Output Regulation of Non-Minimum Phase Nonlinear Systems Using Extended High-Gain Observers

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Abstract: In this paper, a robust output feedback controller for systems in the normal form—which could potentially include unstable zero dynamics—is presented. The control scheme we present incorporates an extended high-gain observer to estimate an unknown function in the system dynamics, in addition to the derivatives of the output. This design scheme is not dependent on any specific type of control law, but continuously implemented sliding mode control was chosen for this paper due to its robustness and the fact that it tends to provide good transient performance. It is shown that regulation can be achieved in the case of an unknown control coefficient and uncertain constant parameters.

Keywords: Nonlinear control, robust control, non-minimum phase system, high-gain observer, sliding mode control, output regulation

1. INTRODUCTION

A lot of research has been conducted in the past couple of decades on the problem of output regulation of nonlinear systems, and papers on this topic are ubiquitous. Most of the available results have in fact been collected in well known books on nonlinear control, such as Huang (2004); Isidori (1999). Despite this wealth of literature, one area of nonlinear control is still very much an open avenue of research in many respects, due mainly to the challenging nature of the problem—and this is the problem of robust output feedback regulation of non-minimum phase systems. A major breakthrough was achieved on the stabilization problem by the pioneering work of Isidori (1999). This work was later followed up by Marconi et al. (2004); Celani et al. (2008), in which the the same technique was extended and applied to the problem of nonlinear output regulation of non-minimum phase systems.

Both of the aforementioned approaches utilize high-gain feedback techniques. The results contained therein do not consider uncertain control coefficients, and this particular issue is addressed in this paper. This work is an extension, in analogous fashion, of the stabilization problem investigated by Nazrulla and Khalil (2011). The approach utilizes an extended high-gain observer (EHGO) to estimate a "virtual output" in addition to the system output (or tracking error in the case of regulation) and its first ρ derivatives (this number being the relative degree of the system), in conjunction with a continuous implementation of sliding mode control. The key idea of Isidori (1999); Marconi et al. (2004) is retained in both these approaches, and consequently, the results take advantage of the knowl-

edge of a dynamic controller for the auxiliary subsystem associated with the original problem, and the definition of the latter in Nazrulla and Khalil (2011) and in this work is identical to the original work of Isidori (1999); Marconi et al. (2004); Celani et al. (2008), respectively.

Despite the aforementioned similarities between this paper and prior work by Isidori (1999); Marconi et al. (2004); Celani et al. (2008), we should note that our approach to the output feedback design is different. The papers Isidori (1999); Marconi et al. (2004); Celani et al. (2008) use high-gain feedback to implement the output feedback control in such a way that the closed-loop system can be represented as the feedback connection of an asymptotically stable system with a memoryless high-gain system. In the current paper, we do not use high-gain feedback. Instead, an EHGO is used to estimate all of the signals that are needed to implement the state feedback controller, thus allowing for us to obtain results in keeping with the spirit of the Separation Principle. Consequently, our approach does not confine us to any specific control design method such as high-gain feedback, sliding mode control (SMC), Lyapunov redesign, etc., although continuously implemented SMC was chosen for this paper. Moreover, high-gain feedback, by its nature, would cause a large spike in the control signal in the initial transient, which will not be observed with the proposed controller. Another technical difference between the results of the current paper and those of Marconi et al. (2004); Celani et al. (2008) is that our design allows for uncertainty in the control coefficient, which was not addressed in the former works. Handling this uncertainty dominates our analysis in Section 2.2.

A preliminary version of this paper appears in Nazrulla and Khalil (2009). The main difference between the aforementioned paper and this one is that we use the standard servocompensator driven by the tracking error in this work, whereas in Nazrulla and Khalil (2009), the servo is driven by the virtual output obtained from the extended high-gain observer. The approach adopted in this paper simplifies some of the technical details in the analysis of the state feedback system, as well as the design of the stabilizing controller for the auxiliary system. The example used in this paper is a linearized pendulum model, which is also different from the second order system utilized in Nazrulla and Khalil (2009).

2. PROBLEM DESCRIPTION AND ASSUMPTIONS

2.1 Background and Preliminaries

In this paper, we consider a single-input, single-output system that is in the normal form and has a well-defined relative degree, and that could potentially include unstable zero dynamics. The system model is assumed to be transformable into the form [1]

$$\left.\begin{aligned}
\dot{\eta} &= f(\eta, \xi, e_\rho, w, \theta), \\
\dot{\xi} &= P\xi + Qe_\rho, \\
\dot{e}_\rho &= b(\eta, \xi, e_\rho, w, \theta) + a(\eta, \xi, e_\rho, w, \theta)[u - \chi(w, \theta)], \\
y &= R\xi = \xi_1,
\end{aligned}\right\} \quad (1)$$

where the state $(\eta, \xi, e_\rho) \in \mathbb{R}^n$, the variables $\xi \in \mathbb{R}^{\rho-1}$ and $e_\rho \in \mathbb{R}$, the system output $y \in \mathbb{R}$ is the regulation error with respect to a sinusoidal or constant reference signal, $\theta \in \Theta \subset \mathbb{R}^p$ is a vector of constant parameters, where Θ is a compact set, and the disturbance $w : [0, \infty) \to \mathbb{R}^d$ is generated by the following neutrally stable exosystem,

$$\dot{w} = S_0 w, \quad (2)$$

where $S_0 \in \mathbb{R}^{d \times d}$ has simple eigenvalues on the imaginary axis. The control coefficient $a(\eta, \xi, e_\rho, w, \theta) \neq 0$ and the functions $f(\cdot)$ and $b(\cdot)$ are such that, for all $(w, \theta) \in W \times \Theta$, $f(0, 0, 0, w, \theta) = 0$ and $b(0, 0, 0, w, \theta) = 0$. The matrices $P \in \mathbb{R}^{(\rho-1) \times (\rho-1)}$, $Q \in \mathbb{R}^{(\rho-1) \times 1}$ and $R \in \mathbb{R}^{1 \times (\rho-1)}$ have the following structure.

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

and $R = (1 \ 0 \ \cdots \ 0)$.

When we examine the error system (1), we notice that in order for regulation to occur, i.e., for $y(t) \to 0$, the control $u(t)$ would have to asymptotically converge to a steady state given by $\chi(w, \theta)$. This is not directly implementable due to w, θ being unknown. Nevertheless, this problem has been solved in the case of minimum phase systems: Huang (2004); Nikiforov (1998). However, when the zero dynamics of (1) are unstable, the high gain feedback approach would most certainly fail because it would force the dynamics of the closed loop system to approach those

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[1] The details regarding the relevant class of systems and its transformation into the form of (1) is provided, for instance, in papers by Seshagiri and Khalil (2005); Marconi et al. (2004).

of the unstable zero dynamics. This problem was dealt with using the design tool introduced by Isidori (1999), where the coupling term $b(\eta, \xi, e_\rho, w, \theta)$ is exploited in the design of the dynamic output feedback stabilizer. This tool was extended to the regulation problem by Marconi et al. (2004).

Assumption 1. There exist real constants $c_0, \ldots, c_{m-1}$, independent of θ, such that the matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{m-1} \end{pmatrix}$$

has simple eigenvalues on the imaginary axis and $\chi(w, \theta)$ is generated by the internal model

$$\left.\begin{aligned}
\frac{\partial \tau(w, \theta)}{\partial w} S_0 w &= S\tau(w, \theta), \\
\chi(w, \theta) &= \Gamma\tau(w, \theta),
\end{aligned}\right\} \quad (3)$$

where $\tau^\top = (\chi, \ L_S\chi, \ \cdots, \ L_S^{m-1}\chi)$, $L_S\chi = (\partial\chi/\partial w)S_0 w$, $\Gamma = (1 \ 0 \ \cdots \ 0) \in \mathbb{R}^{1 \times m}$, and the pair (S, Γ) is observable.

In the next subsection we present the state feedback control design approach that addresses the above-mentioned goal.

2.2 State Feedback Regulator Design

Firstly, in order to achieve output regulation, we augment the system (1) with the servocompensator

$$\dot{\zeta} = S\zeta + J\xi_1, \quad (4)$$

where $\zeta \in \mathbb{R}^m$. As in the stabilization case, the state feedback design for the augmented system (1), (4) is carried out by utilizing virtual signals $\xi_1, \ldots, \xi_{\rho-1}$ and

$$\sigma = b(\cdot) - a_n(\cdot)K_1\zeta - a(\cdot)\chi(\cdot) + \Delta_a(\eta, \xi, e_\rho, w, \theta)u, \quad (5)$$

where $\Delta_a(\eta, \xi, e_\rho, w, \theta) = a(\eta, \xi, e_\rho, w, \theta) - a_n(\xi, e_\rho)$. These signals can be estimated by introducing the following extended-high gain observer (EHGO).

$$\begin{aligned}
\dot{\hat{\xi}} &= P\hat{\xi} + Q\hat{e}_\rho + H(\varepsilon)(y - \hat{\xi}_1), \\
\dot{\hat{e}}_\rho &= \hat{\sigma} + a_n(\hat{\xi}, \hat{e}_\rho)(u + K_1\zeta) + (\hat{\alpha}_\rho/\varepsilon^\rho)(y - \hat{\xi}_1), \\
\dot{\hat{\sigma}} &= (\hat{\alpha}_{\rho+1}/\varepsilon^{\rho+1})(y - \hat{\xi}_1),
\end{aligned}$$

where

$$H(\varepsilon) = \left(\hat{\alpha}_1/\varepsilon, \ \hat{\alpha}_2/\varepsilon^2, \ \cdots \ \hat{\alpha}_{\rho-1}/\varepsilon^{\rho-1}\right)^\top,$$

and $a_n(\xi, e_\rho)$ is a nominal function used in the subsequent state feedback control design in place of $a(\eta, \xi, e_\rho, w, \theta)$. In the above observer equations, we have followed the standard notational convention of labeling the estimates of the virtual signals with the hatted versions of the corresponding symbols. It can be shown [Freidovich and Khalil (2008)] using high gain observer theory [Atassi and Khalil (2001)] in conjunction with singular perturbation analysis [Kokotović et al. (1999)] that the signals $\hat{\xi}_1, \ldots, \hat{\xi}_{\rho-1}, \hat{e}_\rho, \hat{\sigma}$ become arbitrarily close to $\xi_1, \ldots, \xi_{\rho-1}, e_\rho$ and σ, respectively. In the case of minimum phase systems, the regulation problem for a plant with well-defined relative degree can always be reduced to one of regulating a relative degree one system by utilizing a high-gain observer. For example, a virtual output could be defined as $s = e_\rho + \sum_{i=1}^{\rho-1} k_i\xi_i$,

with the constants k_i chosen so as to render the system minimum phase with respect to the output s. Then, a static control law utilizing the virtual output s could be designed in accordance with standard techniques, and this design could be implemented by replacing s with its estimate from a high-gain observer. This technique will not work if the system is non-minimum phase, however.

Since we are considering non-minimum phase systems in this paper, we introduce, in analogous fashion to the stabilization case of Nazrulla and Khalil (2011), the dynamic compensator given by

$$\dot{\phi} = L(\xi, \phi, \zeta) + M(\xi, \phi, \zeta)\sigma. \tag{6}$$

Next, we define a new virtual output

$$s = e_\rho - N(\xi, \phi, \zeta). \tag{7}$$

Our objective is to design the triple $\{L, M, N\}$ such that the augmented system (1), (4), (6), (7) with s taken as its output is minimum phase. In order to continue with the analysis and bearing in mind the goal of reducing the regulation problem to one of stabilization, we introduce the change of variables $e_\rho \mapsto s$, $\zeta \mapsto \psi \triangleq \zeta - X\tau(w, \theta)$ and $u \mapsto v \triangleq u + K_1\zeta = u + K_1\psi - \chi(w, \theta)$ to obtain the following equations describing the augmented system.

$$\left.\begin{aligned}
\dot{\eta} &= f(\eta, \xi, s + N(\xi, \phi, \psi + X\tau), w, \theta), \\
\dot{\psi} &= S\psi + J\xi_1, \\
\dot{\xi} &= P\xi + Q[s + N(\xi, \phi, \psi + X\tau)], \\
\dot{s} &= b(\cdot) - a(\cdot)K_1\psi + a(\cdot)v - \frac{\partial N}{\partial \zeta}(S\psi + J\xi_1) \\
&\quad - \frac{\partial N}{\partial \zeta}SX\tau - \frac{\partial N}{\partial \xi}[P\xi + Q(s + N(\cdot))] \\
&\quad - \frac{\partial N}{\partial \phi}[L + M(b - aK_1\psi + \Delta_a v)], \\
\dot{\phi} &= L(\cdot) + M(\cdot)[b(\cdot) - a(\cdot)K_1\psi + \Delta_a(\cdot)v].
\end{aligned}\right\} \tag{8}$$

We note that if $\rho = 1$, then the $\dot{\xi}$-equation in (8) will not exist, and ξ_1 would be replaced by $s + N(\cdot)$. Now, let

$$g_s(\eta, \xi, s, \phi, \psi, w, \theta) = a(\cdot) - \frac{\partial N}{\partial \phi}M(\cdot)\Delta_a(\cdot).$$

The system (8) will have a relative degree of one with respect to s if

$$\frac{g_s(\eta, \xi, s, \phi, \psi, w, \theta)}{a(\eta, \xi, s, \phi, \psi, w, \theta)} \geq k^\dagger > 0, \tag{9}$$

for some positive real constant $k^\dagger$. When $\Delta_a \neq 0$, we transform (8) into the normal form by the change of variable $T : \phi \mapsto q$ that satisfies the partial differential equation

$$\frac{\partial T}{\partial s}g_s + \frac{\partial T}{\partial \phi}M\Delta_a = 0. \tag{10}$$

The zero dynamics of the transformed system shall have trajectories confined to the set $\{(\eta, \xi, s, q, \psi)|s = 0\}$, and are given by

$$\left.\begin{aligned}
\dot{\eta} &= f(\eta, \xi, N(\xi, q + \varphi, \psi + X\tau, w, \theta), w, \theta), \\
\dot{\psi} &= S\psi + J\xi_1, \\
\dot{\xi} &= P\xi + QN(\cdot), \\
\dot{q} &= \left(1 + \frac{\partial \Phi}{\partial \phi}\right)f_\phi + \frac{\partial \Phi}{\partial \eta}f + \frac{\partial \Phi}{\partial \xi}(P\xi + QN) \\
&\quad + \frac{\partial \Phi}{\partial \psi}(S\psi + J\xi_1) + \frac{\partial \Phi}{\partial w}S_0 w.
\end{aligned}\right\} \tag{11}$$

When $\Delta_a = 0$, this equation can be viewed as a feedback interconnection between the extended auxiliary plant of Marconi et al. (2004) corresponding to the first three equations in (1), and an auxiliary controller given by the $\dot{q}$-equation of (1). With $x_s = \left(\eta^\top \; \psi^\top \; \xi^\top \; q^\top\right)^\top$, (8) can be rewritten in terms of the new variables as

$$\dot{x}_s = F(x_s, s, w, \theta), \tag{12}$$
$$\dot{s} = f_s(x_s, s + N, w, \theta) + g_s(x_s, s + N, w, \theta)v, \tag{13}$$

for some functions $F(\cdot)$ and $f_s(\cdot)$. We note that the system (12)–(13) is relative degree one with s viewed as the output. Our goal now is to design the dynamic compensator $\{L, M, N\}$ such that the resulting augmented system is minimum phase, and the internal dynamics (12) are input-to-state stable with s viewed as an input to the latter. Finally, the stabilizing control input v for the augmented system can be designed using any viable robust control technique for relative degree one minimum phase systems such as high gain feedback, Lyapunov redesign, etc. We choose continuously implemented sliding mode control (SMC) in this paper, but it is important to note that we are not confined to any particular technique. SMC offers several advantages such as control saturation at a predetermined level—which mitigates the effects of the peaking phenomenon caused by the high gain observer—trajectory convergence to a region inside the boundary layer in finite time, and high gain feedback inside the aforementioned region. The control input v is thus designed as

$$v = -\frac{\beta(\xi, e_\rho, \phi, \zeta)}{a_n(\xi, e_\rho)}\text{sat}\left(\frac{s}{\mu}\right), \tag{14}$$

where μ is a constant design parameter that specifies the thickness of the boundary layer, and $\beta(\cdot)$ is chosen in accordance with Assumption 2 and inequality (15) below.

Assumption 2. It is assumed that

$$a(x_s, s + N(x_s), w, \theta)/a_n(x_s, s + N(x_s)) \geq k_0 > 0,$$

for all $(x_s, s) \in D$ and $(w, \theta) \in W \times \Theta$. Moreover, $a_n(x_s, s + N(x_s))$ is locally Lipschitz in (x_s, s) over the domain of interest, and globally bounded in ξ.

The function $\beta(\cdot)$ is also chosen to satisfy

$$\beta(\xi, e_\rho, \phi, \zeta) \geq \beta_0 + \left|\frac{a_n(\xi, e_\rho)f_s(x_s, s + N, w, \theta)}{g_s(x_s, s + N, w, \theta)}\right|, \tag{15}$$

where $\beta_0 > 0$. Assumption 2 and (15) ensure that when $s \geq \mu$, we have $s\dot{s} \leq -k^\dagger k_0 \beta_0 |s|$, and so whenever $|s(0)| > \mu$, $|s(t)|$ will decrease until it reaches the set $\{|s| \leq \mu\}$ in finite time and remain inside thereafter.

The closed-loop state feedback system is given by (12), (13), (14). We require this system to satisfy the following assumptions.

Assumption 3.

1) Let $D \subset \mathbb{R}^{m+n+r}$ be a domain containing the origin, and let $(x_s, s) \in D$. Moreover, suppose there exists a continuously differentiable Lyapunov function $V(x_s, w, \theta)$ such that

$$\left.\begin{aligned}
\alpha_2(\|x_s\|) &\geq V(x_s, w, \theta) \geq \alpha_1(\|x_s\|), \\
-\alpha_3(\|x_s\|) &\geq \frac{\partial V}{\partial x_s}F(x_s, s + N, w, \theta) \\
&\quad + \frac{\partial V}{\partial w}S_0 w, \quad \forall \|x_s\| \geq \gamma(|s|),
\end{aligned}\right\} \tag{16}$$

for all $(x_s, s) \in D \subset \mathbb{R}^{m+n+r}$, and $(w, \theta) \in W \times \Theta$, where $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot)$ and $\gamma(\cdot)$ are class $\mathcal{K}$ functions.

2) With $s = 0$, the origin $x_s = 0$ of (12) is locally exponentially stable.

Remark 1.

(1) The set D is well-defined for any sufficiently smooth $N(\cdot)$ in the definition of the variable $s = e_\rho - N(\cdot)$.

(2) Inequality (16) is equivalent to regional input-to-state stability of (12) with s viewed as an input.

Assumption 4.

1) $L(x_s)$, $M(x_s)$ and $\beta(x_s)$ are locally Lipschitz functions in their arguments over the domain of interest, and $L(0) = 0$.

2) $L(x_s)$ and $M(x_s)$ are globally bounded functions of ξ.

Remark 2. We require the global boundedness assumption on $L(x_s)$ and $M(x_s)$ to protect the system from the peaking phenomenon that may occur in the output feedback implementation when the extended high-gain observer is introduced [Atassi and Khalil (2001); Khalil (2002)].

Under the above assumptions, we obtain the following result pertaining to the stability of this state feedback error system.

Proposition 1. Consider the closed loop state feedback error system (12), (13). Let Assumptions 1 through 4 hold. Then, there exists a positive constant μ^* such that (12), (13) has an exponentially stable equilibrium at $(x_s = 0, s = 0)$ for all $\mu \in (0, \mu^*]$. Moreover, the set Ω defined below is an estimate of the region of attraction.

$$\Omega \triangleq \{V(x_s, w, \theta) \leq c_0\} \times \{|s| \leq c\}, \quad (17)$$

where $c > \mu$, $c_0 \geq \alpha_2(\gamma(c))$, and c, c_0 are chosen such that Ω is a compact subset of D.

Remark 3.

1) Assumption 3 requires the compensator $\{L, M, N\}$ to be designed such that (12) is input to state stable with s viewed as the input. This problem is similar to the auxiliary design problem of Marconi et al. (2004) when $\Delta_a = 0$, with the difference being in the structure of the servocompensator employed in each paper. In Marconi et al. (2004), the interconnection of the extended auxiliary system with its dynamic compensator was assumed to be globally asymptotically stable and locally exponentially stable, with a Lyapunov function independent of the unknown system parameters.

2) If all the assumptions hold globally, then we could take $D = \mathbb{R}^{m+n+r}$, and with the appropriate choice of controller parameter $\beta(\cdot)$ (see (15)), the positively invariant set $\Omega \subset D$ that depends on this parameter can be made arbitrarily large, and thus our design allows for semi-global stabilization of the state feedback error system.

2.3 Output Feedback Control Using an Extended High Gain Observer

Output Feedback Regulator Design In order to estimate the states e of the error system (1) and the virtual output σ, an extended high-gain observer with the following structure is introduced.

$$\left.\begin{aligned}
\dot{\hat{\xi}} &= P\hat{\xi} + Q\hat{e}_\rho + H(\varepsilon)(y - \hat{\xi}_1), \\
\dot{\hat{e}}_\rho &= \hat{\sigma} + a_n(\hat{\xi}, \hat{e}_\rho)(u + K_1\zeta) + (\hat{\alpha}_\rho/\varepsilon^\rho)(y - \hat{\xi}_1), \\
\dot{\hat{\sigma}} &= (\hat{\alpha}_{\rho+1}/\varepsilon^{\rho+1})(y - \hat{\xi}_1),
\end{aligned}\right\} \quad (18)$$

where $H(\varepsilon) = \left(\hat{\alpha}_1/\varepsilon, \, \hat{\alpha}_2/\varepsilon^2, \, \cdots \, \hat{\alpha}_{\rho-1}/\varepsilon^{\rho-1}\right)^\top$, and the constants $\hat{\alpha}$ are chosen so as to make the polynomial $\lambda^{\rho+1} + \hat{\alpha}_1\lambda^\rho + \ldots + \hat{\alpha}_\rho\lambda + \hat{\alpha}_{\rho+1}$ Hurwitz. The structure of this observer is very similar to the one used by Freidovich and Khalil (2008), with the only difference being the fact that this observer is driven by the signal $v = u + K_1\zeta$ instead of by u.

The state and virtual output estimates obtained from this observer are used to replace the unavailable states and unmeasured signals in the state feedback design, and the extended high-gain observer-based output feedback regulator is given by (18) and

$$\left.\begin{aligned}
\dot{\zeta} &= S\zeta + Jy, \\
\dot{\phi} &= L(\hat{\xi}, \phi, \zeta) + M(\hat{\xi}, \phi, \zeta)\hat{\sigma}, \\
u &= K \, \text{sat}(l(\hat{\xi}, \hat{e}_\rho, q, \zeta)/K) - K_1\zeta, \\
l(\cdot) &= \frac{-\beta(\hat{\xi}, \hat{e}_\rho, \phi, \zeta)}{a_n(\hat{\xi}, \hat{e}_\rho)} \, \text{sat}\left(\frac{\hat{e}_\rho - N(\hat{\xi}, \phi, \zeta)}{\mu}\right),
\end{aligned}\right\} \quad (19)$$

where

$$K > \max_{(x_s, s) \in \Omega} \left|\frac{\beta(\hat{\xi}, \hat{e}_\rho, \phi, \zeta)}{a_n(\hat{\xi}, \hat{e}_\rho)}\right|. \quad (20)$$

The sliding mode control component v is saturated at $\pm K$ outside Ω in order to protect the system from peaking during the observer's transient response.

Exponential Stability and Trajectory Convergence Let $x_o = \left(x_s^\top \ s\right)^\top$, and let $(x_o(t, \varepsilon), \hat{\xi}(t, \varepsilon), \hat{e}_\rho(t, \varepsilon), \hat{\sigma}(t, \varepsilon))$ denote the trajectory of the system (8), (18), (19) starting from $(x_o(0), \hat{\xi}(0), \hat{e}_\rho(0), \hat{\sigma}(0))$. Also, let $x_r(t)$ be the solution of (12), (13) starting at $x_o(0)$. We know that the trajectory $(x_s(t), s(t))$ of the system (12), (13) is exponentially stable with respect to $\{(x_s, s) = (0, 0)\}$, so let us suppose that $\mathcal{R}$ is its region of attraction. Let $\mathcal{S}$ be a compact set in the interior of $\mathcal{R}$, and let $\mathcal{Q}$ be a compact subset of $\mathbb{R}^{\rho+1}$. The recovery of exponential stability of the zero error manifold, and the fact that $x_o(t)$ converges to $x_r(t)$ as $\varepsilon \to 0$, uniformly in t, for all $t \geq 0$, is established by the following theorem, (Atassi and Khalil, 2001, Theorems 1, 2, 3 and 5), provided that $x_o(0) \in \mathcal{S}$ and $(\hat{\xi}(0), \hat{e}_\rho(0), \hat{\sigma}(0)) \in \mathcal{Q}$.

Theorem 1. Let Assumptions 1 through 4 hold. Moreover, assume $x_o(0) \in \mathcal{S}$ and $(\hat{\xi}(0), \hat{e}_\rho(0), \hat{\sigma}(0)) \in \mathcal{Q}$. Then,

(1) there exists $\varepsilon_1^* > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_1^*$, the system (8), (18), (19) is exponentially stable with respect to the manifold $\{(x_o(0), \hat{\xi}(0), \hat{e}_\rho(0), \hat{\sigma}(0)) = (0, 0, 0, 0)\}$, and $\mathcal{S} \times \mathcal{Q}$ is included in the region of attraction.

(2) given any $\delta > 0$, there exists $\varepsilon_2^* > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_2^*$, we have

$$\|x_o(t, \varepsilon) - x_r(t)\| \leq \delta, \quad \forall t \geq 0. \quad (21)$$

3. EXAMPLE

We illustrate the design procedure for an extended high-gain observer based sliding mode output feedback regulator using a linearized inverted pendulum on a cart model as an example. The system is given by [Celani et al. (2008)]

$$\left.\begin{aligned}
\dot{z}_1 &= z_2 - \frac{x_2}{h}, \\
\dot{z}_2 &= \frac{g}{h} z_1, \\
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= (u + w_1 - m_p g z_1)\frac{1}{m_c}, \\
y &= x_1,
\end{aligned}\right\} \quad (22)$$

where z_1 is the angle of the pendulum with respect to the vertical, x_1 is the position of the cart, u is the control input, w_1 is a sinusoidal disturbance of known frequency Ω, m_c is the mass of the cart, m_p is the mass of the pendulum (assumed to be concentrated in the bob), h is its length, and g is the acceleration due to gravity. m_p and h are assumed to be uncertain parameters in the ranges 0.19 kg $\leq m_p \leq$ 0.23 kg and 0.55 m $\leq h \leq$ 0.67 m. An exosystem generates the sinusoidal disturbance signal w_1, and this system is modeled by $\dot{w} = S_0 w$, where

$$S_0 = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}.$$

The goal is to utilize the output measurement y to drive a feedback controller designed to make this system output converge to zero despite the presence of the input disturbance w_1 and uncertain parameters m_p and h.

We conclude from the steady-state analysis of (22) that the steady state control is $\chi = -w_1$, and this allows us to determine the S matrix of the servocompensator and an appropriate gain matrix K_1 to make $S - JK_1$ Hurwitz. [2]

We now proceed with the first step in our design by augmenting (22) with the servocompensator

$$\dot{\zeta} = S\zeta + Jy, \quad (23)$$

where

$$S = \begin{pmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Next, we choose a matrix K_1 such that the eigenvalues of $(S - JK_1)$ are -1 and -2. The matrix is thus given by $K_1 = \begin{pmatrix} 3 - \Omega^2, & 2 \end{pmatrix}$.

3.1 The Auxiliary Problem

After applying the change of variable $\zeta \mapsto \psi = \zeta - X\tau$, the auxiliary system associated with the augmented system (22), (23) is given by

$$\left.\begin{aligned}
\begin{pmatrix} \dot{\psi} \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{x}_1 \end{pmatrix} &= \underbrace{\begin{pmatrix} S & 0 & 0 & J \\ 0 & 0 & 1 & 0 \\ 0 & g/h & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} \psi \\ z_1 \\ z_2 \\ x_1 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ -1/h \\ 0 \\ 1 \end{pmatrix}}_{B} u_a, \\
y_a &= \underbrace{\frac{-1}{m_c}\left(-K_1, \; m_p g, \; 0, \; 0\right)}_{C} \begin{pmatrix} \psi^\top & z_1 & z_2 & x_1 \end{pmatrix}^\top.
\end{aligned}\right\} \quad (24)$$

[2] The details on how to compute S can be found in Seshagiri and Khalil (2005) or Marconi et al. (2004).
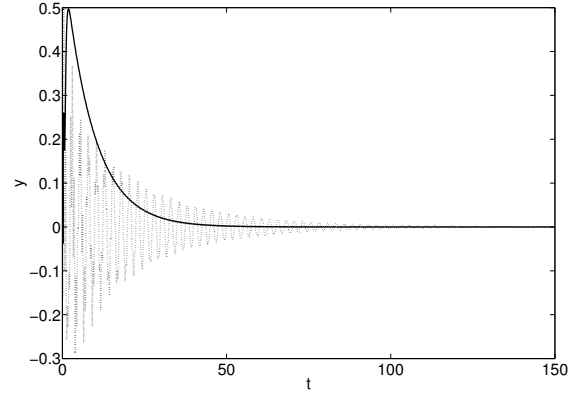
Fig. 1. Trajectories of the output for the nominal value of m_c—the solid curve corresponds to the EHGO + SMC design, while the dashed curve is the result of Celani et al. (2008).

We now design an observer based controller for (24), and it is given by

$$\left.\begin{aligned}
\dot{\phi} &= (A - HC_n - BK)\phi + Hy_a), \\
u_a &= -K\phi,
\end{aligned}\right\} \quad (25)$$

where $C_n = \frac{-1}{\hat{m}_c}\left(-K_1, \; m_p g, \; 0, \; 0\right)$ is a nominal version of the matrix C, $\hat{m}_c$ is a nominal parameter used in lieu of m_c, and the gain matrices K and H are designed using the LQR technique to make $A - BK$ and $A - HC_n$ Hurwitz. These gain matrices were designed to be

$$K = \left(-0.022, \; 0.100, \; -17.099, \; -4.265, \; -10.000\right),$$

$$H = \left(0.082, \; 0.568, \; -4.763, \; -19.097, -0.100\right)^\top.$$

3.2 The Output Feedback Regulator

The EHGO-based output feedback regulator for (22) is designed in accordance with Section 2.3, and is thus given by

$$\left.\begin{aligned}
\dot{\hat{x}}_1 &= x_2 + (\hat{\alpha}_1/\varepsilon)(y - \hat{x}_1), \\
\dot{\hat{x}}_2 &= \hat{\sigma} + v/\hat{m}_c + (\hat{\alpha}_2/\varepsilon^2)(y - \hat{x}_1), \\
\dot{\hat{\sigma}} &= (\hat{\alpha}_3/\varepsilon^3)(y - \hat{x}_1), \\
\dot{\zeta} &= S\zeta + Jy, \\
\dot{\phi} &= (A - HC_n - BK)\phi + H\hat{\sigma}, \\
v &= -\beta\hat{m}_c \, \text{sat}\left(\frac{\hat{x}_2 + K\phi}{\mu}\right), \\
u &= v - K_1\zeta.
\end{aligned}\right\} \quad (26)$$

3.3 Numerical Simulations

Simulations were performed for the output feedback design given in Section 3.2, and the results were compared with that of the design provided in Celani et al. (2008). The results are depicted in the figures that follow.

Discussion   A comparison was made with the design presented in this paper and the one based on Celani et al. (2008) by means of simulations. The results showed that both the high-gain feedback design method of Celani et al. (2008) and the EHGO-based design can achieve regulation for the targeted range of uncertainties in the
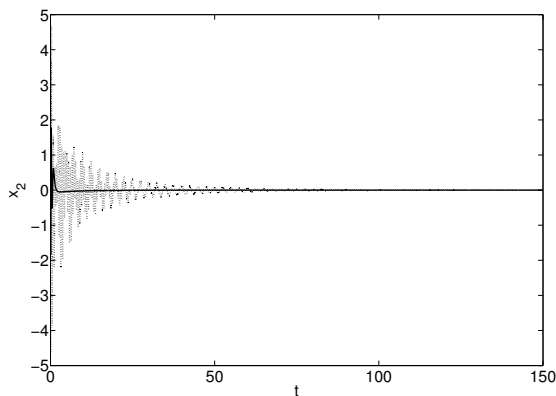
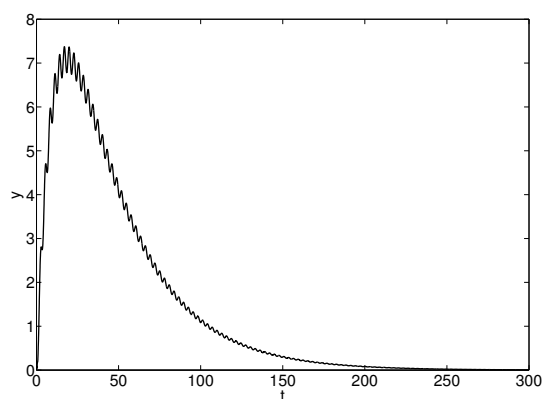Fig. 2. Trajectories of the derivative of the output for the nominal value of m_c.

Fig. 3. The response of the EHGO design when $m_c = 0.777$ kg.

plant parameters m_p and h. Figure 1 and Figure 2 show the state and output trajectories for the case when $\hat{m}_c = m_c = 0.82$ kg. It can be seen from these figures that the EHGO design provides better transient performance and shorter settling times. Another key difference is the performance in the presence of uncertainty in the nominal plant parameter m_c, which also affects the control coefficient, which is basically $1/m_c$, and the range of uncertainty allowed for by each design. The high-gain feedback design of Celani et al. (2008) achieves regulation when $m_c \in [0.819, 0.8206]$ kg, which is a range of uncertainty from -0.12% to $+0.0732\%$, while the EHGO design allows for $m_c \in [0.777, 0.84]$ kg, which is a range of -5.24% to $+2.44\%$. Moreover, the control effort corresponding to the EHGO method is consistently lower than the method of Celani et al. (2008) by a factor of about 5. Figure 3 shows the output trajectory of the EHGO design when $m_c = 0.777$ kg—the design of Celani et al. (2008) is unstable for this particular parameter value.

4. CONCLUDING REMARKS

In this paper, an output feedback regulator design was presented for non-minimum phase systems with uncertainties in the control coefficient, and in the constant system parameters. This design utilizes an extended high-gain observer and continuously-implemented sliding mode control. The fundamental idea is to use the knowledge of a stabilizing controller for an associated auxiliary problem in designing the controller for the full system. The contribution of this paper is that it provides a design procedure that reduces the problem of controlling a non-minimum phase nonlinear system to one of designing a controller for a relative degree one minimum phase augmented system under certain assumptions, and uncertainty in the control coefficient is considered, as well. It is shown that this design is able to solve the regulation problem in the presence of the aforementioned uncertainty, provided it is small enough. Some numerical simulations of a linearized inverted pendulum on a cart model are included to illustrate the procedure and the resulting performance, as well as a comparison between this paper's design and the high-gain feedback method of Celani et al. (2008).

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