Swing-Up and Stabilization of Rotary Inverted Pendulum using Sliding Modes

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Abstract: Control of Rotary Inverted Pendulum (RIP) is a classic control problem. It represents a broader class of underactuated systems. The dynamical equations of RIP are nonlinear coupled and complex. The nonlinear feature and the complex internal dynamics makes the design of the control challenging. This paper presents swing up and stabilization both via sliding modes. Tracking control of rotary actuator using sliding modes is developed for swing up of RIP till the pendulum reaches in the linear region around the vertical upright position. Once it reaches in the linear region, a stabilization control using sliding mode is switched on. A linear model of the RIP is used for developing the SMC law for stabilization. Effectiveness of the method is validated in simulation as well as by experiment.

Keywords: SMC, RIP, Stabilization, Swing-Up

1. INTRODUCTION

Control of Rotary Inverted Pendulum (RIP) is one of the challenging control problems. It has been investigated by many researchers. Early studies of RIP was motivated by the need of the design of the controllers to balance the rockets during a vertical take off. RIP represents a broader class of systems. Nonlinear feature and complex internal dynamics makes the design of the controller challenging. Most of the methods reported in the literature focus on linear control theory to stabilize the pendulum about its unstable equilibrium point considering deviation about it to be very small. The nonlinear equations are linearized about the equilibrium point of interest in order to fit them to the linear scheme of state space control. However, this approach is not robust. Additional control is required to bring the pendulum from vertically downward position to upward position close to unstable equilibrium point.

Most of the controllers reported in the literature are based on using the combination of the two control strategies. The two control strategies include swing up control and stabilization control. The swing up and a stabilization control problem has been studied by many researchers. Astrom and Furuta (2000) used pseudo state feedback for swing up control. Energy based approach has been used for swing up of the RIP and further stabilization about upright position by LQR approach. Energy based swing up with local stabilization controller has been used by many researchers see for example Astrom and Furuta (2000), Sukontanakarn and Parnichkun (2009), Muskinja and Tovornik (2006), S. Suzuki (2004), Mingcong Deng (2007) and the references therein. In Astrom and Furuta (2000), the authors used LQR approach for swing up and stabilization both. Muskinja and Tovornik (2006) used adaptive optimal balancing control along with switching mechanism approach for stabilization of RIP about vertical upright position. Sukontanakarn and Parnichkun (2009) used PD control for swing up and LQR based controller for stabilization. Zhong and Rock used LQR to optimize the control gains used in the feedback controller Zhong and Rck (2001).

The system dynamics being complex and nonlinear, nonlinear methods also have been investigated for the design of the control of RIP. S. Suzuki (2004) proposed a Lyapunov based control for swing up and stabilization both. Non linear back stepping approach has been used by Yan and Edwards (2008), exploiting differential flatness and small gain theorem. However, robustness issue has not been discussed.

Sliding mode control (SMC) is one of the best known robust control Utkin et al. (1999), Edwards and Spurgeon (1998), Hung et al. (1993). Park and Chwa (2009) have used coupled sliding surfaces to synthesize SMC law. SMC has been used for periodic orbit generation and target orbit stabilization of RIP.

This paper presents SMC strategies for swing up and stabilization both. While developing the swing up control, pendulum motion is considered as a disturbance for a second order position control system. SMC is used to make the motor to track the desired reference which will results swing up such that pendulum will be brought in the linear region about vertical upright equilibrium point. Further SMC is used for stabilizing the pendulum in the linear region. In the linear region, the linearized dynamical system of complex coupled non linear system is considered for the design of the controller. The organization of the paper is as follows: Section II describes the control problem. The control development using sliding mode is illustrated in Section III. Simulation results are presented in Section IV followed by the experimental results in Section V. Finally Section VI concludes the work.

2. PROBLEM DESCRIPTION

In this section dynamical equations for rotary pendulum system are developed. RIP includes a horizontal arm coupled to a motor shaft. The pendulum rod which is free to rotate is hinged to the horizontal arm. Fig. 1 shows the Quanser's RIP system. The parameters of the system are



Fig. 1. Rotary Pendulum system

l: Length of Pendulum center of mass (meters), m: mass of Pendulum (kg),



Fig. 2. Schematic of Rotary Pendulum system.

h: height from ground in (meters),

 α : pendulum angle in (rad),

 $\dot{\alpha}$: pendulum velocity (rad/s),

r: length of horizontal arm (meters),

 J_{cm} : moment of inertia of pendulum link about its center of mass, which is given by $ml^2/3$,

 R_m : motor armature resistance (Ohm),

 θ : motor shaft position (rad),

 K_m : motor back emf constant (V.s/rad),

 J_m : motor Inertia (kgm^2) ,

 T_{output} : output torque of dc motor(N.m),

 η_g : gearbox efficiency,

 \check{K}_g : total gear ratio,

 B_{eq} : the viscous damping coefficient,

 η_m : motor efficiency,

 K_t : motor torque constant(N.m/A). A second order motor dynamics is described by

$$J_l\ddot{\theta} + \eta_g K_g^2 J_m \ddot{\theta} + B_{eq} \dot{\theta} = \eta_g \eta_m K_t K_g I_m.$$
(1)

A state space model is

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u},$$
$$y = C\mathbf{x} + D\mathbf{u},$$

where the state space matrices are obtained by substituting the parameter values, and are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -31.67 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 55.75 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ D = 0.$$

Fig.2 shows the schematic of RIP. From the Fig.2

The potential energy of the system is

$$V = mgl\cos\alpha. \tag{2}$$

The kinetic energies in the system arise from the moving motor, the velocity of the point mass in the x-direction, the velocity of the point mass in the y direction and the rotating pendulum about its center of mass. The total kinetic energy is

$$T = KE_{motor} + KE_{vx} + KE_{vy} + KE_{pendulum}.$$
 (3)

From Fig. 2 position of pendulum is

$$r = -l\sin\alpha x + l\cos\alpha y. \tag{4}$$

Velocity of the pendulum:

$$v = -l\cos\alpha\dot{\alpha}x + l\sin\alpha\dot{\alpha}y.\tag{5}$$

The pendulum arm is also moving with the rotating horizontal arm at a rate $V_{arm} = r\theta$. The x and y velocity components $v_x = r\dot{\theta} - l\cos\alpha\dot{\alpha}$ and $v_y = -l\sin\alpha\dot{\alpha}$. Considering all this information the K.E. can be represented as below:

$$T = \frac{J_{eq}\dot{\theta}^2}{2} + \frac{m(r\dot{\theta} - l\cos\alpha\dot{\alpha})^2}{2} + \frac{m(-l\sin\alpha\dot{\alpha})^2}{2} + \frac{J_{cm}\dot{\alpha}^2}{2}$$
$$T = \frac{J_{eq}\dot{\theta}^2}{2} + \frac{m(r^2\dot{\theta}^2 - 2rl\cos\alpha\dot{\alpha}\dot{\theta})}{2} + \frac{2ml^2\dot{\alpha}^2}{3}.$$
(6)

The Lagrangian is

$$\begin{split} L &= T - V \\ L &= \frac{J_{eq}\dot{\theta}^2}{2} + \frac{m(r^2\dot{\theta}^2 - 2rl\cos\alpha\dot{\alpha}\dot{\theta})}{2} + \frac{2ml^2\dot{\alpha}^2}{3} - mgl\cos\alpha. \end{split}$$

The Euler-Lagrange equations are

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$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = T_{output} - B_{eq}\dot{\theta}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} - \frac{\partial L}{\partial \alpha} = 0.$$

Substituting the respective values

 $\frac{d}{dt} \left(J_{eq} \dot{\theta} + mr^2 \dot{\theta} - mrl \cos \alpha \dot{\alpha} \right) - 0 = T_{output} - B_{eq} \dot{\theta}$ $J_{eq}\ddot{\theta} + mr^2\ddot{\theta} - mrl\cos\alpha\ddot{\alpha} + mrlsin\alpha\dot{\alpha}^2 = T_{output} - B_{eq}\dot{\theta}.$ Neglecting the higher order terms and simplifying to get.

$$J_{eq}\ddot{\theta} + mr^2\ddot{\theta} - mrl\cos\alpha\ddot{\alpha} = T_{output} - B_{eq}\dot{\theta}$$
(7)

$$mrl\cos\alpha\ddot{\theta} + \frac{4ml^2\ddot{\alpha}}{3} - mgl\sin\alpha = 0.$$
(8)

These are complex nonlinear coupled equations constituting an underactuated system.

Linear Model: There are two equilibrium points, $\alpha =$ π (pendulum down, stable) and $\alpha = 0$ (pendulum up, unstable). The linearized model about unstable point is

$$(J_{eq} + mr^2)\ddot{\theta} - mrl\ddot{\alpha} + B_{eq}\dot{\theta} = T_{output}$$
(9)

$$\frac{4ml^2\alpha}{3} - mrl\ddot{\theta} - mgl\alpha = 0.$$
(10)

Torque from dc servo motor

$$T_{output} = \frac{\eta_g \eta_m K_g K_t (V_m - K_g K_m \dot{\theta})}{R_m}.$$
 (11)

The state space representation of the above system is

$$\begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \\ \ddot{\theta} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{bd}{E} & \frac{-cG}{E} & 0 \\ 0 & \frac{ad}{E} & \frac{-bG}{E} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \alpha \\ \dot{\theta} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{c\eta_m\eta_g K_t K_g}{R_m E} \\ \frac{b\eta_m\eta_g K_t K_g}{R_m E} \end{bmatrix} V_m(12)$$

where
$$a = J_{ea} + mr^2$$

$$b = mlr$$

$$c = \frac{4ml^2}{3}$$

$$d = mgl$$

$$E = ac - b^2$$

$$G = \frac{\eta_m \eta_g K_g^2 K_m K_t + B_{eq} R_m}{R_m}.$$

Substituting the values of the parameters,

$$\begin{bmatrix} \theta \\ \dot{\alpha} \\ \ddot{\theta} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 33.29 & -16.73 & 0 \\ 0 & 67.48 & -11.83 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \alpha \\ \dot{\theta} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 29.28 \\ 20.56 \end{bmatrix} V_m.$$
(13)

Eigenvalues of the above are at 0, -18.0993, 7.1140, and -5.6447. One pole of the system is in right half of s-plane therefore open-loop system is unstable. Fig.3 shows that the linear model quite accurately describes the system for the first 25 degrees and then began to diverge from the actual motion.



Fig. 3. Linear model validation

3. CONTROL DEVELOPMENT

The control objective is to bring the pendulum from downward position to vertically upright position and maintain it there. We consider the two control strategies one for swing up and the other for stabilization. Initially swing up control action to generate that much energy to bring the pendulum link in the linear region and then switch over to stabilization control which maintains the pendulum in upright position. Both the swing up and the stabilization control are designed using sliding modes.

3.1 Swing-up Control

For the design of the swing-up control a second order position control system with a pendulum link motion acting as a disturbance is considered. A tracking controller is designed to follow a reference trajectory. This gives swing to pendulum link to bring it in region wherein another stabilizing control is used to stabilize pendulum about unstable equilibrium point. The reference trajectory used is a square wave of amplitude ± 30 . A state space equation for position control system is

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}\mathbf{u} + \mathbf{d}.\tag{14}$$

where A is System matrix and \mathbf{b} is input matrix and \mathbf{d} is lumped disturbance which includes disturbance due to

swinging pendulum. This is matched disturbance. Define s_1 as the sliding surface

$$s_1 = \mathbf{c}^T \mathbf{x}_e,\tag{15}$$

where $\mathbf{x}_e = \begin{bmatrix} e_x & \dot{e_x} \end{bmatrix}$ is the error state vector. If \mathbf{x}_d is the desired state vector and \mathbf{x} is the actual state vector then $\mathbf{x}_e = \mathbf{x} - \mathbf{x}_d$. It is assumed that $\dot{\mathbf{x}}_d = A\mathbf{x}_d$. Gao's power rate reaching law is used which guarantees finite time reaching, Hung et al. (1993).

$$\dot{s_1} = -k_1 |s_1|^{\alpha} \operatorname{sign}(s_1).$$
 (16)

Differentiating (15)

$$\dot{s_1} = \mathbf{c}^T \dot{\mathbf{x}_e}.$$
$$\dot{s_1} = \mathbf{c}^T \dot{\mathbf{x}} - \mathbf{c}^T \dot{\mathbf{x}_d}$$

Using (15) and nominal part of (14) to synthesize a tracking control

$$\mathbf{u} = -(\mathbf{c}^T b)^{-1} (\mathbf{c}^T A \mathbf{x} + k_1 |s_1|^{\alpha} \operatorname{sign}(s_1) - \mathbf{c}^T \dot{\mathbf{x}}_d).$$
(17)

Here the desired vector is $\mathbf{x}_d = [\pm 30 \ 0]$ and its derivative vector is $\dot{\mathbf{x}}_d = [0 \ 0]$. Substituting this in (17) to synthesize the control

$$\mathbf{u} = -(\mathbf{c}^T b)^{-1} (\mathbf{c}^T A \mathbf{x} + k_1 |s_1|^{\alpha} \operatorname{sign}(s_1)).$$
(18)

This is a swing up control. The switching gain k_1 is chosen greater than the maximum bound of the $\mathbf{c}^T \mathbf{d}$ to ensure sliding, Edwards and Spurgeon (1998).

3.2 Stabilization control

Linear model of inverted pendulum system is valid up to 25 degree of pendulum angle. A linear model of the inverted pendulum system is already discussed in previous Section. Inverted Pendulum system is 4th order system and state space equation is

$$\dot{\mathbf{z}} = A\mathbf{z} + b\mathbf{u},\tag{19}$$

where $\mathbf{z} \in \Re^{4 \times 1}$ and is $[\theta \ \alpha \ \dot{\theta} \ \dot{\alpha}], A \in \Re^{4 \times 4}, b \in \Re^{4 \times 1}$. Let the sliding surface be

$$s = c^T \mathbf{z}_e,\tag{20}$$

where $c^T \in \Re^{1 \times 4}$, and $\mathbf{z}_e = [e_{\theta} \ e_{\alpha} \ \dot{e_{\theta}} \ \dot{e_{\alpha}}]$ is the error state vector. If \mathbf{z}_d is the desired state vector and \mathbf{z} is the actual state vector then $\mathbf{z}_e = \mathbf{z} - \mathbf{z}_d$. Gao's power rate reaching law which guarantees finite time reaching is

$$\dot{s} = -k|s|^{\alpha} \operatorname{sign}(s). \tag{21}$$

Differentiating (20) to get

$$\dot{s} = c^{T} \dot{\mathbf{z}}_{e}.$$

$$\dot{s} = c^{T} \dot{\mathbf{z}} - c^{T} \dot{\mathbf{z}}_{d}.$$

$$\dot{s} = c^{T} A \mathbf{z} + c^{T} b \mathbf{u} - c^{T} \dot{\mathbf{z}}_{d}.$$

$$-k|s|^{\alpha} \operatorname{sign}(s) = c^{T} A \mathbf{z} + c^{T} b \mathbf{u} - c^{T} \dot{\mathbf{z}}_{d}.$$

$$\mathbf{u} = -(c^{T} b)^{-1} (c^{T} A \mathbf{z} + k|s|^{\alpha} \operatorname{sign}(s) - c^{T} \dot{\mathbf{z}}_{d}). \qquad (22)$$

The desired vector $\mathbf{z}_d = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and its derivative vector is $\mathbf{z}_d = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. Substituting this in (22), The synthesized control law becomes

$$\mathbf{u} = -(c^T b)^{-1} (c^T A \mathbf{z} + k |s|^{\alpha} \operatorname{sign}(s)).$$
(23)

This is a stabilizing control.

4. SIMULATION RESULTS

In simulations the performance of the controller was tested. The design parameters are: $s_1 = [-63 \ 1]\mathbf{x}_e, k_1 = 110, \alpha = 0.5, s = [17.076 \ -76.9016 \ 7.3774 \ -12.1598]\mathbf{z}, k = 10$. A sigmoid function was used to alleviate the chattering. Fig.4 shows the controller performance without disturbance. A continuous sinusoidal disturbance of amplitude one and frequency one radian per second was added to check the disturbance rejection capability. Fig.5 shows the controller performance. It is evident that SMC exhibits excellent disturbance rejection capability.



Fig. 4. Simulation results: Without disturbance

5. EXPERIMENTAL RESULTS

The performance of the controller was validated experimentally. The SM controller was compared with PD-LQR controller wherein PD control was used for swing up and LQR based control was used for stabilization. The design parameter for PD-LQR controller are PD gains are: $k_p = 17$, $k_d = 0.6$. To design the gains of LQR based controller $Q = diag([3.5 \ 14 \ 0 \ 0])$, R = 1 were used to get the gain matrix $K = [-1.8257 \ 23.9850 \ -1.8109 \ 3.3997]$. The design parameter for SM control are the same as used in simulation. Fig.6 shows the controller performance of the two controllers without disturbance. Fig.7 shows the controller performance of same sinusoidal disturbance.

It is found that SMC exhibits excellent disturbance rejection capability. Following table illustrates the better performance of SMC than PD-LQR in terms of control energy and control quality.



(b) Evolution of control u

Fig. 5. Simulation results: With continuous sinusoidal disturbance



(b) Evolution of control u

Fig. 6. Experimental results: Without disturbance

Control	$ u _{\infty}$	$\parallel u \parallel_2$	$\ \alpha \ _1$	$\parallel heta \parallel_1$				
PD-LQR	15.40	257.27	5.50e + 005	1.907e + 005				
SMC-SMC	12.76	164.90	1.509e + 005	1.226e + 005				
Table 1. Comparison of the controller perfor-								
mance:Without disturbance								

6. DISCUSSIONS AND CONCLUSIONS

A benchmark control problem viz control of RIP has been investigated using sliding modes. A DC motor has been



Fig. 7. Experimental results:With continuous sinusoidal disturbance

ſ	Control	21						
l	Control	$ \ u \ _{\infty}$	$\ u \ _2$					
	PD-LQR	12.52	220.27	5.01e + 005	3.307e + 005			
	SMC-SMC	13.17	162.52	1.529e + 005	2.209e+005			
	Table 2. Comparison of the controller perfor-							
	mance:With sinusoidal continuous disturbance							

used as a rotary actuator. Position control of the arm coupled to the shaft of this actuator has been examined using sliding modes. Tracking using SMC has been used to get the desired swing up from downward position of RIP. This has been used to bring the pendulum in the linear region wherein the linear model is valid for the control development. Further in the linear region a stabilization control has been used. This stabilization control has been developed using sliding modes. Following are the concluding remarks:

- Sliding mode controller yield excellent swing up control as well as stabilization about vertical upright position.
- The performance of control of RIP using SMC for swing up and SMC for stabilization is better than the control which use PD control for swing up and LQR controller for stabilization. Swing up time has been reduced to less than 2 seconds in case of the proposed method compared to 4 to 5 seconds in case of the other method.
- SMC exhibits excellent disturbance rejection capabilities while controlling RIP.
- Design of the controller using SMC is simple and implementable.
- Experimental results are in close agreement with the simulation results.
- The method can be extended to a class of underactuated systems.

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