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On the Convergence Rate of the Leake-Liu Algorithm for Solving Hamilton-Jacobi-Bellman Equation ⋆

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Abstract: Although many iterative algorithms have been proposed for solving Hamilton-Jacobi-Bellman equation arising from nonlinear optimal control, it remains open how fast those algorithms can converge. The convergence rate of those algorithms is of great importance in concluding whether they offer practical benefit. This paper presents a study on how fast the well-known Leake-Liu algorithm in Leake and Liu (1967) can converge. The relationship between the sequence of approximate solutions to the HLB equation and the corresponding sequence of control laws is first established. Based on this relation, several results are provided on the convergence rate of the Leake-Liu algorithm. These results demonstrate that the convergence rate of the Leake-Liu algorithm can be quadratic in the domain of interest under favorable conditions. Further, they include the well known quadratic convergence results in Kleinman (1968) and in Reid (1972) as special cases, which have been established for linear time-invariant and time-varying systems (with quadratic performance index) respectively. Under weaker conditions, the convergence rate of the Leake-Liu algorithm is shown to be quadratic locally i.e. in the neighborhood of the origin.

1. INTRODUCTION

Nonlinear optimal control often boils down to the problem of solving the well known Hamilton-Jacobi-Bellman (HJB) equation, which is a nonlinear partial differential equation. Except for a few special cases, obtaining the explicit exact solution of Hamilton-Jacobi-Bellman equation has been proved to be extremely hard if not impossible. This difficulty has led to the idea of using *successive approximations* to approach the exact solution of the HJB equations. Along this line of research, many iterative algorithms for solving the HJB equations have been proposed in the literature, see for example algorithms in Leake and Liu (1967); Saridis and Lee (1979); Beard and McLain (1998); Abu-Khalaf and Lewis (2005). For a more detailed review on iterative algorithms for solving the HJB equations, the readers are referred to Beard and McLain (1998), Abu-Khalaf and Lewis (2005), and Feng et al. (2009).

Although many iterative algorithms for solving the HJB equation have been shown to be able to converge to the exact solution of Hamilton-Jacobi-Bellman equations, it remains open on how fast they can converge. Because of the numerical nature of those algorithms, it is of great importance to know their convergence rate in order to determine properly the stopping criterion. However, in the literature, very little research has been done on establishing the convergence rate of the proposed algorithms. The aim of this paper is to make some contribution in this direction. Specifically, an answer will be provided on how fast the Leake-Liu algorithm proposed in Leake and Liu (1967) can converge. We are not proposing a new algorithm.

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2. THE LEAKE-LIU ALGORITHM AND ITS RELATED RESULTS

For the convenience of analysis, some materials drawn from Leake and Liu (1967) will be cited in this section.

2.1 Systems and problem of interest

The system under consideration in Leake and Liu (1967) is described as

$$\dot{x} = f(x, u, t), x(t_0) = x_0 \tag{1}$$

where $x \in R^n$ is the state, $f \in R^n$ is a continuously differentiable vector function, and $u(x, t)$ is an $r-$dimensional vector defined on $R^n \times R^1$. The solution of (1) is denoted as $\phi_u(t) = \phi_u(t; x_0, t_0)$.

Let $G \subseteq R^n \times R^1$ be a closed subset of $R^n \times R^1$, which is the domain of interest, and let the target set S be a closed subset of G.

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Admissible feedback control law: u will be called an **admissible feedback control law** if (a) it is continuously differentiable with $u(x,t)$ belonging to a locally compact set $U \subset R^r$ for all t; (b) it has the property that when substituted into (1), any motion beginning in $G-S$ reaches S, or approaches S, in a uniform asymptotic manner without leaving G. The set of all **admissible feedback control laws** will be denoted as $\mathcal{K}^0$.

The terminal time $t_1 = t_1(x_0, t_0)$ is defined as the first time instant after t_0 when the motion $(\phi_u(t), t)$ becomes a member of S. If instead, the motion $(\phi_u(t), t)$ never enters S but asymptotically approaches it, the terminal time is defined as $t_1 = \infty$.

The problem is to find the optimal control u (if it exists) to minimize the following cost function:

$$J(x_0, t_0; u) = \lambda[\phi_u(t_1; x_0, t_0), t_1] + \int_{t_0}^{t_1} L[\phi_u(\alpha; x_0, t_0), u(\phi_u(\alpha; x_0, t_0), \alpha), \alpha] d\alpha \quad (2)$$

where L is nonnegative and continuously differentiable, and λ is positive definite and continuously differentiable.

Define
$$V^0(x_0, t_0) = inf_{u \in \mathcal{K}^0} J(x_0, t_0; u).$$

Let

$$H(x, p, t, u) = < f(x, u, t), p > + L(x, u, t). \quad (3)$$

The following two assumptions are needed.

- Assumption A1: H has a unique absolute minimum for each x, p, t with respect to the values in $u \in U$. Let the associated location of the minimum be denoted as $c(x, p, t)$.
- Assumption A2: $c(x, p, t)$ is a continuously differentiable function of x, p, t.

Define the Hamiltonian as

$$H^0(x, p, t) = H(x, p, t, c(x, p, t)) = min_{u \in U} H(x, p, t, u). \quad (4)$$

Then, the Hamilton-Jacobi equation is defined as

$$V_t + H^0(x, V_x, t) = 0 \quad (5)$$

where $V(x,t)$ is a scalar function defined on $R^n \times R^1$, $V_t = \partial V / \partial t$ and $V_x = grad\, V$.

According to Leake and Liu (1967), we have along the motion of (1)

$$\frac{dJ(\phi_u(t; x_0, t_0), t; u)}{dt} = -L[\phi_u(t; x_0, t_0), u(\phi_u(t; x_0, t_0), t), t]. \quad (6)$$

Denote $\phi_u(t; x_0, t_0)$ by x, we have

$$\frac{dJ(x, t; u)}{dt} = -L(x, u, t), \quad (7)$$

where x satisfies $\dot{x} = f(x, u, t)$.

2.2 The Leake-Liu algorithm

Define υ as a set of all continuously differentiable scalar functions such that $V(x,t) = \lambda(x,t)$ on S, and let υ^0 be the subset of υ such that if $u(x,t) = c(x, V_x(x,t), t)$, then $u \in \mathcal{K}^0$, i.e, u is admissible.

In Leake and Liu (1967), the following three transformations were introduced.

$T_1 : \upsilon^0 \to \mathcal{K}^0$ is defined by $T_1(V) = u$, where $u(x,t) = c(x, V_x, t)$.

$T_2 : \mathcal{K}^0 \to \upsilon$ is defined by $T_2(u) = J(x, t; u)$, where $J(x, t; u)$ is continuously differentiable with $J(x, t; u) = \lambda(x,t)$ on S.

$T : \upsilon^0 \to \upsilon$ is defined by $T(V) = T_2(T_1(V)) = J(x, t; u)$ with $u(x,t) = c(x, V_x, t)$.

Based on these transformations, the following sequence was constructed in Leake and Liu (1967) to provide a sequence of approximate solutions to the HJB equation:

Step 1: Choose $V^1 \in \upsilon^0$.

Step 2: For $k = 1, 2, 3, \cdots$, define $u^{k+1} = c(x, V_x^k, t)$, check whether $T(V^k) \in \upsilon^0$. If yes, let $V^{k+1} = T(V^k)$. Otherwise, stop.

In this paper, we call the above sequence construction procedure the Leake-Liu algorithm. The following result from Leake and Liu (1967) summarizes the main property of this algorithm.

Theorem 1. For the optimal problem given by (2) subject to (1), if the Leake-Liu algorithm is applied with initialization $V^1 \in \upsilon^0$ and if $T(V^k) \in \upsilon^0$ for all $k = 1, 2, \cdots$, then $V^0(x,t) \leq V^{k+1}(x,t) \leq V^k(x,t) \leq V^1(x,t), (x,t) \in G$. Let $V^*(x,t) = lim_{k \to \infty} V^k(x,t)$, if $V^*(x,t) \in \upsilon^0$ and T is continuous in $\upsilon^0 \subset \upsilon$, $lim_{k \to \infty} V^k(x,t) = V^0(x,t)$ pointwise on G. If G is bounded, the convergence is uniform.

In fact, suppose that $V^1(x,t)$ has been found such that the system $\dot{x} = f(x, c(x, V_x^1, t), t)$ is uniformly asymptotically stable, then the sequence $V^k(x,t)$ can be recursively computed by solving the following sequence of linear PDEs

$$< V_x^{k+1}, f(x, c(x, V_x^k, t), t) > + V_t^{k+1} = -L(x, V_x^k, t) \quad (8)$$

with the boundary condition $V^k(x, t_1) = \lambda(x, t_1)$.

In the next section, an answer will be provided on how fast the convergence is.

3. THE CONVERGENCE RATE OF THE LEAKE-LIU ALGORITHM

The convergence of the Leake-Liu algorithm has been established in Theorem 1. In this section, the convergence rate of the algorithm will be studied. We only consider the case that G is bounded, where t_1 is finite.

3.1 A relationship between $V^k(x,t)$ and $u^k(x,t)$

To establish the relationship between $V^k(x,t)$ and $u^k(x,t)$, the technique in Kalman (1960) will be used here with some slight modifications.

Define $L^*(x, u, t) = L(x, u, t) + V_t^0(x, t) + V_x^0(x, t) f(x, u, t)$.

Since $H(x, V_x^0, t, u)$ has a unique and absolute minimum at $u^0 = c(x, V_x^0(x,t), t)$, we have

$$L_u(x, u^0, t) + V_x^0 f_u(x, u^0, t) = 0. \tag{9}$$

Since $V^0(x,t)$ satisfies (5), we obtain

$$-V_t^0(x,t) = L(x, u^0, t) + V_x^0 f(x, u^0, t). \tag{10}$$

Using (10), we obtain

$$\begin{aligned} L^*(x,u,t) &= L(x,u,t) + V_t^0(x,t) + V_x^0(x,t) f(x,u,t) \\ &= L(x,u,t) + V_t^0(x,t) + V_x^0 f(x,u^0,t) \\ &\quad - V_x^0 f(x,u^0,t) + V_x^0 f(x,u,t) \\ &= L(x,u,t) - L(x,u^0,t) \\ &\quad + V_x^0[f(x,u,t) - f(x,u^0,t)] \end{aligned} \tag{11}$$

It is immediate to see that $L^*(x,u^0,t) = 0$. Noting that $L_u^*(x,u,t) = L_u(x,u,t) + V_x^0(x,t) f_u(x,u,t)$, it follows from (9) that $L_u^*(x,u^0,t) = 0$. Therefore, there exists a matrix $\Phi(x,t)$ such that

$$L^*(x,u,t) = (u - u^0)^T \Phi(x,t)(u - u^0), \tag{12}$$

where $\Phi(x,t) = L_{uu}^*(x, u^0 + \theta(u - u^0), t)$ with $0 \le \theta \le 1$ and θ depending on u (which is an admissible control law and is continuous differentiable with respect to x, t) and u^0 (which is the optimal control law) and thus on x, t.

If $V_t^0(x,t)$ and $V_x^0(x,t)$ are continuous with respect to x, t, the smoothness of $f(x,u,t), L(x,u,t)$ and the admissible control law imply that $\Phi(x,t)$ is continuous on G.

Now $V^k(x,t) - V^0(x,t)$ will be characterized by making use of $L^*(x,u,t)$. Note that $V^k(x,t) = J(x,t;u^k)$ with u^k defined as $u^k(x,t) = c(x, V_x^{k-1}, t)$.

When $u^k(x,t) = c(x, V_x^{k-1}, t)$ is applied to system (1), the state $x(t)$ satisfies

$$\dot{x} = f(x, u^k, t), x(t_0) = x_0 \tag{13}$$

Along the solution of (13), it follows from (7)

$$\frac{dV^k(x,t)}{dt} = -L(x, u^k, t). \tag{14}$$

For $V^0(x,t)$, it is obvious that, along the solution of (13), one can obtain

$$\frac{dV^0(x,t)}{dt} = V_t^0(x,t) + V_x^0(x,t) f(x, u^k, t). \tag{15}$$

It follows from (14), (15) and (12) that

$$\begin{aligned} &\frac{dV^0(x,t)}{dt} - \frac{dV^k(x,t)}{dt} \\ &= L(x,u^k,t) + V_t^0(x,t) + V_x^0(x,t) f(x,u^k,t) \\ &= L^*(x,u^k,t) \\ &= (u^k - u^0)^T \Phi^k(x,t)(u^k - u^0) \end{aligned} \tag{16}$$

where $\Phi^k(x,t) = L_{uu}^*(x, u^0 + \theta(u^k - u^0), t)$.

Integrate (16) from t_0 to t_1, the following relationship between $V^k(x,t)$ and $u^k(x,t)$ can be established:

$$\begin{aligned} &V^k(x_0,t_0) - V^0(x_0,t_0) \\ &= \int_{t_0}^{t_1} (u^k - u^0)^T \Phi^k(x,t)(u^k - u^0) dt. \end{aligned} \tag{17}$$

3.2 The convergence rate of the Leake-Liu algorithm in G

In this subsection, we will provide some results on the convergence rate of the Leake-Liu algorithm.

We have the following result.

Theorem 2. Suppose that all the conditions in Theorem 1 are met. Then there exist a constant K for $k = 1, 2, \cdots$ such that

$$\begin{aligned} &max_{(x,t)\in G}[V^k(x,t) - V^0(x,t)] \\ &\le K\, max_{(x,t)\in G} \|u^k - u^0\|^2. \end{aligned} \tag{18}$$

Proof: It follows from (17) that we have

$$\begin{aligned} &V^k(x_0,t_0) - V^0(x_0,t_0) \\ &\le \int_{t_0}^{t_1} \|u^k - u^0\|^2 \|\Phi^k(x,t)\| dt \\ &\le \int_{t_0}^{t_1} \|\Phi^k\| \|u^k - u^0\|^2 dt \\ &\le max_{(x,t)\in G} \|u^k - u^0\|^2 \int_{t_0}^{t_1} \|\Phi^k\| dt. \end{aligned} \tag{19}$$

Since (19) is true for any $(x_0, t_0) \in G$ and any $k = 1, 2, \cdots$, it follows that

$$\begin{aligned} &max_{(x,t)\in G}[V^k(x,t) - V^0(x,t)] \\ &\le \int_{t_0}^{t_1} \|\Phi^k\| dt\; max_{(x,t)\in G} \|u^k - u^0\|^2. \end{aligned} \tag{20}$$

Since $V_t^0(x,t)$ and $V_x^0(x,t)$ are continuous with respect to x, t, it can be shown that $\Phi^k(x,t), k = 0, 1, 2, \cdots$ are continuous with respect to x, t on G. Thus, $\Phi^k(x,t), k = 0, 1, 2, \cdots$ are bounded on G. Because $lim_{k\to\infty} V_x^k = V_x^0$ in G, it follows from the smoothness of $L(x,u,t), f(x,u,t)$ and the definition of $\Phi^k(x,t)$ that $lim_{k\to\infty}\Phi^k(x,t) = \Phi^0(x,t) = L_{uu}^*(x,u^0,t)$ in G. Noting that G is compact, this convergence is uniform, which implies that $\Phi^k(x,t)$ are bounded for all k and $(x,t) \in G$. Hence, there exists a constant K independent of k such that for $k = 1, 2, \cdots$, it holds $\int_{t_0}^{t_1} \|\Phi^k\| dt \le K$. This completes the proof. ¶

If the control law sequence converges, the result in the above theorem shows that the convergence rate of the

Leake-Liu algorithm is at least equal to the square of the convergence rate of the control law sequence. If the convergence rate of the control law sequence is the same as or faster than the convergence rate of the Leake-Liu algorithm, the convergence rate of the Leake-Liu algorithm will be quadratically fast.

In the following, we shall establish the relation between the convergence rate of the Leake-Liu algorithm and the convergence rate of the sequence V_x^k.

Note that $u^k = c(x, V_x^{k-1}, t)$ and $u^0 = c(x, V_x^0, t)$, the smoothness of function $c(.,.,.)$ implies that there exists a continuous matrix function $\rho^k(x,t) \geq 0$ such that

$$u^k - u^0 = \rho^k(x,t)(V_x^{k-1} - V_x^0). \tag{21}$$

Then we have the following result, which can be proved in a similar way as Theorem 2.

Theorem 3. Suppose that all the conditions in Theorem 1 are met. Then there exist a constant K for $k = 1, 2, \cdots$ such that

$$\begin{aligned} &max_{(x,t)\in G}[V^{k+1}(x,t) - V^0(x,t)] \\ &\leq K \; max_{(x,t)\in G}\|V_x^k - V_x^0\|^2. \end{aligned} \tag{22}$$

If the sequence V_x^k converges, the result in the above theorem shows that the convergence rate of the Leake-Liu algorithm is at least equal to the square of the convergence rate of the sequence V_x^k. If the convergence rate of V_x^k is the same as or faster than the convergence rate of the Leake-Liu algorithm, the convergence rate of the Leake-Liu algorithm will again be quadratically fast.

It should be noted that the point for having Theorem 2 and Theorem 3 is to provide theoretical results on the convergence rate of the Leake-Liu algorithm rather than techniques that can be used in practical computation. The results are of significant importance because they show that the Leake-Liu algorithm can converge quadratically fast in theory under certain conditions. No such results have been provided although the Leake-Liu algorithm has been proposed for over 40 years. Although these results cannot be directly used in computation, the quadratic convergence rate certainly offers theoretical guidance on the determination of the stopping criterion of the Leake-Liu algorithm.

Further significance of the result in Theorem 3 is that it includes the well known quadratic convergence results in Kleinman (1968) and in Reid (1972) for linear time-invariant and time-varying systems with quadratic performance index as special cases. This point will be seen clearly in the next subsection.

3.3 Special cases

Consider the following time varying system

$$\dot{x} = A(t)x(t) + B(t)u(t), x(t_0) = x_0. \tag{23}$$

Compare it with (1), we have $f(x,u,t) = A(t)x + B(t)u$. Consider the following cost function

$$\begin{aligned} &J(x_0, t_0; u) = x^T(t_1)Wx(t_1) \\ &+ \int_{t_0}^{t_1} x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)dt \end{aligned} \tag{24}$$

where $\lambda[x(t_1), t_1] = x^T(t_1)Wx(t_1)$ with W being a non-negative definite matrix, $L[x,u,t] = x^T Q(t)x + u^T R(t)u$.

According to Anderson and Moore (2007), the optimal cost function is $V^0(x,t) = x^T P(t)x$ and the optimal control is $u^0(t) = -R^{-1}(t)B(t)P(t)x$, where $P(t)$ is the solution of the following equation

$$\begin{aligned} &-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) \\ &+ P(t)B(t)R^{-1}(t)B^T(t)P(t) + Q(t) \end{aligned} \tag{25}$$

with $P(t_1) = W$.

To solve the equation (25), a recursive algorithm has been provided in Reid (1972). The idea is to compute a sequence of matrix functions $P^k(t)$ recursively by solving the respective differential equations

$$\begin{aligned} &-\dot{P}^{k+1}(t) = Q(t) - P^k(t)B(t)R^{-1}(t)B^T(t)P^k(t) \\ &+P^{k+1}(t)[A(t) + B(t)R^{-1}(t)B^T(t)P^k(t)] \\ &+[A^T(t) + P^k(t)B(t)R^{-1}(t)B^T(t)]P^{k+1}(t) \end{aligned} \tag{26}$$

with $P^{k+1}(T) = W$.

Since $V^1 = x^T P^1(t)x$ and $u^2(t) = -R^{-1}(t)B(t)P^1(t)x$, using the fact the system is linear, it is not hard to show that the Leake-Liu algorithm will ensure that $V^k(x,t) = x^T P^k(t)x, k = 2, \cdots$. Then $u^{k+1}(t) = c(x, V_x^k, t) = -R^{-1}(t)B(t)P^k(t)x, k = 1, 2, \cdots$. Now, using $L[x,u,t] = x^T Q(t)x + u^T R(t)u$, it is easy to check that (8) leads to (26). Hence, the recursive algorithm in Reid (1972) is a special case of the Leake-Liu algorithm.

When $A(t) = A, B(t) = B, R(t) = R, Q(t) = Q, P^k(t) = P^k, W = 0, t_1 = \infty$ and under controllability and observability requirements, then the optimal cost function is $V^0(x) = x^T Px$ and the optimal control is $u^0(t) = -R^{-1}BPx$, where P is the solution of the following equation

$$0 = PA + A^T P + PBR^{-1}B^T P + Q. \tag{27}$$

In Kleinman (1968), a recursive algorithm has been proposed to compute P. The idea is to compute a sequence of matrix functions P^k recursively by solving the respective equations

$$\begin{aligned} &0 = Q - P^k BR^{-1}B^T P^k \\ &+P^{k+1}[A + BR^{-1}B^T P^k] \\ &+[A^T + P^k BR^{-1}B^T]P^{k+1} \end{aligned} \tag{28}$$

with P^1 chosen such that $u^2(t) = -R^{-1}BP^1x$ makes the closed-loop system asymptotically stable.

Since $V^1(x) = x^T P^1 x$ and $u^2(t) = -R^{-1}BP^1x$, using the fact the system is linear time-invariant, it is not hard to show that the Leake-Liu algorithm will ensure that $V^k(x) = x^T P^k x, k = 2, \cdots$. Then $u^{k+1}(t) = c(x, V_x^k, t) = -R^{-1}BP^k x, k = 1, 2, \cdots$. Now, using $L[x,u,t] = x^T Qx + u^T Ru$, it is easy to check that (8) leads to (28). Hence, the well-known Kleinman algorithm in Kleinman (1968) is also a special case of the Leake-Liu algorithm.

For the time-varying case, since $V^k(x,t) = x^T P^k(t)x, k = 1, 2, \cdots$, the fact that $lim_{k\to\infty} V^k(x,t) = V^0(x,t) = x^T P(t)x$ implies that $lim_{k\to\infty} P^k(t) = P(t)$. Noting that $V_x^k = 2P^k(t)x$ and $V_x^0 = 2P(t)x$, it follows that $lim_{k\to\infty} V_x^k = V_x^0$. Since all the conditions in Theorem 3 are met and the convergence rate of V_x^k is the same as V^k, it follows that $V^k(x,t)$ converges to $V^0(x,t)$ quadratically fast. By the definitions of $V^k(x,t)$ and $V^0(x,t)$, $P^k(t)$ must converge to $P(t)$ quadratically fast, that is, $max_{t\in[t_0,t_1]} \|P^{k+1}(t) - P(t)\| \le M max_{t\in[t_0,t_1]} \|P^k(t) - P(t)\|^2$, which is the result obtained in Reid (1972). Similarly, for the time-invariant case, it can be shown that $\|P^{k+1} - P\| \le M\|P^k - P\|^2$, which is the result obtained in Kleinman (1968). The above argument shows that the result in Theorem 3 includes the results in Reid (1972) and Kleinman (1968) as its special cases.

3.4 Local quadratic convergence of the Leake-Liu algorithm

In the previous subsection, it is shown that the Leake-Liu algorithm can converge quadratically fast under favorable conditions. However, these favorable conditions may be too strong or difficult to check. In this subsection, it will be shown that these assumptions can be removed if only the local convergence rate of $V^k(x,t), k = 1, 2, \cdots$ (i.e. convergence restricting x to a neighborhood of the origin) is of concern.

The following lemma is useful.

Lemma 1. Suppose $V(x,t)$ is a smooth function of x and t with $V(x,t) > 0$ for $x \neq 0$ and all t and $V(0,t) = 0$ for all t. Then, $V(x,t)$ can be written as

$$V(x,t) = x^T \Phi(x,t)x \tag{29}$$

where $\Phi(x,t) = V_{xx}(\theta x, t)$ with $0 \le \theta \le 1$ and θ depending on x.

Proof: Expand $V(x,t)$ at $x = 0$, we obtain

$$V(x,t) = V(0,t) + V_x(0,t)x + E(x,t), \tag{30}$$

where $E(x,t)$ is the quadratic remainder in the Taylor series of $V(x,t)$ at $x = 0$.

Using the well known estimate for the remainder, we have

$$E(x,t) = x^T V_{xx}(\theta x, t)x, \tag{31}$$

where $0 \le \theta \le 1$ and θ depends on x.

Since θ depends on x,$V_{xx}(\theta x,t)$ is a function of x, t and thus can be denoted as $\Phi(x,t)$. This proves $V(x,t) = x^T\Phi(x,t)x$.

Note that $V(0,t) = 0$ and also that $V(x,t) > 0$ for $x \neq 0$ and all t, it follows (30) and (31) that $V_x(0,t)$ must be zero for all t. This leads to the conclusion of the lemma immediately.¶

The following assumption, almost costless in terms of loss of generality, is made on $V^k(x,t), k = 1, 2, \cdots$.

- Assumption A4: $V^1(0,t) = 0$ for all t, and $V^0(x,t) > 0$ for $x \neq 0$ and all t.

Lemma 2. Suppose that all the conditions in Theorem 1 are met. If assumption A4 is satisfied, then $V^k(x,t) > 0, k = 1, 2, \cdots$ for all $x \neq 0$ and all t, and $V^k(0,t) = 0, k = 0, 2, 3, \cdots$ for all t.

Proof: This lemma follows from Lemma 1 and Theorem 1 immediately.¶

Also, the following result can be proven.

Lemma 3. Suppose that all the conditions in Theorem 1 are met and that assumption A4 is satisfied. Then there exists a sequence of matrices $\Phi^k(x,t), k = 0, 1, 2, \cdots$ with continuous elements such that

$$V^k(x,t) = x^T\Phi^k(x,t)x, k = 0, 1, 2, \cdots. \tag{32}$$

Proof: The conclusion follows from Lemma 2 and Lemma 1. ¶

Because of Lemma 3, $V^k(x,t)$ can be written in the following form:

$$V^k(x,t) = x^T P_k(t)x + O(x),, k = 0, 1, 2, \cdots \tag{33}$$

where $P_k(t), k = 0, 1, 2, \cdots$ are symmetric and $O(x)$ denotes the high order terms.

The following assumptions, which are without significant loss of generality, are also needed.

- Assumption A5: $P_0(t) > 0$.
- Assumption A6: For $k = 2, 3, \cdots$, u^k is a control law such that the solutions of $\dot{x} = f(x, u^k, t)$ satisfy $(x,t) \in G$, $\|x(t)\| \le M_1\|x_0\|$ and $\|V_x^k\| \le M_2$ with M_1 and M_2 independent of k.

Regarding $P_k(t)$, the following result can be obtained.

Theorem 4. Suppose that all the conditions in Theorem 1 are met. If assumption A4 is satisfied, then

$$P_0(t) \le P_{k+1}(t) \le P_k(t) \le P_1(t); \tag{34}$$

and $lim_{k\to\infty} P_k(t) = P_0(t)$. If in addition, assumption A5 is also satisfied, then $P_k(t) > 0, k = 1, 2, \cdots$ for all t;

Proof: According to Theorem 1, we have $V^{k+1}(x,t) \le V^k(x,t)$. This together with (33) implies that $P_{k+1}(t) \le P_k(t)$. Use this argument, the first conclusion is proved to be correct. Since $lim_{k\to\infty} V^k(x,t) = V^0(x,t)$, we must have $lim_{k\to\infty} P_k(t) = P_0(t)$. The first conclusion together with assumption A5 proves that $P_k(t) > 0, k = 1, 2, \cdots$ for all t. ¶

The above theorem actually shows the quadratic terms $x^T P_k(t)x, k = 1, 2, \cdots$ in $V^k(x,t), k = 1, 2, \cdots$ are non-increasing and converge to the quadratic term $x^T P_0(t)x$ in $V^0(x,t)$.

A further result is presented in the following theorem.

Theorem 5. Suppose that all the conditions in Theorem 1 are met. If assumptions A4, A5 and A6 are satisfied, there exists a constant K such that

$$\begin{aligned} &max_{t\in[t_0,t_1]} \|P_{k+1}(t) - P_0(t)\| \\ &\le K\ max_{t\in[t_0,t_1]} \|P_k(t) - P_0(t)\|^2, k = 1, 2, \cdots. \end{aligned} \tag{35}$$

Proof: Using (33), $\Phi^k(x,t)$ can be written as

$$\Phi^k(x,t) = P_k(t) + O(x), k = 0, 1, 2, \cdots. \quad (36)$$

Substitute (36) into (17) and use (21), one obtains

$$\begin{aligned} & x_0^T(P_{k+1}(t_0) - P_0(t_0))x_0 + O(x_0) \\ & \leq M_1\|x_0\|^2 \int_{t_0}^{t_1} \rho^{k+1}\|\Phi^{k+1}\|\|P_k(t) - P_0(t)\|^2 dt \\ & +O(x_0), k = 1, 2, \cdots. \end{aligned} \quad (37)$$

Since G is bounded and closed, it is not hard to show that Assumption A6 implies that there exists a constant K such that

$$M_1 \int_{t_0}^{t_1} \rho^{k+1}\|\Phi^{k+1}\|dt \leq K.$$

The above fact together with (37) implies that there exists a positive constant δ small enough such that for any $x_0 \in B_\delta$, the following holds

$$\begin{aligned} & x_0^T(P_{k+1}(t_0) - P_0(t_0))x_0 \\ & \leq K\|x_0\|^2 max_{t\in[t_0,t_1]}\|P_k(t) - P_0(t)\|^2, k = 1, 2, \cdots \end{aligned} \quad (38)$$

Since $P_k(t_0) > 0, k = 0, 1, 2, \cdots$, (38) implies that

$$\begin{aligned} & \|P_{k+1}(t_0) - P_0(t_0)\| \\ & \leq K\ max_{t\in[t_0,t_1]}\|P_k(t) - P_0(t)\|^2, k = 1, 2, \cdots \end{aligned} \quad (39)$$

Since (39) is true for any $t_2 \in [t_0, t_1]$, we have

$$\begin{aligned} & \|P_{k+1}(t_2) - P_0(t_2)\| \\ & \leq K\ max_{t\in[t_2,t_1]}\|P_k(t) - P_0(t)\|^2 \\ & \leq K\ max_{t\in[t_0,t_1]}\|P_k(t) - P_0(t)\|^2, k = 1, 2, \cdots \end{aligned} \quad (40)$$

It follows from (40) immediately $max_{t\in[t_0,t_1]}\|P_{k+1}(t) - P_0(t)\| \leq K\ max_{t\in[t_0,t_1]}\|P_k(t) - P_0(t)\|^2, k = 1, 2, \cdots$, which completes the proof. ¶

The result presented in Theorem 5 proves that the convergence rate of the quadratic terms $x^T P_k(t)x, k = 1, 2, \cdots$ in $V^k(x,t), k = 1, 2, \cdots$ is actually quadratic in a sense that $max_{t\in[t_0,t_1]}\|P_{k+1}(t) - P_0(t)\| \leq K\ max_{t\in[t_0,t_1]}\|P_k(t) - P_0(t)\|^2, k = 1, 2, \cdots$.

With the help of Theorem 5, the following local result on the convergence rate of $V^k(x,t), k = 1, 2, \cdots$ can be proved.

Theorem 6. Suppose that all the conditions in Theorem 1 are met. If assumptions A4, A5 and A6 are satisfied, there exists a positive constant δ such that for any $\|x\| < \delta$, the sequence $V^k(x,t), k = 1, 2, \cdots$ converges quadratically.

Proof: It follows from (33) that

$$\begin{aligned} & \|V^{k+1}(x,t) - V^0(x,t)\| \\ & = \|x^T(P_{k+1}(t) - P_0(t))x\| + O(x), \end{aligned} \quad (41)$$

where $O(x)$ is of order higher than the term $\|x\|^2$.

Note that $O(x)$ is of order higher than the term $\|x\|^2$, there exists a positive constant δ such that for any $\|x\| < \delta$,

$$\begin{aligned} & \|x^T(P_{k+1}(t) - P_0(t))x\|/2 \\ & \leq \|V^{k+1}(x,t) - V^0(x,t)\| \\ & \leq 2\|x^T(P_{k+1}(t) - P_0(t))x\|, \end{aligned} \quad (42)$$

This proves that for any $\|x\| < \delta$, the convergence rate of $V^k(x,t), k = 1, 2, \cdots$ is the same as the sequence $P_k(t), k = 1, 2, \cdots$, which proves the theorem. ¶

It is obvious that the result would be more attractive if one could provide an estimate on the constant δ. However, given the fact that general nonlinear systems are under consideration, such an estimate is highly nontrivial and will be left as a future topic.

4. CONCLUSIONS

In this paper, the convergence rate of the Leake-Liu algorithm for solving Hamilton-Jacobi-Bellman equation has been investigated. It has been shown the convergence rate of the Leake-Liu algorithm can be quadratically fast in a compact domain of interest under favorable conditions. With less restrictive conditions, it is shown that the convergence rate of the Leake-Liu algorithm is quadratic locally.

In this paper, only the case that G is compact was considered. The case that G is not compact, which is more challenging, is left as a future research topic.

REFERENCES

M. Abu-Khalaf and F.L. Lewis. Nearly optimal control laws for nonlinear systems with saturating actuators using a neural network HJB approach. *Automatica,* 41: 779-791, 2005.

B.D.O. Anderson and J.B. Moore. *Optimal Control: Linear Quadratic Methods, Augmented edition.* Dover Publications, New York, 2007.

R.W. Beard and T. W. McLain. Successive Galerkin approximation algorithms for nonlinear optimal and robust control. *International Journal of Control*, 71: 717-743, 1998.

Y. Feng, B.D.O. Anderson, and M. Rotkowitz. A game theoretic algorithm to compute local stabilizing solutions to HJBI equations in nonlinear H_∞ control. *Automatica,* 45: 881-888, 2009.

R.E. Kalman. Contributions to the theory of optimal control. *Reimpreso del Boletin de la Sociedad EMathematica Mexicana*, 102-119, 1960.

D.L. Kleinman. On an iterative technique for Riccati equation computations. *IEEE Trans. Automatic Control*, 13: 114-115, 1968.

R.J. Leake and Ruey-Wen Liu. Construction of suboptimal control sequences. *SIAM J. Control*, 5: 54-63, 1967. frank2005

W. T. Reid. *Riccati Differential Equations.* Academic Press, New York and London, 1972.

G. N. Saridis and Chun-Sing G. Lee. An approximation theory of optimal control for trainable manipulatyors. *IEEE Trans. Systems, Man, and Cybernetics*, 9: 152-159, 1979.