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Mixed Strategies in Maxi-min Testing of the Quality of Robust Stabilization Algorithms ⋆

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Abstract: Mixed strategies in maxi-min testing of the quality of robust stabilization algorithms are considered. Testing procedure is based on transformation of initial dynamic game to the geometric game. The transformation to a mixed expansion of the geometric game is proposed in case of absence of saddle points. An optimal mixed perturbation strategy is shown to be applicable. Maxi-min testing procedure is illustrated by the planar problem of convergence of the Space Rescue Module (SRM "Saver") and the space station.

Keywords: Maxi-min testing, stabilization quality, testing strategy, optimal mixed strategy

1. INTRODUCTION

An important stage in the development of complicated control algorithms for dynamic objects is a stage of testing their quality. The testing is especially actual for controlled systems with high cost of risk, such as control systems in space. For these purposes we are proposing to use the method of maxi-min testing.

2. MAXI-MIN TESTING PROCEDURE

2.1 Statement of the problem

Consider the problem of testing the quality of robust stabilization algorithms for dynamic system which is written down in the following form:

$$\begin{array}{l} \dot{x} = A_q(t)x + B_q(t)u + C_q(t)v_r(t) \\ x(t_0) = x_p \\ r = 1, 2, .., R \\ p = 1, 2, .., P \\ q = 1, 2, .., Q \end{array} \quad (1)$$

Here $x(t)$ is a n-dimensional state vector; $u(\cdot) \in U = \{L_2[t_0, t_k] | |u_i(t)| \leq u^{\max}\}$ is a s-dimensional function of controls; $w = (r, p, q) \in W$ is a perturbation, where W is a finite set of perturbations, containing $R \cdot P \cdot Q$ elements.

Lets define the quality functional in the following form:

$$J(w, u(\cdot)) = x^T(t_k)Sx(t_k). \quad (2)$$

Here S is a constant, symmetric positive semidefinite matrix, t_0, t_k are the fixed moments of time , $w \in W$, $u(\cdot) \in U$.

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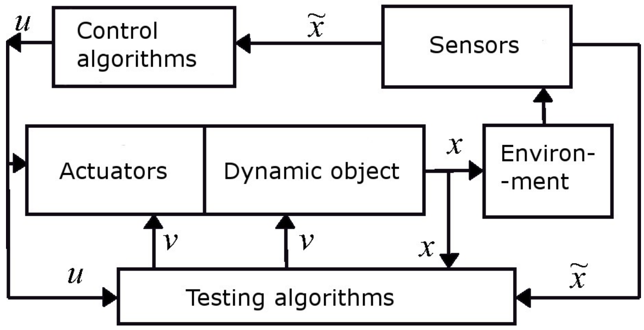

Fig. 1. Functional scheme of the testing bench

It is necessary to objectively evaluate the quality of control applied to the the system (1) from the viewpoint of the quality functional (2). Evaluation will be performed on the special test bench, which can be represented by the functional scheme. See Fig. 1

Blocks of actuators, moving object, sensors and the environment can be presented by computer model. Block of testing algorithms generates worst perturbations acting on the dynamic system and thus forms measures of the quality of control algorithm. The method of maxi-min testing is proposed. This technique allows to obtain objective measures of the control algorithm accuracy under extreme conditions. See Alexandrov (1997) and Alexandrov et al. (2005).

2.2 Maxi-min testing procedure

The maxi-min testing procedure consists of three stages.

1-st stage The preparation stage. The best estimation of the quality functional is realized. Optimal perturbation strategy is found according to the solution of maxi-min problem.

2-nd stage The basic stage. Computer testing process is realized by modeling of the process (1) which is controlled by operator (or control algorithm) and exposed to an optimal perturbation strategy, which is found on the first stage. Real estimation of the control algorithm quality is being found on this stage.

3-rd stage The final stage. At this stage, the comparison of the best and the real estimates and recommendations for further training and diagnostics, calibration and correction are done.

2.3 Basic assumption

To successfully implement the maxi-min testing procedure let us consider disturbance w and control u as independent players with conflicting interests. The control u aims to reduce the value of quality functional J while disturbance w tends to increase it.

Consider a zero-sum dynamic game $\Gamma = (W, U, J)$ based on system (1) with two players – disturbance w and control u, which are considered independent. Thus, we have staged two extremum problems:

$$\min_{u(\cdot)\in U} J(w, u(\cdot)) \to \max_{w\in W}, \tag{3}$$

$$\max_{w\in W} J(w, u(\cdot)) \to \min_{u(\cdot)\in U}. \tag{4}$$

Quality functional $J(w, u(\cdot))$ is considered as a payoff function of player 2 (controls). See Petrosyan (1998).

For the game Γ, as for all of zero-sum games, we have the following chain of inequalities:

$$\begin{aligned} J_0 = \max_{w\in W} \min_{u(\cdot)\in U} J(w, u(\cdot)) \le \min_{u(\cdot)\in U} \max_{w\in W} J(w, u(\cdot)) \le \\ \le \max_{w\in W} J(w, \tilde{u}) = J(w^0(\tilde{u}), \tilde{u}) = \tilde{J}. \end{aligned} \tag{5}$$

Here $\tilde{u} \in U$ is some control strategy.

Further we consider the lower bound J_0 as the best quality index of the stabilization algorithm $u(\cdot)$. Perturbations strategy w^0 will be considered as the maxi-min testing strategy, which will be realized on the second stage.

According to (5), the lower bound of quality functional J_0 is reachable by control $\tilde{u}$ only when equilibrium situation in the game Γ takes place:

$$\begin{aligned} J_0 = J(w^0, u^0) = \max_{w\in W} \min_{u(\cdot)\in U} J(w, u(\cdot)) = \\ = \min_{u(\cdot)\in U} \max_{w\in W} J(w, u(\cdot)). \end{aligned} \tag{6}$$

Thus, the relations (3), (4), (5), (6) allow to estimate the best value of the quality index J_0, and the optimal testing strategy w^0. In this case, control $u(\cdot)$ is able to realize the best quality of robust stabilization. See Alexandrov et al. (2005).

2.4 Reduction to geometric game

Let consider the expansion of the state vector $x(t)$ in order to find the lowest value of the quality functional J_0 :

$$x(t) = x_w(t) - x_u(t).$$

Here $x_w(t)$ and $x_u(t)$ satisfy the equations:

$$\dot{x}_w = A_q(t)x_w + C_q(t)v_s(t) \quad x_w(t_0) = x_m, \tag{7}$$

$$\dot{x}_u = A_q(t)x_u - B_q(t)u \quad x_u(t_0) = 0. \tag{8}$$

System (7) at the final moment of time t_k corresponds to the set G_w, containing $R \cdot P \cdot Q$ points — at one point on each value of the perturbation $w = (r, p, q)$. System (8) corresponds to Q convex, centrally symmetric reachability sets with zero origin.

Let us consider the reduction of original game Γ to geometric game.

Instead of the reachability sets $G_u(q)$, we consider their intersection $G_u = \bigcap_{q=1,..,Q} G_u(q)$ (which is not empty and contains at least zero point). Instead of control set U we consider its contraction, bringing the state of the system (8) into the set G_u at the time moment t_k. Thus we have the reduction of the original dynamic zero-sum game Γ to the geometric zero-sum game $\Gamma_1 = (G_w, G_u, J)$ built on the reachability sets of the systems (7) and (8) at the time moment t_k.

Problem of finding a lowest value (3) of the quality functional J_0 reduces to enumeration of values $x_w \in G_w$ with the search for $\min_{x_u\in G_u} J(x_w, x_u)$ at each step.

The following theorem will be used to check the existence of equilibrium point in the game Γ (which is equivalent to the presence of a saddle point in geometry game Γ_1).

Theorem 1. It is necessary and sufficient for a couple of strategies (w^*, u^*) to be an equilibrium point of a zero-sum game (W, U, J) that $\exists \max_{w\in W} \min_{u\in U} J(w, u)$ and

$$J(w^*, u^*) = \max_{w\in W} \min_{u\in U} J(w, u) \ge J(w, u^*), \forall w \in W. \tag{9}$$

This theorem is easily proved by using the definition and criterion for the existence of an equilibrium point of a zero-sum game. See Petrosyan (1998).

To determine the existence of equilibrium point we need to solve the problem (3) and to check the conditions of the theorem 1 for all perturbations $w \in W$. If the equilibrium situation take place, we can use perturbation w^0 that corresponds to the solution of the problem (3) as a testing strategy, and then proceed to the second stage of maxi-min testing procedure.

2.5 Transition to mixed strategies

In case when the equilibrium situation does not exist, you can change the set of perturbations W so that a new zero-sum game (W^*, U, J) and, consequently, the geometric game (G_w^*, G_u, J) have a saddle point. In cases where the modification is not acceptable, transformation to the mixed extension $\bar{\Gamma}$ of geometry game Γ_1 is needed. Such transition is possible if the game Γ_1 is convex. See Petrosyan (1998).

The game $\Gamma_1 = (G_w, G_u, J), G_w \subset R^m, G_u \subset R^m$ is convex and has the equilibrium value

$$K_0 = \min_{x_u\in G_u} \max_{x_w\in \Omega_w} J(x_w, x_u). \tag{10}$$

Player 1 (perturbations) have an optimal mixed strategy μ_0 with finite spectrum, containing not more than $(m+1)$ points of set G_w (here m is the dimension of a geometric game Γ_1, defined by metrics $J(x_w, w_u) = (x_w - x_u)^T S(x_w - x_u)$).

All pure strategies x_u^M which lead to $\min\limits_{x_u \in G_u} \max\limits_{x_w \in G_w} J(x_w, x_u)$ are optimal to player 2 (controls). Moreover, the chain of inequalities corresponding to the definition of equilibrium in mixed strategies is valid:

$$\begin{aligned} K(x_w, x_u^M) &\leq K(\mu_0, x_u^M) \leq \\ &\leq K(\mu_0, x_u), \forall x_u \in G_u, \forall x_w \in G_w, \end{aligned} \tag{11}$$

Where

$$K(\mu_0, x_u) = \sum_{w \in W} \mu_w J(x_w, x_u) \tag{12}$$

is mathematical expectation of winning of the player 1 (perturbations) in point $x_u \in G_u$ (here μ_w are the probabilities corresponding to the mixed strategy μ_0), $K(\mu_0, x_u^M) = K_0$, $K(x_w, x_u^M) = J(x_w, x_u^M)$.

As we can see in (11), K_0 value is a lowest estimate of the mathematical expectation $K(\mu_0, x_u)$. Moreover, existence of equilibrium in mixed expansion $\bar{\Gamma}$ of the geometric game Γ_1 leads to attainability of this point.

2.6 Realization of the testing procedure in case of mixed strategies

To realize the second stage of the testing procedure it is necessary to obtain the lowest value $K(\mu_0, x_u) = K_0$ and to obtain the mixed perturbations strategy μ_0. Also it is necessary to satisfy the conditions (11). The following example will show how to determine μ_0 in the case of simple geometric game.

Determination of min-max value in geometric game Γ_1 To determine value K_0 it is necessary to use one of the straight algorithms of searching the minimum of nonsmooth convex function on a convex set. See Demyanov (1981). Another way is to use necessary conditions for a mini-max, which can be summarized as follows:

Theorem 2. If x_u^M is absolute minimum of the function $\varphi_0(x_u) = \max\limits_{w \in W} J(x_w, x_u)$ on the set G_u, then the vector $a \in K_u^*$ exists and constant scalar values $\lambda_1, \lambda_2 ..., \lambda_{R \cdot P \cdot Q}$ exist and the following conditions are true:

(1) $a + \sum\limits_{r=1}^{R \cdot P \cdot Q} \lambda_r \left(\frac{\partial \rho_r(x_u^M)}{\partial x} \right)^\top = 0$

(2) $\lambda_r \geq 0,\ r = 1, ..., R \cdot P \cdot Q$

(3) $\lambda_r \left(\varphi_0(x_u^M) - \rho_r(x_u^M) \right) = 0$

(4) $\sum\limits_{r=1}^{R \cdot P \cdot Q} |\lambda_r| + |a| \neq 0$

Here $\rho_r(x_u) = J(x_r, x_u),\ x_u \in G_u$. K_u^* is the dual cone to the approximating cone (tent) of the set G_u at the point x_u^M.

This theorem is a consequence of the separation theorem for convex cones. See Boltyanskiy (1973). Theorem 2 allows us to find the point x_u^M and value $K_0 = K(\mu_0, x_u^M)$.

Determination of optimal mixed testing strategy μ_0 After finding the point x_u^M (solution of the problem (10)) it is necessary to find an optimal mixed strategy μ_0 according to conditions (11). Spectrum of strategy μ_0 contains only those points from G_w which lie on the surface of the sphere with center at x_u^M and radius K_0. This sphere contains all other points of G_w. Consequently, the left-hand side of inequalities (11) is fulfilled: $K(x_w, x_u^M) \leq K(\mu_0, x_u^M) = K_0, \forall x_w \in G_w$. Distribution of probabilities μ_0 should be chosen so that the right-hand side of inequalities (11) is fulfilled: $K(\mu_0, x_u^M) \leq K(\mu_0, x_u), \forall x_u \in G_u$. This can be done by selecting such probabilities μ_0, that a minimum of a convex function (12) is attained at point x_u^M on the set G_u.

Realization of second stage in case of mixed testing strategy At the second stage of testing is necessary to hold a series of tests when testing strategies are acting on controlled object. These testing strategies are perturbations selected in accordance with the probability distribution μ_0 found on the first stage. Each test is a process of mathematical modeling of the controlled object. Perturbations and control (players 1 and 2) are acting on this object during this process. Control function $\tilde{u}$ is obtained in real time from on-board computer or manual control.

The value of mathematical expectation $\hat{K}(\mu_0, x_u) = \frac{1}{N} \sum\limits_{i=1}^{N} K(\mu_0, x_u^i)$ can be approximately computed after finishing of these series of tests. Here N is the amount of tests, x_u^i is implementation of the control action during i-th test.

We can conclude that $K(\mu_0, x_u)$ satisfies the estimate $\hat{K} - \alpha \frac{\tilde{\sigma}}{\sqrt{N}} < K(\mu_0, x_u) < \hat{K} + \alpha \frac{\tilde{\sigma}}{\sqrt{N}}$ with probability p_α due to assumption of a sufficiently large number of tests and taking into account the central limit theorem. Here $\tilde{\sigma}^2 = \frac{1}{N-1} \sum\limits_{i=1}^{N} (K(\mu_0, x_u^i) - \hat{K})$ is unbiased estimate of variance. P_α and α are related by $p_\alpha = \frac{2}{\sqrt{2\pi}} \int\limits_0^\alpha \exp \frac{u^2}{2} du$.

Consequently, for $\epsilon = \alpha \frac{\tilde{\sigma}}{\sqrt{N}}$, the inequality corresponding to the situation of ϵ - equilibrium is fullfilled (13). See Petrosyan (1998).

$$K(\mu_0, x_u^M) \leq \hat{K}(\mu_0, x_u) + \epsilon \tag{13}$$

The value $\hat{K}(\mu_0, x_u)$ must be compared with the lower bound of the mathematical expectation $K(\mu_0, x_u^M) - \epsilon$ on the third stage of testing procedure, according to the point of ϵ - equilibrium. The estimate obtained as a result of the testing process will be objective because the best result K_0 is reachable.

3. EXAMPLE

We illustrate the first stage of testing procedure for the planar problem of the space rescue module (SRM "Saver") convergence with the orbital station. See Sadovnichy et al. (2007).

Space rescue module is designed for short-term movements in the vicinity of the orbital station. The module is a

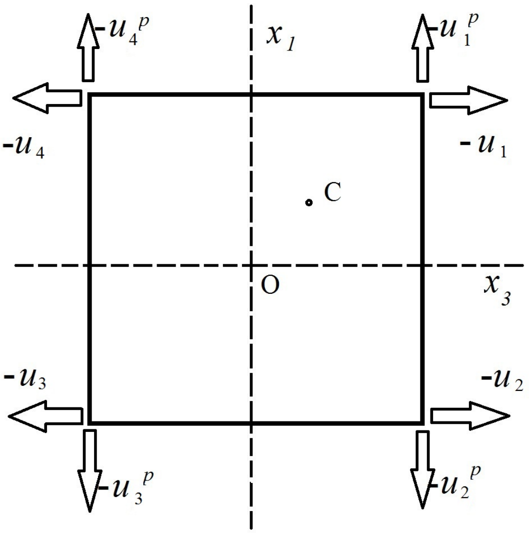

Fig. 2. Planar mathematical model of space rescue module

rectangular frame that is mounted on the spacesuit. Gas thrusters and control system are mounted on the frame. The control system is activated during the emergency separation of astronaut and the space station. The angular velocity resulting from an emergency situation is extinguished by the automatic control algorithm. After that, the astronaut need to turn around face-to-station. Then he comes close to the station to recover a lost contact using the sustainer mode. Let consider the last phase of the movement: a straightforward approach to the orbital station.

3.1 Problem statement

Consider the space module in the form of rectangular frame in the orbital plane with sides $2a$ and $2b$. Two gas thrusters are located in each corner of the frame directed along its sides. See Fig. 2.

The ideal trajectory is a straight line. Linearized equations in deviations from the ideal trajectory becomes:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (-v_1^r(t) + v_2^r(t) + v_3^r(t) - v_4^r(t))/M \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = (-x_5(u_2^p(t) + u_3^p(t) - u_1^p(t) - u_4^p(t)) \\ \quad -u_1 - u_2 + u_3 + u_4)/M \\ \dot{x}_5 = x_6 \\ \dot{x}_6 = ((b+\delta_2)(u_1 - u_4) + (b-\delta_2)(u_3 - u_2) + \\ \quad (a+\delta_1)(u_2^p(t) - u_1^p(t) + v_2^r(t) - v_1^r(t)) + \\ \quad (a-\delta_1)(u_4^p(t) - u_3^p(t) + v_4^r(t) - v_3^r(t)))/B \end{cases} \tag{14}$$

Here x_1, x_3 are deviations from the ideal trajectory in the plane of motion of a space module, x_5 is a deviation from the ideal angle, $u_i^p(t)$ is program control (prescribed functions of time), $u_i(\cdot) \in U = \{u(\cdot) \in L_\infty[t_0,t_1] | 0 \leq u_i(t) \leq f, f = const\}$ is a stabilizing control, $v_j^i(t)$ is a set of known functions, $j = 1,..,4$, $i = 1,2$, δ_1, δ_2 is a shifting of the frame mass center C of space module with respect to its geometric center O, B is a moment of inertia of the frame, M is a mass of the module.

Assume that the initial condition $x(t_0) = x_0$ is fixed. System (14) has the form (1), where the perturbations are represented in the form of $w = r \in \{1,2\}$.

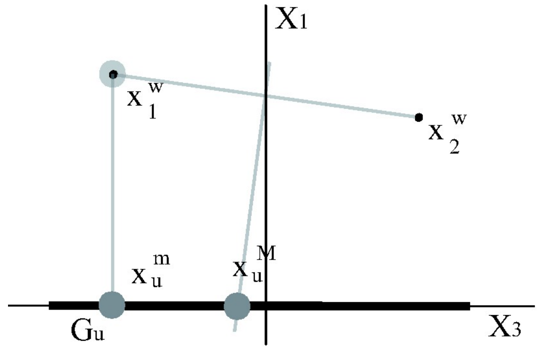

Fig. 3. Destination zones of subsystems (7) and (8)

Consider the following criterion for control quality: $J = x_1^2(t_k) + x_3^2(t_k)$ which has type (2). For simplicity, we assume that criterion contains only linear deviation in the plane $x_1(t_k), x_3(t_k)$ at the moment t_k. But it is also possible solution in the problem with higher dimension, for example, when the quality criterion contains the deviation of the velocity or the angle.

3.2 First stage of maxi-min testing procedure

The transformation of the original system (14) on perturbed subsystem (15) and controlled subsystem (16) is made by replacing $x = x_w - x_u$:

$$\begin{cases} \dot{x}_1^w = x_2^w \\ \dot{x}_2^w = (-v_1^r(t) + v_2^r(t) + v_3^r(t) - v_4^r(t))/M \\ \dot{x}_3^w = x_4^w \\ \dot{x}_4^w = -x_5^w(u_2^p(t) + u_3^p(t) - u_1^p(t) - u_4^p(t))/M \\ \dot{x}_5^w = x_6^w \\ \dot{x}_6^w = ((a+\delta_1)(u_2^p(t) - u_1^p(t) + v_2^r(t) - v_1^r(t)) + \\ \quad (a-\delta_1)(u_4^p(t) - u_3^p(t) + v_4^r(t) - v_3^r(t)))/B \end{cases} \tag{15}$$

$$\begin{cases} \dot{x}_1^u = x_2^u \\ \dot{x}_2^u = 0 \\ \dot{x}_3^u = x_4^u \\ \dot{x}_4^u = (-x_5^u(u_2^p(t) + u_3^p(t) - u_1^p(t) - u_4^p(t)) \\ \quad +u_1 + u_2 - u_3 - u_4)/M \\ \dot{x}_5^u = x_6^u \\ \dot{x}_6^u = (-(b+\delta_2)(u_1 - u_4) - (b-\delta_2)(u_3 - u_2))/B \end{cases} \tag{16}$$

Reachability set G_u of system (16) is a straight line on the axis x_3. The perturbed system (15) corresponds to two points x^w. We assume for simplicity that the points x_1^w and x_2^w lie on one side of the axis x_3, and their projections onto the axis x_3 belong to G_u. See Fig. 3.

The lower bound of the quality index J_0 can be found by solving the maxi-min problem (3) and checking the existence of equilibrium point. The existence of equilibrium point of a geometric game Γ_1 can be checked using the theorem 1. We turn to the second stage of testing procedure using a testing strategy wich corresponds to equilibrium in the case of equilibrium situation.

Consider the case where it is known that equilibrium point in geometric game Γ_1 does not exist. In this case we construct mixed extension $\bar{\Gamma}$ of the game Γ_1.

Pure control strategy is a point x_u^M which is corresponding to $\min_{x_u \in G_u} \max_{x_w \in G_w} J(x_w, x_u)$. This point is intersection of median perpendicular and set G_u. Using Theorem 2 it is easy to show that point x_u^M is a solution of problem (10).

Next, we find a mixed strategy of perturbations μ_0, provided that none of the points x_w lies on the set G_u. For this we define the relation between the probabilities μ_1, μ_2 corresponding to the mixed strategy that allow us to find these probabilities.

Mathematical expectation of winning at x_u defined by the relation $K(\mu_0, x_u) = \mu_1|x_u - x_w^1| + \mu_2|x_u - x_w^2|$. Gradient of this convex function for x projected on the set G_u has the form: $K' = (\mu_1 \frac{x_u - x_1^w}{|x_u - x_1^w|} + \mu_2 \frac{x_u - x_2^w}{|x_u - x_2^w|}, \overrightarrow{e_3})$ (dot product on $\overrightarrow{e_3}$ denotes the projection onto the axis x_3, ie, on the set G_u).

If we take into account the fact that the equation $K' = 0$ has a unique root at G_u for fixed probabilities μ_1, μ_2 ($\mu_1 + \mu_2 = 1$), then we come to the following statement:

Probabilites μ_1 and μ_2 corresponding to optimal mixed testing strategy μ_0 can be found from solution of linear equations:

$$\begin{cases} (\mu_1 \dfrac{x_u^M - x_1^w}{|x_u^M - x_1^w|} + \mu_2 \dfrac{x_u^M - x_2^w}{|x_u^M - x_2^w|}, \overrightarrow{e_3}) = 0 \\ \mu_1 + \mu_2 = 1 \end{cases} \tag{17}$$

3.3 Numeric results

Consider following numeric parameters: $M = 230kg$, $B = 38.8N \cdot m \cdot sec^2$, $f = 0.5N$, $a = 0.65m$, $b = 0.29m$, $\delta_1 = 0.01m$, $\delta_2 = 0.02m$. Initial conditions for system (14) (initial deviation from ideal trajectory): $x(t_0) = (0.2m, 0, -1m, 0, 0, 0)$.

Geometric game is considered in projection on the plane x_1, x_3. Set of perturbations is defined by two elements: $v^1(t) = (0, 0, 0, 0) = const$, $v^2(t) = (0, f/100, 0, 0) = const$. These perturbations brings system (15) in two points of set G_w: $x_1^w = (0.2, 0.84)$, $x_2^w = (0.47, 2.75)$. Easy to conclude using Theorem 1 that is no equilibrium point in geometric game Γ_1.

Next we find mini-max point $x_u^M = (0, 1.8416)$. Optimal pure control strategy that corresponds with x_u^M can be written in following form:

$$u_1(t) = u_2(t) = \begin{cases} f, t < 7.693 \\ 0, t >= 7.693 \end{cases} \quad u_3(t) = u_4(t) = 0$$

Optimal value of control quality criterion is $K_0 = 1.0214$. Let us integrate system (16) with found optimal control. Result of such integration is represented on fig 4.

System (17) for our case takes the form:

$$\begin{cases} -\mu_1 0.89 + \mu_2 0.98 = 0 \\ \mu_1 + \mu_2 = 1 \end{cases}$$

It has unique solution: $\mu_1 = 0.524$, $\mu_2 = 0.476$.

The probabilities μ_1 and μ_2 are treated as frequencies of selection of the perturbations $v^1(t)$ and $v^2(t)$ during the second stage of maxi-min testing procedure.

4. CONCLUSION

Maxi-min testing procedure can be fully implemented after finding the game equilibrium point, cost of game $J_0 = \min_{x_u \in G_u} \max_{x_w \in \Omega_w} J(x_w, x_u)$, optimal pure control strategy x_u^M and optimal mixed perturbation strategy μ_0. The existence of equilibrium point in mixed strategies guarantees the objectiveness of the result of maxi-min testing procedure. The above example illustrates the constructive use of testing procedure even in the absence of the equilibrium point in pure strategies.

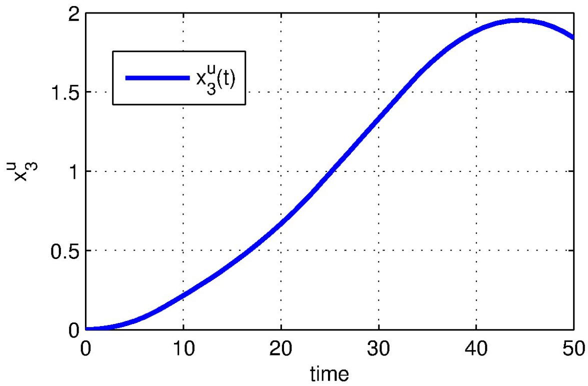

Fig. 4. Process $x_u^3(t)$ during the action of optimal control.

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