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Guidance Laws in Target–Missile–Defender Scenario with an Aggressive Defender

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Abstract: An encounter among a target, an intercepting missile and a defending missile is studied in a linear quadratic game setting. The purpose of the defending missile is to destroy the intercepting missile, before the latter reaches the target. The limiting values of the three participants optimal strategies is studied as the quadratic weight on the defending missiles acceleration command tends to zero. It is shown that in the limit the intercepting missiles and the targets optimal strategies are identical in form to that obtained in the game without the defending missile.

Keywords: Differential game, Team game, Missile defense, Cooperative game, Non-cooperative game

1. INTRODUCTION

This paper discusses the game formulation of a three participant engagement among an attacked target (the target), a missile launched from the target (the defender), and an attacking missile (the missile), whose aim is to intercept and destroy the target. This general problem has been studied in Shinar and Silberman (1995), Boyell (1976), Boyell (1980), Shneydor (1977). The problem has also been formulated as a two team LQ differential game between the target and the defender team on one hand and the missile on the other hand in Rusnak (2004, 2005a,b, 2006, 2007a,b, 2010), Perelman et al. (2010a,b). In this formulation the aim of the missile is to maximize the miss distance between himself and the defender, while minimizing the miss distance between himself and the target. Conversely, the aim of the target and the defender is to minimize the miss distance between the defender and the missile, while maximizing the miss distance between the missile and the target. All the participants desire to accomplish their goals with a minimum of control energy expenditure.

At the appropriate moment after detecting the incoming missile, the target fires the defender and the defender turns towards the missile. It is assumed that after the defender completes its turn toward the missile, that

(1) The variation in the three bodies' velocities is negligible until the end of the engagement.
(2) The missile and the target are close to a collision course.
(3) The defender and the missile are close to a collision course.

Under these assumptions both the missile–target and the missile–defender trajectories may be linearized perpendicular to each pairs' initial line of sight direction. Also, there are two intercept times: The first one between the defender and the missile; and a second one between the missile and the target. It is assumed that the intercept between the defender and the missile occurs *before* the intercept between the missile and the target.

The present paper deals with the limit as the weighting on the defender's control effort tends to zero. Although setting the defender's weighting on his control effort in the criterion function precludes a solution, the solution is well defined for any finite weight on the defenders control effort and as the weight tends to zero, all three players control strategies tend to well defined limits. The paper shows the surprising result that the missiles and the targets control laws in the limit tend to the same values as would be obtained in a game in which only they participated. That is, in the limit *both* ignore the existence of the defender.

The definition and the solution of the game are repeated here for the sake of completeness, but the details of the solution derivation are omitted. In section 2 the problem is posed, in section 3 the solution for the case of finite weighting on the defenders control effort is presented. The main results may be found in section 4, where the limit is taken as the weight on the defenders control effort goes to zero. The paper concludes with an example and a summary section.

2. PROBLEM SETUP

The scenario is described in Fig. 1. The letters **T**, **M**, and **D** indicate the positions of the target, the missile and the defender, respectively, a short time after the

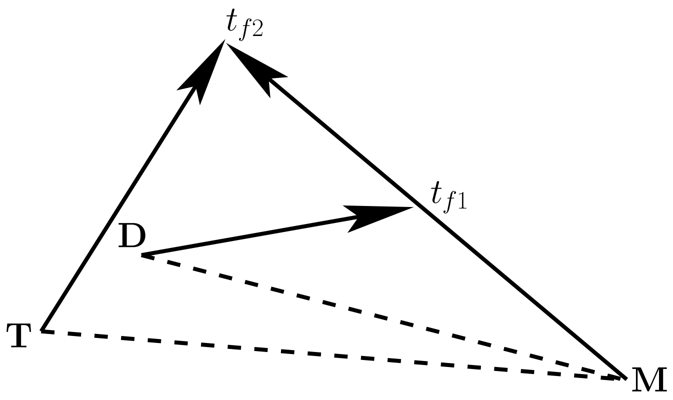

Fig. 1. The Target–Missile–Defender scenario

defender is launched. There are two interceptions taking place, one after the other. The first one takes place at time t_{f1} between the defender and the missile; and later a second one at t_{f2}, between the missile and the target. In general there are two collision triangles: the one between the defender and the missile ($\mathbf{D}$, t_{f1}, $\mathbf{M}$), and second one between the missile and the target ($\mathbf{M}$, t_{f2}, $\mathbf{T}$). If the defender is capable of changing his velocity vector instantaneously to that required for him to be on a collision course with the missile then as was shown in Boyell (1976) the line of sight directions of the two collision triangles coincide.This is not assumed here, but it is assumed that sufficient time has elapsed after its launch for the defender to have reached collision course with the target.

The initial line of sight directions of the two collision triangles are indicated by dashed lines in the figure. Here, only the linearized dynamics perpendicular to the two respective line of sight directions are studied. The details of this linearization may be found in Perelman et al. (2010a). Let $p_{\mathbf{DM}}$ and $v_{\mathbf{DM}}$ be the relative position and velocity of the defender and the missile perpendicular to their initial line of sight direction, $\mathbf{DM}$ in Fig. 1. The variables $p_{\mathbf{DM}}$ and $v_{\mathbf{DM}}$ satisfy the equations

$$\dot{p}_{\mathbf{DM}} = v_{\mathbf{DM}} \tag{1}$$

$$\dot{v}_{\mathbf{DM}} = b_{\mathbf{DM}} a_{\mathbf{M}} - a_{\mathbf{D}} \tag{2}$$

where $a_{\mathbf{M}}$ is the missiles acceleration in its coordinate system, $b_{\mathbf{DM}}$ is a factor that projects the missiles acceleration in its coordinate system to a line perpendicular to the defender–missile initial line of sight direction, and $a_{\mathbf{D}}$ is the defenders acceleration perpendicular to the same line of sight direction. Similarly, the relative position and velocity perpendicular to the initial line of sight direction between the missile and the target, $p_{\mathbf{MT}}$ and $v_{\mathbf{MT}}$ satisfy

$$\dot{p}_{\mathbf{MT}} = v_{\mathbf{MT}} \tag{3}$$

$$\dot{v}_{\mathbf{MT}} = a_{\mathbf{T}} - b_{\mathbf{MT}} a_{\mathbf{M}} \tag{4}$$

where $a_{\mathbf{M}}$ is the missile acceleration perpendicular to the initial line of sight direction $\mathbf{TM}$, and $b_{\mathbf{MT}}$ is a factor that projects the missiles acceleration in its coordinate system to a line perpendicular to the missile–target initial line of sight direction. It is assumed that all the participants accelerations are related to their respective command inputs by first order dynamics,

$$a_i = 1/\tau_i a_i + 1/\tau_i a_{ic} \quad i = \mathbf{T}, \mathbf{M}, \mathbf{D} \tag{5}$$

and a_{ic} for $i = \mathbf{T}$, $\mathbf{M}$, $\mathbf{D}$ are the respective acceleration commands.

To define the criterion function the engagement is divided into two parts: the first part starts at time 0 and ends at time t_{f1}; and the second part starts at time t_{f1} and ends at time t_{f2}. The criterion function is the sum of the criterion for the first time interval, J_1 and the criterion for the second time interval, J_2 . In the second interval only the missile and the target participate. For this time interval the criterion function, J_2 is

$$J_2 = g_{\mathbf{MT}} p_{\mathbf{MT}}^2(t_{f2}) + \int_{t_{f1}}^{t_{f2}} \left(r_{\mathbf{M}} a_{\mathbf{Mc}}^2 - r_{\mathbf{T}} a_{\mathbf{Tc}}^2 \right) \, dt \tag{6}$$

For the first time interval, the criterion, J_1 is

$$\begin{aligned} J_1 &= -g_{\mathbf{DM}} p_{\mathbf{DM}}^2(t_{f1}) \\ &+ \int_0^{t_{f1}} \left(r_{\mathbf{M}} a_{\mathbf{Mc}}^2 - r_{\mathbf{D}} a_{\mathbf{Dc}}^2 - r_{\mathbf{T}} a_{\mathbf{Tc}}^2 \right) \, dt \end{aligned} \tag{7}$$

The criterion for the game is

$$J = J_1 + J_2 \tag{8}$$

The aim of the defender and the target is to jointly choose their control signals, $a_{\mathbf{Mc}}$ and $a_{\mathbf{Dc}}$ to maximize J; while the missile's aim is to minimize J.

It is also possible to express the criterion function as a single integral and terminal cost if generalized functions are used, as was done in Rusnak (2004).

$$\begin{aligned} J &= g_{\mathbf{MT}} p_{\mathbf{MT}}^2(t_{f2}) \\ &+ \int_0^{t_{f2}} \left(r_{\mathbf{M}} a_{\mathbf{Mc}}^2 - r_{\mathbf{D}} a_{\mathbf{Dc}}^2 - r_{\mathbf{T}} a_{\mathbf{Tc}}^2 \right. \\ &\left. - g_{\mathbf{DM}} p_{\mathbf{DM}}^2 \delta(t - t_{f1}) \right) \, dt \end{aligned} \tag{9}$$

3. THE SOLUTION OF THE GAME

The solution of the game posed in the previous section is presented here. Since the subject of this paper is the limit of the strategies as the parameter $r_{\mathbf{D}}$ tends to zero, only the results are presented. The details may be found in Rusnak (2010).

As in many guidance problems the zero effort miss plays a central role here also. There are two zero effort misses in the game: $Z_{\mathbf{DM}}$ for the first intercept between the defender and the missile; and $Z_{\mathbf{MT}}$, between the missile and the target.

$$\begin{aligned} Z_{\mathbf{DM}} &= p_{\mathbf{DM}} + v_{\mathbf{DM}}(t_{f1} - t) \\ &+ a_{\mathbf{M}} b_{\mathbf{DM}} \tau_{\mathbf{M}}^2 \psi\left(\frac{t_{f1} - t}{\tau_{\mathbf{M}}}\right) - a_{\mathbf{D}} b_{\mathbf{DM}} \tau_{\mathbf{D}}^2 \psi\left(\frac{t_{f1} - t}{\tau_{\mathbf{D}}}\right) \end{aligned} \tag{10}$$

$$\begin{aligned} Z_{\mathbf{MT}} &= p_{\mathbf{MT}} + v_{\mathbf{MT}}(t_{f2} - t) \\ &+ a_{\mathbf{T}} b_{\mathbf{MT}} \tau_{\mathbf{T}}^2 \psi\left(\frac{t_{f2} - t}{\tau_{\mathbf{T}}}\right) - a_{\mathbf{M}} b_{\mathbf{MT}} \tau_{\mathbf{M}}^2 \psi\left(\frac{t_{f2} - t}{\tau_{\mathbf{M}}}\right) \end{aligned} \tag{11}$$

and

$$\psi(\xi) = e^{-\xi} + \xi - 1 \tag{12}$$

In the second interval, $t_{f1} \le t \le t_{f2}$ the solution was presented in Rusnak and Hexner (2008). The solution is

$$a_{\mathbf{Mc}} = \frac{S_{\mathbf{M2}} b_{\mathbf{MT}}}{r_{\mathbf{M}} \left(\frac{1}{g_{\mathbf{MT}}} + \boldsymbol{I}_{\mathbf{MT}}(t, t_{f2}) \right)} Z_{\mathbf{MT}} \tag{13}$$

$$a_{\mathbf{Tc}} = \frac{S_{\mathbf{T}}}{r_{\mathbf{T}} \left(\frac{1}{g_{\mathbf{MT}}} + \boldsymbol{I}_{\mathbf{MT}}(t, t_{f2}) \right)} Z_{\mathbf{MT}} \tag{14}$$

where

$$S_{\mathbf{T}} = \tau_{\mathbf{T}} \left(e^{-(t_{f2}-t)/\tau_{\mathbf{T}}} + (t_{f2}-t)/\tau_{\mathbf{T}} - 1 \right) \tag{15}$$

$$S_{\mathbf{Mi}} = \tau_{\mathbf{M}} \left(e^{-(t_{fi}-t)/\tau_{\mathbf{M}}} + (t_{fi}-t)/\tau_{\mathbf{M}} - 1 \right) \quad i = 1,2 \tag{16}$$

$$\boldsymbol{I}_{\mathbf{MT}}(t_1,t_2) = \int_{t_1}^{t_2} \left(\frac{S_{\mathbf{M2}}^2 b_{\mathbf{MT}}^2}{r_{\mathbf{M}}} - \frac{S_{\mathbf{T}}^2}{r_{\mathbf{T}}} \right) dt \tag{17}$$

and, in addition a similar expression involving the defender is required to define his optimal acceleration command,

$$S_{\mathbf{D}} = \tau_{\mathbf{D}} \left(e^{-(t_{f1}-t)/\tau_{\mathbf{D}}} + (t_{f1}-t)/\tau_{\mathbf{D}} - 1 \right) \tag{18}$$

As is common practice Zarchan (1997), in guidance problems the acceleration commands can be expressed in terms of the navigation coecients, $N'_{\mathbf{M}}$ and $N'_{\mathbf{T}}$.

$$a_{\mathbf{Mc}} = \frac{N'_{\mathbf{M}}(t_{f2}-t)}{(t_{f2}-t)^2} Z_{\mathbf{MT}} \tag{19}$$

and

$$a_{\mathbf{Tc}} = \frac{N'_{\mathbf{T}}(t_{f2}-t)}{(t_{f2}-t)^2} Z_{\mathbf{MT}} \tag{20}$$

The cost-to-go for this second game when starting from time t_{f1} is

$$J_2 = g_2 Z_{\mathbf{MT}}^2(t_{f1}) \tag{21}$$

where

$$g_2 = \frac{1}{\frac{1}{g_{\mathbf{MT}}} + \boldsymbol{I}_{\mathbf{MT}}(t_{f1},t_{f2})} \tag{22}$$

To express the solution in the first time interval, $0 < t < t_{f1}$ the dynamics and the criterion are expressed in matrix vector notation. The dynamics are,

$$\dot{\boldsymbol{Z}} = \boldsymbol{B}_E \boldsymbol{u} - \boldsymbol{B}_{\mathbf{M}} a_{\mathbf{Mc}} \tag{23}$$

$$\boldsymbol{Z} = \begin{bmatrix} Z_{\mathbf{DM}} \\ Z_{\mathbf{MT}} \end{bmatrix} \tag{24}$$

$$\boldsymbol{B}_E = \begin{bmatrix} -S_{\mathbf{D}} & 0 \\ 0 & S_{\mathbf{T}} \end{bmatrix} \tag{25}$$

and the criterion function is

$$J = Z'(t_{f1})\boldsymbol{G}Z(t_{f1}) + \int_0^{t_{f1}} \left(r_{\mathbf{M}} a_{\mathbf{Mc}}^2 - \boldsymbol{u}'\boldsymbol{R}_E\boldsymbol{u} \right) dt \tag{26}$$

The symbol g_2 appears in (21)

$$\boldsymbol{R}_E = \begin{bmatrix} r_{\mathbf{D}} & 0 \\ 0 & r_{\mathbf{T}} \end{bmatrix} \tag{27}$$

and $\boldsymbol{u}$ is the control of the target–defender team

$$\boldsymbol{u} = \begin{bmatrix} a_{\mathbf{Dc}} \\ a_{\mathbf{Tc}} \end{bmatrix} \tag{28}$$

The solution is

$$a_{\mathbf{Mc}} = r_{\mathbf{M}}^{-1} \boldsymbol{B}'_{\mathbf{M}} \boldsymbol{G} \boldsymbol{A}^{-1} \boldsymbol{Z} \tag{29}$$

$$\boldsymbol{u} = \boldsymbol{R}_E^{-1} \boldsymbol{B}'_E \boldsymbol{G} \boldsymbol{A}^{-1} \boldsymbol{Z} \tag{30}$$

where

$$\boldsymbol{A} = \boldsymbol{I} + \int_t^{t_{f1}} \left(\boldsymbol{B}_{\mathbf{M}} r_{\mathbf{M}}^{-1} \boldsymbol{B}'_{\mathbf{M}} - \boldsymbol{B}_E \boldsymbol{R}_E^{-1} \boldsymbol{B}'_E \right) dt \tag{31}$$

The optimal acceleration commands in terms of the navigation coecients are,

$$a_{\mathbf{Mc}} = \frac{N'_{\mathbf{M}}(t_{f2}-t)}{(t_{f2}-t)^2} Z_{\mathbf{MT}} + \frac{\Gamma'_{\mathbf{M}}(t_{f2}-t)}{(t_{f2}-t)^2} Z_{\mathbf{DM}} \tag{32}$$

$$a_{\mathbf{Tc}} = \frac{N'_{\mathbf{T}}(t_{f2}-t)}{(t_{f2}-t)^2} Z_{\mathbf{MT}} + \frac{\Gamma'_{\mathbf{T}}(t_{f2}-t)}{(t_{f2}-t)^2} Z_{\mathbf{DM}} \tag{33}$$

$$a_{\mathbf{Dc}} = \frac{N'_{\mathbf{D}}(t_{f1}-t)}{(t_{f1}-t)^2} Z_{\mathbf{DM}} + \frac{\Gamma'_{\mathbf{D}}(t_{f1}-t)}{(t_{f1}-t)^2} Z_{\mathbf{MT}} \tag{34}$$

(Note the different time scale in the example in figures 2–3 for $N'_{\mathbf{D}}$ and $\Gamma'_{\mathbf{D}}$.)

4. MAIN RESULTS

In the present section a limit is taken of the optimal strategies as $r_{\mathbf{D}} \to 0$, and the limiting strategies are examined. Up to this point $r_{\mathbf{D}}$ was possibly a time varying function. From this point on it is assumed that $r_{\mathbf{D}}$ is a constant. An explicit formula for the matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{35}$$

$$a_{11} = 1 - g_{\mathbf{DM}} \int_t^{t_{f1}} \left(\frac{S_{\mathbf{M1}}^2 b_{\mathbf{DM}}^2}{r_{\mathbf{M}}} - \frac{S_{\mathbf{D}}^2}{r_{\mathbf{D}}} \right) dt \tag{36}$$

$$a_{12} = -g_2 \int_t^{t_{f1}} \left(\frac{S_{\mathbf{M1}} S_{\mathbf{M2}} b_{\mathbf{DM}} b_{\mathbf{MT}}}{r_{\mathbf{M}}} \right) dt \tag{37}$$

$$a_{21} = g_{\mathbf{DM}} \int_t^{t_{f1}} \left(\frac{S_{\mathbf{M1}} S_{\mathbf{M2}} b_{\mathbf{DM}} b_{\mathbf{MT}}}{r_{\mathbf{M}}} \right) dt \tag{38}$$

$$a_{22} = 1 + g_2 \int_t^{t_{f1}} \left(\frac{S_{\mathbf{M2}}^2 b_{\mathbf{MT}}^2}{r_{\mathbf{M}}} - \frac{S_{\mathbf{T}}^2}{r_{\mathbf{T}}} \right) dt \tag{39}$$

A sufficient condition for (29) and (30), which were obtained by applying the first order necessary conditions, to actually be a solution of the problem is that $\boldsymbol{A}$ defined in (31),

$$\boldsymbol{A} > \boldsymbol{0} \tag{40}$$

Let $r_{\mathbf{D0}} > 0$ be a value for $r_{\mathbf{D}}$ such that for all $0 < r_{\mathbf{D}} < r_{\mathbf{D0}}$ the solution of the game exists, and let $\boldsymbol{A}_0$ be the matrix $\boldsymbol{A}$ corresponding to $r_{\mathbf{D}} = r_{\mathbf{D0}}$. Since $r_{\mathbf{D}}$ appears only in the a_{11} term,

$$\boldsymbol{A} = \boldsymbol{A}_0 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x \, [\, 1 \;\; 0 \,] \tag{41}$$

$$x = g_{\mathbf{DM}} \left(\frac{1}{r_{\mathbf{D}}} - \frac{1}{r_{\mathbf{D0}}} \right) \int_t^{t_{f1}} S_{\mathbf{D}}^2 \, dt \tag{42}$$

Using the matrix inversion lemma, the $\boldsymbol{A}^{-1}$ that appears in the optimal strategies in (29) and (30) can be calculated as

$$\boldsymbol{A}^{-1} = \boldsymbol{A}_0^{-1} - \frac{\boldsymbol{A}_0^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{A}_0^{-1}}{\frac{1}{x} + \left(\boldsymbol{A}_0^{-1}\right)_{11}} \tag{43}$$

where $\left(\boldsymbol{A}_0^{-1}\right)_{11}$ denotes the $(1,1)$ element of $\boldsymbol{A}_0^{-1}$. Evaluating $1/x$

$$\frac{1}{x} = \frac{r_{\mathbf{D}} r_{\mathbf{D0}}}{g_{\mathbf{DM}}(r_{\mathbf{D0}} - r_{\mathbf{D}})} \frac{1}{\int_t^{t_{f1}} s_{\mathbf{D}}^2 \, dt} \tag{44}$$

so that $\frac{1}{x} \to 0$ as $r_{\mathbf{D}} \to 0$. Hence for $r_{\mathbf{D}}$ sufficiently small

$$\boldsymbol{A}^{-1} = \boldsymbol{A}_0^{-1} - \frac{\boldsymbol{A}_0^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{A}_0^{-1}}{\left(\boldsymbol{A}_0^{-1}\right)_{11}} \left\{ 1 - \frac{1}{x} \frac{1}{\left(\boldsymbol{A}_0^{-1}\right)_{11}} + O\left(\frac{1}{x^2}\right) \right\} \tag{45}$$

The missile optimal strategy (29),

$$\begin{aligned} a_{\mathbf{Mc}} &= r_{\mathbf{M}}^{-1}\boldsymbol{B}_{\mathbf{M}}'\boldsymbol{G}\boldsymbol{A}^{-1}\boldsymbol{Z} \\ &= \begin{bmatrix} \dfrac{b_{\mathbf{DM}}S_{\mathbf{M1}}g_{\mathbf{DM}}}{r_{\mathbf{M}}} & \dfrac{b_{\mathbf{MT}}S_{\mathbf{M2}}g_2}{r_{\mathbf{M}}} \end{bmatrix} \boldsymbol{A}^{-1}\boldsymbol{Z} \\ &\xrightarrow{r_{\mathbf{D}}\to 0} \begin{bmatrix} \dfrac{b_{\mathbf{DM}}S_{\mathbf{M1}}g_{\mathbf{DM}}}{r_{\mathbf{M}}} & \dfrac{b_{\mathbf{MT}}S_{\mathbf{M2}}g_2}{r_{\mathbf{M}}} \end{bmatrix} \mathcal{A}_0\boldsymbol{Z} \end{aligned} \tag{46}$$

where $\mathcal{A}_0$ is defined as

$$\mathcal{A}_0 = \boldsymbol{A}_0^{-1} - \frac{\boldsymbol{A}_0^{-1}\begin{bmatrix}1&0\\0&0\end{bmatrix}\boldsymbol{A}_0^{-1}}{\left(\boldsymbol{A}_0^{-1}\right)_{11}} \tag{47}$$

Separating out the target's optimal strategy in (30),

$$\begin{aligned} a_{\mathbf{Tc}} &= \begin{bmatrix} 0 & \dfrac{g_2 S_{\mathbf{T}}}{r_{\mathbf{T}}} \end{bmatrix} \boldsymbol{A}^{-1}\boldsymbol{Z} \\ &\xrightarrow{r_{\mathbf{D}}\to 0} \begin{bmatrix} 0 & \dfrac{g_2 S_{\mathbf{T}}}{r_{\mathbf{T}}} \end{bmatrix} \mathcal{A}_0\boldsymbol{Z} \end{aligned} \tag{48}$$

Evaluating $\mathcal{A}_0$ (47) yields

$$\mathcal{A}_0 = \begin{bmatrix} 0 & 0 \\ 0 & \left(\boldsymbol{A}_0^{-1}\right)_{22} - \dfrac{\left(\boldsymbol{A}_0^{-1}\right)_{21}\left(\boldsymbol{A}_0^{-1}\right)_{12}}{\left(\boldsymbol{A}_0^{-1}\right)_{11}} \end{bmatrix} \tag{49}$$

A further straightforward calculation shows that

$$\mathcal{A}_0 = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{(\boldsymbol{A_0})_{22}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{a_{22}} \end{bmatrix} \tag{50}$$

which when substituted in (48) and (46)

$$\begin{aligned} a_{\mathbf{Mc}} &= \frac{b_{\mathbf{MT}}S_{\mathbf{M2}}}{r_{\mathbf{M}}\left(\frac{1}{g_2} + \boldsymbol{I}_{\mathbf{MT}}(t,t_{f1})\right)} Z_{\mathbf{MT}} \\ &= \frac{b_{\mathbf{MT}}S_{\mathbf{M2}}}{r_{\mathbf{M}}\left(\frac{1}{g_{\mathbf{MT}}} + \boldsymbol{I}_{\mathbf{MT}}(t,t_{f2})\right)} Z_{\mathbf{MT}} \end{aligned} \tag{51}$$

$$\begin{aligned} a_{\mathbf{Tc}} &= \frac{S_{\mathbf{T}}}{r_{\mathbf{T}}\left(\frac{1}{g_2} + \boldsymbol{I}_{\mathbf{MT}}(t,t_{f1})\right)} Z_{\mathbf{MT}} \\ &= \frac{S_{\mathbf{T}}}{r_{\mathbf{T}}\left(\frac{1}{g_{\mathbf{MT}}} + \boldsymbol{I}_{\mathbf{MT}}(t,t_{f2})\right)} Z_{\mathbf{MT}} \end{aligned} \tag{52}$$

which are the optimal strategies for the game whose only participants are the target and the missile. Note that these are identical to the optimal strategies from (13) and (14), and in fact are valid for the *entire interval* from 0 to t_{f2} , in the limit as $r_{\mathbf{D}} \to 0$.

Next the limiting defender strategy is calculated. The defenders strategy is

$$a_{\mathbf{Dc}} = \begin{bmatrix} \dfrac{S_{\mathbf{D}}g_{\mathbf{DM}}}{r_{\mathbf{D}}} & 0 \end{bmatrix} \boldsymbol{A}^{-1}\boldsymbol{Z} \tag{53}$$

and $\boldsymbol{A}^{-1}$ is expanded as in (45). Using (47) and regrouping the terms yields,

$$\boldsymbol{A}^{-1} = \mathcal{A}_0 + \left\{ \frac{1}{x\left[\left(\boldsymbol{A}_0^{-1}\right)_{11}\right]^2} + O\left(\frac{1}{x^2}\right) \right\} \boldsymbol{A}_0^{-1}\begin{bmatrix}1&0\\0&0\end{bmatrix}\boldsymbol{A}_0^{-1} \tag{54}$$

Since the only non-zero term in the $\mathcal{A}_0$ matrix is the (2, 2) entry, when (54) is substituted into (53) the first term on the right hand side of (54), $\mathcal{A}_0$ does not contribute anything. Also,

$$\begin{aligned} \frac{1}{r_{\mathbf{D}}x} &= \frac{1}{g_{\mathbf{DM}}\int_t^{t_{f1}} S_{\mathbf{D}}^2\,dt} \frac{r_{\mathbf{D0}}}{r_{\mathbf{D0}} - r_{\mathbf{D}}} \\ &\xrightarrow{r_{\mathbf{D}}\to 0} \frac{1}{\int_t^{t_{f1}} S_{\mathbf{D}}^2\,dt} \end{aligned} \tag{55}$$

and

$$\frac{1}{r_{\mathbf{D}}x^2} = \frac{1}{\left(g_{\mathbf{DM}}\int_t^{t_{f1}} S_{\mathbf{D}}^2\,dt\right)^2} \frac{r_{\mathbf{D}}r_{\mathbf{D0}}^2}{(r_{\mathbf{D0}} - r_{\mathbf{D}})^2} \xrightarrow{r_{\mathbf{D}}\to 0} 0 \tag{56}$$

so that the defenders limiting strategy is

$$\begin{aligned} a_{\mathbf{Dc}} &= \begin{bmatrix} \dfrac{S_{\mathbf{D}}}{\int_t^{t_{f1}} S_{\mathbf{D}}^2\,dt} & 0 \end{bmatrix} \frac{\boldsymbol{A}_0^{-1}\begin{bmatrix}1&0\\0&0\end{bmatrix}\boldsymbol{A}_0^{-1}}{\left[\left(\boldsymbol{A}_0^{-1}\right)_{11}\right]^2} \boldsymbol{Z} \\ &= \frac{S_{\mathbf{D}}}{\int_t^{t_{f1}} S_{\mathbf{D}}^2\,dt} \begin{bmatrix} 1 & \dfrac{\left(\boldsymbol{A}_0^{-1}\right)_{12}}{\left(\boldsymbol{A}_0^{-1}\right)_{11}} \end{bmatrix} \boldsymbol{Z} \end{aligned} \tag{57}$$

and the rightmost term inside the square bracket is evaluated as

$$\begin{aligned} \frac{\left(\boldsymbol{A}_0^{-1}\right)_{12}}{\left(\boldsymbol{A}_0^{-1}\right)_{11}} &= \frac{\int_t^{t_{f1}}\left(\frac{S_{\mathbf{M1}}S_{\mathbf{M2}}b_{\mathbf{DM}}b_{\mathbf{MT}}}{r_{\mathbf{M}}}\right)dt}{\frac{1}{g_2} + \int_t^{t_{f1}}\left(\frac{S_{\mathbf{M2}}^2 b_{\mathbf{MT}}^2}{r_{\mathbf{M}}} - \frac{S_{\mathbf{T}}^2}{r_{\mathbf{T}}}\right)dt} \\ &= \frac{\int_t^{t_{f1}}\left(\frac{S_{\mathbf{M1}}S_{\mathbf{M2}}b_{\mathbf{DM}}b_{\mathbf{MT}}}{r_{\mathbf{M}}}\right)dt}{\frac{1}{g_{\mathbf{MT}}} + \boldsymbol{I}_{\mathbf{MT}}(t,t_{f2})} \end{aligned} \tag{58}$$

5. DISCUSSION

Recall the steps that were taken to obtain (51) and (52). The Target–Missile–Defender game was solved for finite values of $r_{\mathbf{D}}$, and once the solution was obtained a limit was taken of the optimal strategies as $r_{\mathbf{D}} \to 0$. A limit for the optimal strategies of all the participants was obtained. The existence of this limit does not imply anything about the existence of the solution of the game when $r_{\mathbf{D}} = 0$ is substituted into (8). (In fact there is no solution for this game.) There are at least two reasons that these limits are interesting: the first is the fact that the limiting strategies exist; and the second is the fact that the other participants', namely the target's and the missile's limiting strategies are the same as would be obtained if the defender did not exist.

One may inquire whether similar limits exist in other situations. One such situation is the target–missile encounter that takes place in the portion of the game studied in the present paper during the interval $t_{f1} - t_{f2}$. If in the optimal strategies (13) and (14) $r_{\mathbf{M}} \to 0$, then the optimal strategies become

$$a_{\mathbf{Mc}} = \frac{S_{\mathbf{M2}}b_{\mathbf{MT}}}{\boldsymbol{I}_{\mathbf{M}}(t,t_{f2})} Z_{\mathbf{MT}} \tag{59}$$

$$a_{\mathbf{Tc}} = 0 \tag{60}$$

and

$$\boldsymbol{I}_{\mathbf{M}}(t,t_{f2}) = \int_t^{t_{f2}} S_{\mathbf{M2}}^2 b_{\mathbf{MT}}\,dt \tag{61}$$

Note the target's optimal strategy. Because the missile's energy cost vanish in the limit, the missile can achieve zero miss for any target maneuver of finite energy. This is ensured by the missiles guidance gain in (59) tending to infinity as $t \to t_{f2}$. There is then no point in the target making any effort to out-maneuver the missile. A similar

Parameter	Value	Units
t_{f1}	3	s
t_{f2}	5	s
$\tau_{\mathbf{T}}$	0.2	s
$\tau_{\mathbf{M}}$	0.1	s
$\tau_{\mathbf{D}}$	0.05	s
$r_{\mathbf{M}}$	1	
$r_{\mathbf{T}}$	2	
$r_{\mathbf{D}}$	0.5, 0.05, 0.005	
$g_{\mathbf{MT}}$	10^{12}	
$g_{\mathbf{DM}}$	10^{12}	
$b_{\mathbf{MT}}$	1	
$b_{\mathbf{DM}}$	1	

Table 1. Parameter values used in the simulations

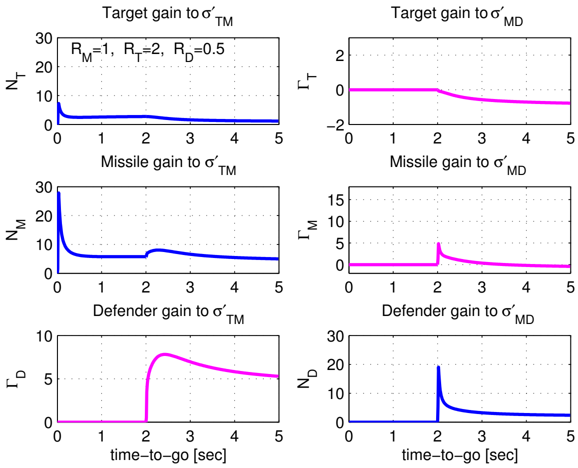

Fig. 2. Navigation Coefficients for $r_{\mathbf{D}} = 0.5$

situation exists for the three participant game discussed in the present paper. Because of the vanishing energy cost, the defenders guidance gain in (57) tends to infinity as $t \to t_{f1}$, the defender can always ensure zero miss, regardless of the missiles or the targets maneuver. There is then no point in the missile making any effort to avoid the defender, so that his only choice is to concentrate his efforts towards decreasing the miss between himself and the target. Accordingly, the targets only concern is to increase the miss at the terminal time t_{f2} between himself and the missile. In other words the best that the missile and target can do is to use the optimal strategies from the game where only the two of them participate.

6. AN EXAMPLE

To illustrate the results of the paper a simple simulation was run. The parameters used in the simulation are shown in Table 1.

In order to illustrate the limiting operations as $r_{\mathbf{D}} \to 0$ the simulations were run for three values of $r_{\mathbf{D}}$: 0.5, 0.05, and 0.005. Since the simulation results are unchanged for $r_{\mathbf{D}} = 0.05$ and $r_{\mathbf{D}} = 0.005$ only the results for $r_{\mathbf{D}} = 0.5, 0.005$ are presented here. The simulation results for $r_{\mathbf{D}} = 0.005$ are indistinguishable from their limiting values for $r_{\mathbf{D}} = 0$. The three participants' optimal strategies were expressed in (29) and (30) for the interval $0 - t_{f1}$ and were also expressed in terms of the navigation coefficients in (32)–

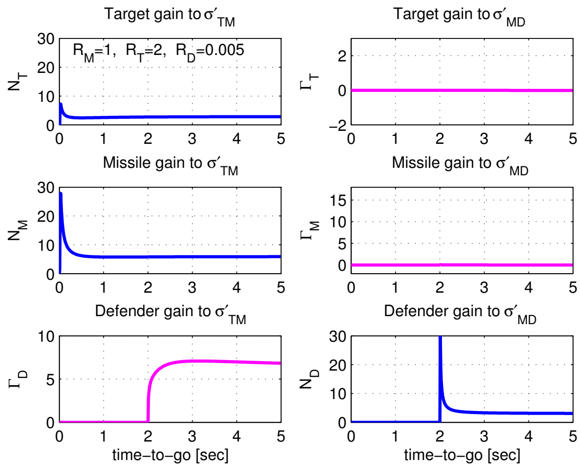

Fig. 3. Navigation Coefficients for $r_{\mathbf{D}} = 0.005$

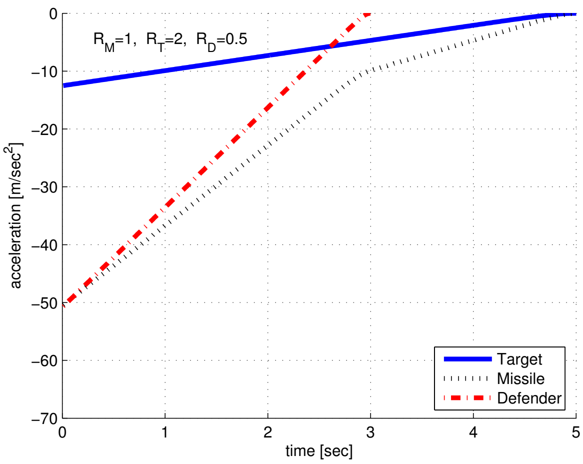

Fig. 4. Acceleration commands for $r_{\mathbf{D}} = 0.5$

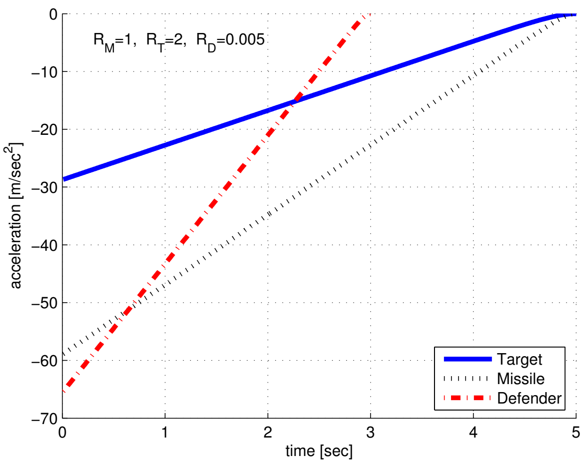

Fig. 5. Acceleration commands for $r_{\mathbf{D}} = 0.005$

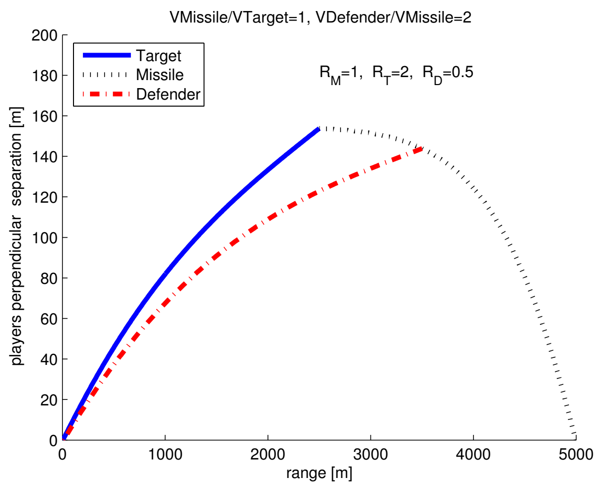

Fig. 6. Trajectories for $r_{\mathbf{D}} = 0.5$

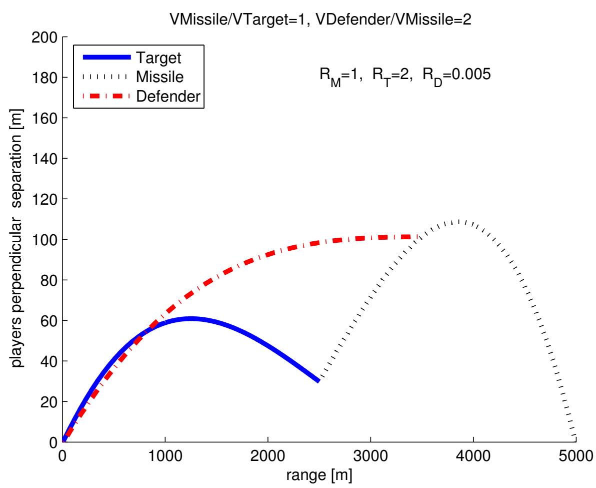

Fig. 7. Trajectories for $r_{\mathbf{D}} = 0.005$

(34). The optimal acceleration commands for the second interval $t_{f1} - t_{f2}$ were (13) and (14), and the corresponding navigation coefficients were defined in (19) and (20). The navigation coefficients are shown in figures 2–3. Note that the defender related coefficients appear as equal to zeros in the interval $t_{f1} - t_{f2}$. The three participants' acceleration commands are shown in figures 4–5, and the paths are shown in figures 6–7. There are two intercepts taking place, one between the defender and the missile and one between the missile and the target. Note the linear variation of the three participants' acceleration commands with time in both scenarios in figures 4 and 5. Also note that the defender's acceleration level is comparable to that of the missile, *even as the parameter* $r_{\mathbf{D}}$ *approaches 0.* This is apparently connected to the fact that the missile maneuvers in response to the target's acceleration, but the target and the defender are acting as a team.

7. SUMMARY

The paper discussed the linear quadratic formulation of a three participant, two team differential game. The differential game is a model for an engagement among a target, a defending missile launched from the target, and an intercepting missile, whose aim is to destroy the target. The purpose of the defending missile was to reach the intercepting missile before it has a chance to reach the target. The particular aspect studied in this paper was the limiting form of the three participants' optimal acceleration commands as the quadratic weight on the defender's acceleration command goes to zero. It was shown that the limiting acceleration commands of the missile and the target are identical in form to what would be obtained if there were no defender in the game.

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