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Fixed Structure Controller Synthesis Using Groebner Bases and Sign-Definite Decomposition ⋆

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Abstract:
This paper presents a new method for computing stabilizing fixed structure/order controllers using Groebner bases and sign-definite decomposition. An application of Routh-Hurwitz stability condition results in a system of polynomial inequalities that must be satisfied by the parameters of any stabilizing controller. Using positive slack variables, the system of polynomial inequalities can be converted to a system of polynomial equations. With the aid of Groebner bases and elimination theory, an equivalent system of polynomial equations can be obtained which simplifies the construction of the set of stabilizing controllers using the sing-definite decomposition. The results of this approach are illustrated by examples provided.

Keywords: Fixed order control, Routh-Hurwitz stability, Groebner bases, elimination theory, sign-definite decomposition.

1. INTRODUCTION

The problem of synthesizing stabilizing controllers of a fixed structure arises in many practical applications and has been open for about five decades. Without any restriction on the structure of the controller, the problem of controller design can be handled through various techniques of modern control theory. However, the constraint on structure yields non-convex constraints and the corresponding set of stabilizing controller parameters is often non-convex and at times, is disconnected.
While the attempts on solving this problem have been numerous, we will restrict here to a subset of these bodies of work for the want of space and focus on some articles that deal with the approximation of the set of controllers of fixed structure or with algebraic techniques dealing with elimination. The problem of deciding the existence of a stabilization with a fixed structure/order controller reduces to the problem of deciding the feasibility of a system of polynomial inequalities and this can be shown to be decidable using a plethora of techniques such as Quantifier Elimination Anderson et al. (1975) or using Groebner bases Cox et al. (2007). Anai and Hara (2000) proposed a method based on sign-definite condition and a special Quantifier Elimination (QE) technique to design robust controllers of a fixed structure. Recently, Shin and Lall (2010a,b) have approached the problem of optimal decentralized controller synthesis using Groebner bases.

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The problem of approximation of sets of stabilizing controllers is important for a practical viewpoint as stabilization is an essential part of any control system. Parametric control design techniques are well suited for this purpose and earlier work concerning these techniques can be found in Siljak (1969). Recently, Bhattacharyya et al. (2009) provided a systematic method for constructing the set of PID controllers; PID controllers are fixed structure controllers that are widely employed in industrial applications. This work exploits the specific structure of PID controllers. Malik et al. (2008) provided a systematic method for arbitrarily tight inner and outer approximation of the set of stabilizing controllers of a fixed order for a single-input or a single-output system. Henrion (2003) used the properties of positive polynomials to obtain a convex inner approximation of the set of stabilizing controllers in the space of controller parameters. In this paper, we plan to use Groebner bases and sign-definite decomposition method for constructing an inner approximation of the set of stabilizing controllers. The latter method was proposed by Elizondo-Gonzalez (2000) and followed up by Knap et al. (2011) for the construction of a set of stabilizing controllers.
This paper is organized in 4 sections. In section 2, we explain our method in a general way. The characteristic polynomial of a closed loop linear system has coefficients that are polynomial functions of control parameters $\mathbf{K} = [k_1, k_2, ..., k_m]^T$. Routh-Hurwitz criterion provides the stability conditions as a set of polynomial inequalities. Introducing strictly positive slack variables, these inequal-

ities can be converted to equalities. These slack variables are dependent on the controller parameters. With the aid of Groebner bases and elimination theory the controller parameters can be decoupled and expressed in terms of the slack variables. Some of the new equations involve only the slack variables (slack constraints) which represent the stability region(s) in the space of the slack variables. Other equations involve both the controller parameters and the slack variables which provide the mapping functions from the space of the slack variables to the space of the controller parameters. Since the slack constraints involve strictly positive variables, the stability region(s) computations via sign-definite decomposition can be simplified. Section 3 provides some examples to show how this approach can be applied to a given problem. Finally we summarize our conclusions in section 4.

2. PROCEDURE

2.1 Routh-Hurwitz criterion and Groebner bases

Consider a unity feedback control system with a known plant transfer function $P(s) = \frac{N_p(s)}{D_p(s)}$ and a controller transfer function $C(s) = \frac{N_c(s,\mathbf{K})}{D_c(s,\mathbf{K})}$ where $\mathbf{K} = [k_1, k_2, ..., k_m]^T$ is the vector of controller design parameters. The set of stabilizing controllers will be all vectors $\mathbf{K}$ that result the closed loop characteristic polynomial Hurwitz.
The closed loop characteristic polynomial can be written as

$$\delta(s,\mathbf{K}) = N_p(s)N_c(s,\mathbf{K}) + D_p(s)D_c(s,\mathbf{K}) \quad (1)$$

or in the general form of

$$\delta(s,\mathbf{K}) = a_n(\mathbf{K})s^n + a_{n-1}(\mathbf{K})s^{n-1} + \cdots + a_0(\mathbf{K}) \quad (2)$$

where $a_n(\mathbf{K}) \neq 0$.
From Routh-Hurwitz stability criterion, the number of RHP roots of the closed loop characteristic polynomial is equal to the number of changes in sign of the elements of the first column of Routh-Hurwitz table. This means that the set

$$f_0(\mathbf{K}) > 0\ ,\ f_1(\mathbf{K}) > 0\ ,\ \ldots\ ,\ f_n(\mathbf{K}) > 0 \quad (3)$$

where f_i's represent the elements of the first column of Routh-Hurwitz table, define the boundaries of stability region(s) in the space of the controller parameters. In general, this is a set of multivariate polynomial inequalities in terms of the controller design parameters $k_1, k_2, ..., k_m$ which is hard to solve and in some cases practically impossible. We convert inequalities (3) to equalities by introducing strictly positive slack variables $s_0, s_1, ..., s_n$; so that

$$\begin{aligned} h_0(\mathbf{K}, s_0) &= f_0(\mathbf{K}) - s_0 = 0 \\ h_1(\mathbf{K}, s_1) &= f_1(\mathbf{K}) - s_1 = 0 \\ &\vdots \\ h_n(\mathbf{K}, s_n) &= f_n(\mathbf{K}) - s_n = 0. \end{aligned} \quad (4)$$

In this set of equations, the slack variables are dependent variables and expressed in terms of the independent variables which are the controller parameters. The controller parameters are coupled in the equations (4). Now an approach that can decouple these parameters which express them in terms of the slack variables is desirable. Such a decoupling can be accomplished using Groebner bases and elimination theory on polynomial rings. We refer the readers to Cox et al. (2007) for a detailed treatment of Groebner bases.
If we were to choose a *Lexicographic ordering* $k_m > k_{m-1} > k_{m-2} > \cdots > k_1 > s_n > s_{n-1} > \cdots > s_1 > s_0$, then the variable k_m is eliminated first, followed by k_{m-1} and so on. As a result, the resulting reduced set of polynomial equations will involve one less variable every time a variable is eliminated as is the case in a Gaussian elimination, i.e., the system of polynomial equations will be triangular. Let the system of polynomial equations after the elimination process be

$$\begin{aligned} g_0(\mathbf{S}) &= 0 \\ g_1(\mathbf{S}) &= 0 \\ &\vdots \\ g_p(\mathbf{S}) &= 0 \end{aligned} \quad (5)$$

$$\begin{aligned} g_{p+1}(\mathbf{K}, \mathbf{S}) &= 0 \\ g_{p+2}(\mathbf{K}, \mathbf{S}) &= 0 \\ &\vdots \\ g_t(\mathbf{K}, \mathbf{S}) &= 0. \end{aligned} \quad (6)$$

We observe that equations (5) may not have the triangular structure because the number of variables is $m + n + 1$ (including the slack variables) and is more than the number of polynomial equations, which are only $n + 1$ in number. By specifying the Lexicographic ordering in the above mentioned manner, we want to treat m of the slack variables to be independent variables (which is given by equations (5)) and the rest of them (including the controller parameters) can be determined for any given value of the m independent variables through the system of equations in (5) and the triangular system of equations in (6). It is possible that for the same set of m independent variables, there may be more than one set of control parameters.
The equations (5), referred to as the *slack constraints*, define an algebraic variety in the space of the slack variables. Since the slack variables are strictly positive, this variety is confined to the first orthant of the space of the slack variables. All the vectors $\mathbf{S} = [s_0, s_1, ..., s_n]^T$, (where $s_i > 0,\ i = 0, 1, ..., n$) satisfying the equations (5) represent the stability region(s) in the space of the slack variables. The computation of the stability region (via the sign-definite decomposition) is simpler in the space of the slack variables than in the space of the controller parameters because the slack variables take positive values; however, the controller parameters take positive and negative values. For a specific vector $\mathbf{S} = [s_0, s_1, ..., s_n]^T$ satisfying the equations (5), one can sequentially find $k_1, k_2, ..., k_m$ using equations (6).
Therefore the procedure described above can be summarized as

(1) Write the Routh-Hurwitz stability inequalities for the closed loop characteristic polynomial,
(2) Convert inequalities to equalities by introducing slack variables,
(3) Find the Groebner bases of the system of polynomials obtained above (which involve controller parameters and slack variables) using Lexicographic ordering.

It should be noted that the necessary condition of the Routh-Hurwitz stability criterion is that all the coefficients of the characteristic polynomial be non-zero and all have the same sign. This can be embedded into the set of equations (4). This induces more slack variables which will increase the number of slack constraints in (5), but may simplify the equations (5) and (6).

2.2 Sign-Definite Decomposition in Determining Positivity (Negativity) of Polynomials

Elizondo-Gonzalez (2000) followed by Knap et al. (2011) have proposed a method to determine the robust positivity (negativity) of a real function $f(\mathbf{x})$ as the real vector $\mathbf{x}$ varies over a box $\mathcal{X} \in R^n$ by only checking a finite number of specially constructed points. Let $f(\mathbf{x})$ with $\mathbf{x} = (x_1, x_2, ..., x_n)$ be a real function of $\mathbf{x}$ and consider the problem of determining if $f(\mathbf{x})$ is positive over the box

$$\mathcal{X} = \{\mathbf{x} : x_i^- \leq x_i \leq x_i^+, \text{ for all } i\}.$$

The function $f(\mathbf{x})$ can be decomposed as

$$f(\mathbf{x}) = f^+(\mathbf{x}) - f^-(\mathbf{x}) \quad (7)$$

where $f^+(\mathbf{x}) \geq 0$, $f^-(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathcal{X}$. Now assume that x_i's take only positive values. Defining $\mathbf{x}^+$ and $\mathbf{x}^-$ as

$$\mathbf{x}^+ = (x_1^+, x_2^+, ..., x_n^+)$$
$$\mathbf{x}^- = (x_1^-, x_2^-, ..., x_n^-)$$

such that

$$\begin{aligned} f^+(\mathbf{x}^+) &= \max_{\mathbf{x}\in\mathcal{X}} f^+(\mathbf{x}) \\ f^-(\mathbf{x}^+) &= \max_{\mathbf{x}\in\mathcal{X}} f^-(\mathbf{x}) \\ f^+(\mathbf{x}^-) &= \min_{\mathbf{x}\in\mathcal{X}} f^+(\mathbf{x}) \\ f^-(\mathbf{x}^-) &= \min_{\mathbf{x}\in\mathcal{X}} f^-(\mathbf{x}). \end{aligned} \quad (8)$$

Therefore

$$\begin{aligned} f^+(\mathbf{x}^-) \leq f^+(\mathbf{x}) \leq f^+(\mathbf{x}^+) \\ f^-(\mathbf{x}^-) \leq f^-(\mathbf{x}) \leq f^-(\mathbf{x}^+). \end{aligned} \quad (9)$$

Now consider the rectangle formed by the following four points in the (f^-, f^+) plane

$$\begin{aligned} A &= \big(f^-(\mathbf{x}^-), f^+(\mathbf{x}^-)\big) \\ B &= \big(f^-(\mathbf{x}^-), f^+(\mathbf{x}^+)\big) \\ C &= \big(f^-(\mathbf{x}^+), f^+(\mathbf{x}^+)\big) \\ D &= \big(f^-(\mathbf{x}^+), f^+(\mathbf{x}^-)\big) \end{aligned} \quad (10)$$

Now it can be shown that for all $\mathbf{x} \in \mathcal{X}$ (see Fig. 1)

$$f(\mathbf{x}) \begin{cases} \geq 0, \text{ if } f^+(\mathbf{x}^-) - f^-(\mathbf{x}^+) \geq 0 \\ \leq 0, \text{ if } f^+(\mathbf{x}^+) - f^-(\mathbf{x}^-) \leq 0. \end{cases} \quad (11)$$

This relation can be used recursively to construct the robustly positive regions. For more details see Elizondo-Gonzalez (2000) and Knap et al. (2011). We use (11) later to plot the stability region in the space of the (free) slack variables.

3. EXAMPLES

3.1 SISO: Second-order Plant and First-order Controller

Now consider a general second-order plant and a general first-order controller in a unity feedback control system.

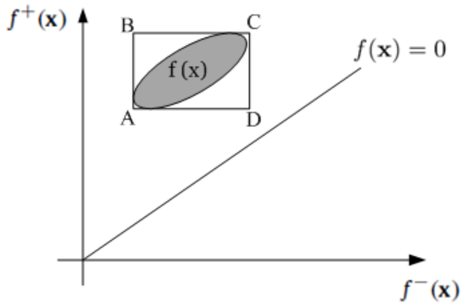

Fig. 1. Condition for Positivity of $f(\mathbf{x})$ in the Sign-Definite Decomposition Method

The corresponding transfer functions for the plant and the controller are

$$\begin{aligned} P(s) &= \frac{q_1 s + q_0}{s^2 + p_1 s + p_0} \\ C(s) &= \frac{k_1 s + k_2}{s + k_3} \end{aligned} \quad (12)$$

where the plant parameters p_0, p_1, q_0, q_1 are known and the controller parameters $\mathbf{K} = [k_1, k_2, k_3]^T$ are unknown. The closed loop characteristic polynomial in this case will be

$$\begin{aligned} \delta(s, \mathbf{K}) = s^3 &+ (q_1 k_1 + p_1 + k_3) s^2 \\ &+ (p_0 + q_0 k_1 + q_1 k_2 + p_1 k_3) s \\ &+ (p_0 k_3 + q_0 k_2). \end{aligned} \quad (13)$$

The elements of the first column of the Routh-Hurwitz array must be strictly positive in order to have a stable closed-loop system, therefore

$$\begin{aligned} f_0(\mathbf{K}) &= q_1 k_1 + p_1 + k_3 > 0 \\ f_1(\mathbf{K}) &= q_0 q_1 k_1^2 + p_1 k_3^2 + q_1^2 k_1 k_2 + (p_1 q_1 + q_0) k_1 k_3 \\ &\quad + q_1 k_2 k_3 + (p_1 q_0 + p_0 q_1) k_1 + (p_1 q_1 - q_0) k_2 \\ &\quad + p_1^2 k_3 + p_0 p_1 > 0 \\ f_2(\mathbf{K}) &= p_0 k_3 + q_0 k_2 > 0 \end{aligned} \quad (14)$$

where the term $f_1(\mathbf{K})$ represents only the numerator of the 3rd element in the Routh-Hurwitz array because its denominator $f_0(\mathbf{K})$ is already assumed to be positive. Also the first element of the array is 1 which is positive and is not included in (14). Defining slack variables $s_0 > 0$, $s_1 > 0$, $s_2 > 0$, one can generate h_0, h_1, h_2 as

$$\begin{aligned} h_0(\mathbf{K}, s_0) &= f_0(\mathbf{K}) - s_0 = 0 \\ h_1(\mathbf{K}, s_1) &= f_1(\mathbf{K}) - s_1 = 0 \\ h_2(\mathbf{K}, s_2) &= f_2(\mathbf{K}) - s_2 = 0. \end{aligned} \quad (15)$$

The Groebner bases of the polynomials in (15) with respect to Lexicographic ordering $k_1 > k_2 > k_3 > s_2 > s_1 > s_0$ are

$$\begin{aligned} g_0(k_3, \mathbf{S}) &= -q_0^2 s_1^2 - q_1^2 s_0 s_1 + q_0 q_1 s_0 \\ &\quad + (q_0^2 p_1 - q_0 p_0 q_1) s_1 + q_0 q_1 s_2 \\ &\quad + (p_0 q_1^2 - q_0 p_1 q_1 + q_0^2) s_1 k_3 \\ g_1(k_2, k_3, \mathbf{S}) &= q_0 k_2 + p_0 k_3 - s_0 \\ g_2(k_1, k_3, \mathbf{S}) &= q_1 k_1 + k_3 - s_1 + p_1. \end{aligned} \quad (16)$$

None of the above Groebner bases are in terms of only the slack variables, i.e. there is no constraint on choosing slack variables, therefore the entire first orthant in the space of s_0, s_1, s_2 is the stability region for this example. One may set the Groebner bases (16) to zero and solve for the controller parameters k_1, k_2, k_3;

$$k_3 = \frac{q_0^2 s_1^2 + q_1^2 s_0 s_1 - q_0 q_1 (s_0 + s_2) + (q_0 p_0 q_1 - q_0^2 p_1) s_1}{(p_0 q_1^2 - q_0 p_1 q_1 + q_0^2) s_1} \tag{17}$$

$$k_2 = \frac{-1}{q_0}(p_0 k_3 - s_0) \tag{18}$$

$$k_1 = \frac{-1}{q_1}(p_1 + k_3 - s_1). \tag{19}$$

This example shows a special case of our method where there is no restriction on choosing slack variables, i.e. the entire first orthant in the space of the slack variables is the stability region. This is analogous to the pole placement problem where the number of controller parameters is the same as the number of closed loop poles.

3.2 SISO: Third-order Plant and First-order Controller

In this example we show a case where a constraint on slack variables exists. Consider the following third-order plant and a general first-order controller as

$$\begin{aligned} P(s) &= \frac{s^2 + s - 1}{s^3 + 2s^2 + s - 1} \\ C(s) &= \frac{k_1 s + k_2}{s + k_3} \end{aligned} \tag{20}$$

where the controller parameters $\mathbf{K} = [k_1, k_2, k_3]^T$ are unknown. The closed loop characteristic polynomial is

$$\begin{aligned} \delta(s, \mathbf{K}) = \; & s^4 + (k_3 + 2 + k_1)s^3 \\ & +(k_1 + k_2 + 1 + 2k_3)s^2 \\ & +(k_2 - k_1 + k_3 - 1)s - k_3 - k_2. \end{aligned} \tag{21}$$

The Routh-Hurwitz array corresponding to the characteristic polynomial (21) can be constructed easily. In this example embedding the positivity of the coefficients of the characteristic polynomial simplifies the Groebner bases equations. Although this increases the number of the slack variables and the slack constraints, the number of the *free slack variables* (introduced later) does not change and therefore the stability region in the space of the free slack variables can still be plotted in a 3-dimensional space. Therefore there are 6 inequalities in this case. Defining strictly positive slack variables $s_0, s_1, ..., s_5$, one can construct $h_0, h_1, ..., h_5$ as

$$\begin{aligned} h_0 = \; & k_3 + 2 + k_1 - s_0 = 0 \\ h_1 = \; & 3k_1k_3 + 4k_1 + {k_1}^2 + k_2k_3 + k_2 + k_2k_1 \\ & +4k_3 + 3 + 2{k_3}^2 - s_1 = 0 \\ h_2 = \; & -3 - 7k_1 + 6k_2 + 3k_3 + k_1k_3 - 5{k_1}^2 \\ & +8k_2k_3 + 6k_2k_1 + 6{k_3}^2 + {k_2}^2 + {k_2}^2k_3 \\ & +{k_2}^2k_1 + 4k_2{k_3}^2 - {k_1}^2k_3 + 3k_1{k_3}^2 \\ & +k_2{k_1}^2 + 5k_2k_3k_1 - {k_1}^3 + 3{k_3}^3 - s_2 = 0 \\ h_3 = \; & -k_3 - k_2 - s_3 = 0 \\ h_4 = \; & k_2 - k_1 + k_3 - 1 - s_4 = 0 \\ h_5 = \; & k_1 + k_2 + 1 + 2k_3 - s_5 = 0. \end{aligned} \tag{22}$$

The Groebner bases of the polynomials in (22) with respect to Lexicographic ordering $k_1 > k_2 > k_3 > s_2 > s_1 > s_3 > s_0 > s_4 > s_5$ are

$$\begin{aligned} g_1 = \; & 1 + s_3 - s_0 + s_5 \\ g_2 = \; & -s_5s_0 + s_4 + s_1 \\ g_3 = \; & {s_4}^2 - {s_0}^2 - s_5s_4s_0 - s_5{s_0}^2 \\ & +s_2 + {s_0}^3 \end{aligned} \tag{23}$$

$$\begin{aligned} g_4 = \; & 2 - 2s_0 - s_4 + s_5 + k_3 \\ g_5 = \; & -3 + 3s_0 + s_4 - 2s_5 + k_2 \\ g_6 = \; & s_0 + s_4 - s_5 + k_1. \end{aligned} \tag{24}$$

The Groebner bases (24) involve the controller parameters and they are decoupled. Setting (24) to zero, one may obtain the controller parameters k_1, k_2, k_3 as

$$k_3 = -2 + 2s_0 + s_4 - s_5 \tag{25}$$

$$k_2 = 3 - 3s_0 - s_4 + 2s_5 \tag{26}$$

$$k_1 = -s_0 - s_4 + s_5. \tag{27}$$

In this example s_0, s_4, s_5 are the *free slack variables*. One may set (23) to zero and solve for s_3, s_1, s_2 respectively as (recall that the slack variables are strictly positive)

$$s_3 = -1 + s_0 - s_5 > 0 \tag{28}$$

$$s_1 = s_5s_0 - s_4 > 0 \tag{29}$$

$$s_2 = -{s_4}^2 + {s_0}^2 + s_5s_4s_0 + s_5{s_0}^2 - {s_0}^3 > 0. \tag{30}$$

Now s_1, s_2, s_3 are the *constrained slack variables* and the inequalities (28)-(30) define the stability region in the first orthant of the space of the free slack variables s_0, s_4, s_5. Any vector $[s_0, s_4, s_5]^T$ satisfying the above inequalities will guarantee the positivity of s_1, s_2, s_3 and can be mapped into the space of the controller parameters by (25)-(27). As mentioned earlier, one important advantage of this approach is that the slack variables are positive. This simplifies the computations involving sign-definite decomposition because all the variables are positive and therefore the approximation boxes should be constructed only in the first orthant of the space of the free slack variables; however, on the other hand, applying the sign-definite decomposition directly to the Routh-Hurtwitz inequalities requires consideration of all orthants because the controller parameters can take negative values as well.
For this example we define the following polynomials

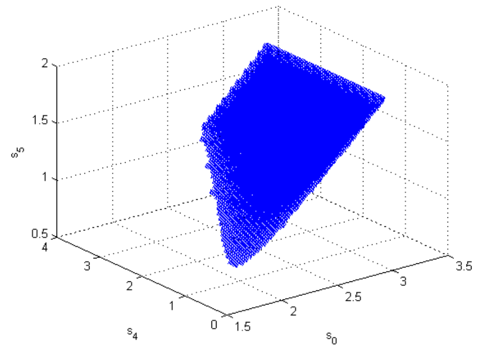

Fig. 2. The Stability Region in the Space of the Free Slack Variables for the Control Feedback System (20)

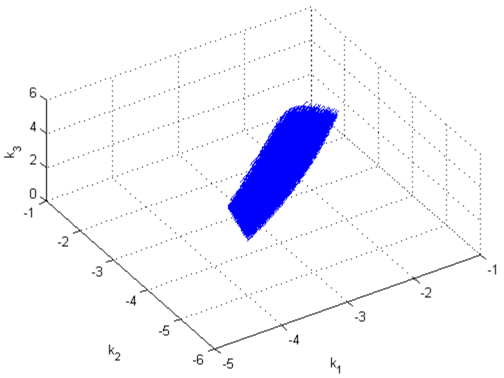

Fig. 3. The Stability Region in the Space of the Controller Parameters for the Control Feedback System (20)

$$
\begin{aligned}
s_3^+ &= s_0 \\
s_3^- &= 1 + s_5 \\
s_1^+ &= s_5 s_0 \\
s_1^- &= s_4 \\
s_2^+ &= s_0^2 + s_5 s_4 s_0 + s_5 s_0^2 \\
s_2^- &= s_4^2 + s_0^3. \qquad (31)
\end{aligned}
$$

Now each pair of (s_i^+, s_i^-), $i = 1, 2, 3$ are treated as $f^+(\mathbf{x}), f^-(\mathbf{x})$, introduced earlier, and the approximation boxes are defined as

$$\mathcal{S} = \{\mathbf{s} : s_i^- \le s_i \le s_i^+,\ i = 0, 4, 5\}.$$

The stability region defined by (28)-(30) in the space of the free slack variables is plotted in Fig. 2 via the sign-definite decomposition method. Each vector $\mathbf{S} = [s_0, s_4, s_5]^T$ in the plot of Fig. 2 corresponds to a vector $\mathbf{K} = [k_1, k_2, k_3]^T$ by (25)-(27). Fig. 3 shows the stability region in the space of the controller parameters k_1, k_2, k_3.

3.3 MIMO Feedback Control System

Consider the following characteristic polynomial corresponding to a MIMO feedback system. The controller parameters are k_1, k_2.

$$
\begin{aligned}
\delta(s, \mathbf{K}) = {} & s^4 + (k_1 - 2 + k_2)\, s^3 \\
& + (k_1 k_2 + 2k_2 + k_1 - 3)\, s^2 \\
& + (4 - 5k_2 - 4k_1 + 5k_1 k_2)\, s \\
& + 4 - 6k_2 + 6k_1 k_2 - 4k_1. \qquad (32)
\end{aligned}
$$

The stability inequalities from the Routh-Hurwitz array and the coefficients of the characteristic polynomial are

$$
\begin{aligned}
f_0 = {} & k_1 - 2 + k_2 > 0 \\
f_1 = {} & {k_1}^2 k_2 - 4k_1 k_2 + k_1 {k_2}^2 - 2k_2 \\
& + 2{k_2}^2 + {k_1}^2 - k_1 + 2 > 0 \\
f_2 = {} & -8 + 22k_2 - 65k_1 k_2 + 48{k_1}^2 k_2 + 46k_1 {k_2}^2 \\
& - 10{k_2}^2 - 12{k_1}^2 - 41{k_1}^2{k_2}^2 - k_1 {k_2}^3 \\
& - 5{k_1}^3 k_2 + 5{k_1}^3{k_2}^2 + 5{k_1}^2{k_2}^3 \\
& - 4{k_2}^3 + 20k_1 > 0 \\
f_3 = {} & 4 - 6k_2 + 6k_1 k_2 - 4k_1 > 0 \\
f_4 = {} & 4 - 5k_2 - 4k_1 + 5k_1 k_2 > 0 \\
f_5 = {} & k_1 k_2 + 2k_2 + k_1 - 3 > 0. \qquad (33)
\end{aligned}
$$

Defining strictly positive slack variables $s_0, s_1, ..., s_5$; the Groebner bases (set to zero) for this example are

$$
\begin{aligned}
g_0 = {} & 20 + 120s_0 + 20s_3 - 56s_5 + 180{s_0}^2 + 28s_0 s_3 \\
& + {s_3}^2 - 168s_5 s_0 - 12s_5 s_3 + 36{s_5}^2 = 0 \\
g_1 = {} & 2 + 6s_0 - 3s_3 - 2s_5 + 4s_4 = 0 \\
g_2 = {} & -2 - 6s_0 + 3s_3 + 2s_5 + 4s_1 - 4s_5 s_0 = 0 \\
g_3 = {} & 8 + 68s_0 - 64s_3 - 8s_5 + 192{s_0}^2 - 156s_0 s_3 \\
& + 40{s_3}^2 - 44s_5 s_0 + 60s_5 s_3 + 180{s_0}^3 + 100s_3{s_0}^2 \\
& + s_0{s_3}^2 - 60{s_0}^2 s_5 - 66s_0 s_3 s_5 + 72s_2 = 0 \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
g_4 = {} & -2 + 8k_2 + 10s_0 + s_3 - 6s_5 = 0 \\
g_5 = {} & -14 - 18s_0 - s_3 + 6s_5 + 8k_1 = 0. \qquad (35)
\end{aligned}
$$

Equations (35) can be solved for the controller parameters k_2, k_1 in terms of the free slack variables s_0, s_3, s_5. Solution to the last 3 equations in (34) for s_4, s_1, s_2 respectively yields (recall that the slack variables are strictly positive)

$$s_4 = -\frac{1}{2} - \frac{3}{2}s_0 + \frac{3}{4}s_3 + \frac{1}{2}s_5 > 0 \qquad (36)$$

$$s_1 = \frac{1}{2} + \frac{3}{2}s_0 - \frac{3}{4}s_3 - \frac{1}{2}s_5 + s_5 s_0 > 0 \qquad (37)$$

$$
\begin{aligned}
s_2 = {} & -\frac{1}{9} - \frac{17}{18}s_0 + \frac{8}{9}s_3 \\
& + \frac{1}{9}s_5 - \frac{8}{3}{s_0}^2 + \frac{13}{6}s_0 s_3 - \frac{5}{9}{s_3}^2 + \frac{11}{18}s_5 s_0 \\
& - \frac{5}{6}s_5 s_3 - \frac{5}{2}{s_0}^3 - \frac{25}{18}s_3{s_0}^2 - \frac{1}{72}s_0{s_3}^2 \\
& + \frac{5}{6}{s_0}^2 s_5 + \frac{11}{12}s_0 s_3 s_5 > 0. \qquad (38)
\end{aligned}
$$

Equation g_0 in (34) involves only the free slack variables s_0, s_3, s_5; thus is an algebraic variety in the first orthant of the space of the free slack variables. Inequalities (36)-(38) and equation g_0 in (34) define the stability region in the space of the free slack variables s_0, s_3, s_5. Fig. 4 shows this stability region plotted using the sign-definite decomposition. The stability region in the space of the controller parameters k_1, k_2 can be plotted using the equations (35) (see Fig. 5).

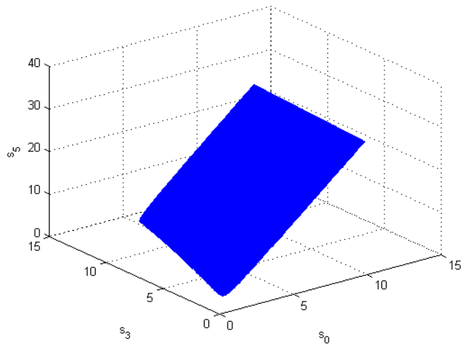

Fig. 4. The Stability Region for the MIMO Example in the Space of the Free Slack Variables

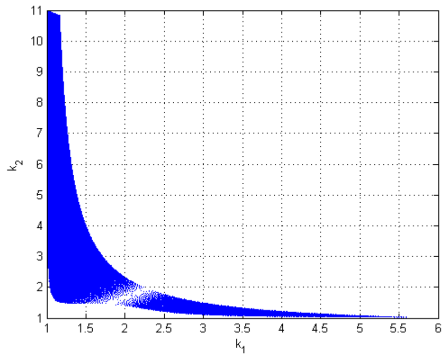

Fig. 5. The Stability Region for the MIMO Example in the Space the Controller Parameters

4. CONCLUSIONS

In this paper we proposed a method to construct a set of stabilizing controllers of fixed structure/order using strictly positive slack variables. This is accomplished through a systematic use of elimination theory on the Routh-Hurwitz stability inequalities which allows for the computation of controller parameters in a sequential manner. The presence of strictly positive slack variables in the equations simplifies the computations of the stability region(s) via the sign-definite decomposition method. Also by introducing free slack variables and constrained slack variables, we showed that the stability region(s) can be plotted in the space of the free slack variables.

Since the Groebner bases change with respect to different monomial order and different initial polynomials, finding an approach to obtain one single connected stability region in the space of the slack variables for the cases where the stability region is disconnected in the space of the controller parameters is an open problem.

It is also possible to add performance to the problem. The performance requirements can be embedded to the initial set of stability inequalities by additional corresponding polynomial inequalities. In this case the region obtained in the space of the slack variables will satisfy both the stability and the performance of the closed loop system.

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