red points are anchors (nodes 1-4) and the blue points are nonanchor nodes

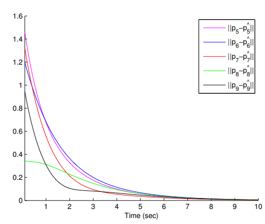
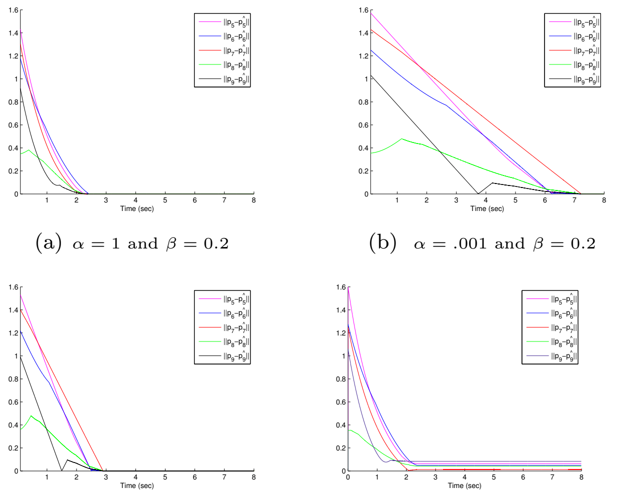


Fig. 3. Estimation error of nonanchors' locations using the continuous version of the algorithm proposed in Khan et al. [2009]



(a) $\alpha = 1$ and $\beta = 0.2$ (b) $\alpha = .001$ and $\beta = 0.2$

(c) $\alpha = .001$ and $\beta = 0.5$ (d) $\alpha = 1$ and $\beta = 0.2$ (normal noise, variance=.001, mean=0)

Fig. 4. Estimation error of nonanchors' locations using the updating equation (19).

6. CONCLUSION

A distributed localization algorithm is proposed that guarantees finite-time convergence. This algorithm uses inter-agent distances to estimate the location of sensors with only local measurements. It is shown in simulations that larger α and β results in faster convergence speed.

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