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Dynamic Lyapunov Functions

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Abstract: Lyapunov functions are a fundamental tool to investigate the stability properties of equilibrium points in linear or nonlinear systems. Unfortunately, even if the existence of Lyapunov functions for asymptotically stable equilibrium points is guaranteed by *converse Lyapunov theorems*, the actual computation of the analytic expression of the function may be difficult or impossible. Herein we propose an approach to avoid the issue of finding an explicit solution of the Lyapunov partial differential inequality, providing a family of Lyapunov functions for linear and nonlinear systems.

Keywords: Lyapunov Functions, Invariant Manifolds, System Immersion

1. INTRODUCTION

Stability analysis of equilibrium points, usually characterized in the sense of Lyapunov, see Lyapunov (1992), is a fundamental issue in systems and control theory. Lyapunov's theory provides a tool to assess (asymptotic) stability of equilibrium points of linear and nonlinear systems. In particular, it is well-known, see *e.g.* Bacciotti (1992) and Khalil (2001), that the existence of a scalar positive definite function, the time derivative of which along the trajectories of the system is negative definite, is a sufficient condition to guarantee asymptotic stability of an equilibrium point. Moreover, several theorems, the so-called *converse Lyapunov theorems*, implying at least conceptually the existence of a strict Lyapunov function defined in a neighborhood of an asymptotically stable equilibrium point, have been established, see *e.g.* La Salle (1976), Khalil (2001) and Lin et al. (1996).

In recent years the development of control techniques requiring the explicit knowledge of a Lyapunov function, such as *backstepping* and *control Lyapunov functions*, see *e.g.* Isidori (1995) and Sontag (1989) respectively, have conferred a crucial role to the issue of computation of Lyapunov functions. Finally, Lyapunov functions are useful to characterize and to estimate the region of attraction of equilibrium points, see for instance Chiang et al. (1988) and Vannelli and Vidyasagar (1985), where the notion of Maximal Lyapunov function is introduced. However, even if the existence of strict Lyapunov functions for asymptotically stable equilibrium points is guaranteed by *converse Lyapunov theorems*, the actual computation of the analytic expression of the function may be difficult or impossible. From a practical point of view this is the main drawback of Lyapunov methods Bacciotti and Rosier (2005). An alternative approach consists in determining, if it exists, a *weak* Lyapunov function, *i.e.* the time derivative of which is only negative semi-definite along the trajectories of the system, and then prove asymptotic stability by means of *Invariance Principle* arguments.

Exploiting the approach developed in Sassano and Astolfi (2010a) and Sassano and Astolfi (2010b), the main contribution of this article consists in the definition of a methodology to *construct* families of Lyapunov functions for linear and nonlinear systems by means of a dynamic extension to the state of the system. To begin with a Lyapunov function is defined for the *augmented* system and then an invariant submanifold in the extended state-space is determined such that the restriction of the flow of the *augmented* system to the manifold is a *copy* of the flow of the original nonlinear system. Finally, it is shown that the explicit solution of the partial differential equation that guarantees invariance of the submanifold can be avoided and an approximate solution is sufficient to determine the family of Lyapunov functions.

The rest of the article is organized as follows. In Section 2 the proposed methodology to construct Lyapunov functions via dynamic extension is introduced for linear time-invariant systems. The extension to nonlinear systems is the topic of Section 3. In the same section, it is shown that the explicit solution of the partial differential equation arising in the invariance condition can be avoided. The paper is concluded with a numerical example and some comments in Sections 4 and 5, respectively.

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2. LINEAR SYSTEMS

Consider a linear, time-invariant, autonomous system described by equations of the form

$$\dot{x} = Ax\,, \tag{1}$$

with $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Suppose that there exists a row-vector defined as $x^T P$, with $P = P^T > 0$ such that

$$\frac{1}{2}x^T PAx + \frac{1}{2}x^T A^T Px = -x^T Qx\,, \tag{2}$$

for some given $Q = Q^T > 0$ and for all $x \in \mathbb{R}^n$.

To present the main ideas of the proposed approach suppose that, instead of integrating the mapping $x^T P$ obtaining the quadratic function

$$V(x) = \frac{1}{2}x^T Px = \int_0^x \zeta^T P d\zeta\,, \tag{3}$$

we exploit the mapping $P(x) = x^T P$ to construct an *extended* function, namely considering the *immersion* of the system (1) into an augmented linear system defined on an *extended* state-space. To be more precise, let

$$V(x,\xi) = \int_0^\xi x^T P d\xi + \frac{1}{2}\|x-\xi\|_R^2\,, \tag{4}$$

where $\xi(t) \in \mathbb{R}^n$ and $R = R^T > 0$. Note that $\|v\|_R^2$ denotes the Euclidean norm of the vector v weighed by the matrix R, *i.e.* $\|v\|_R^2 = v^T Rv$. A Schur complement argument shows that the function $V(x,\xi)$ is globally positive definite for all the matrices $R \geq \bar{R} = \frac{1}{2}P$. Consider now the *augmented* linear system described by

$$\begin{aligned} \dot{x} &= Ax\,, \\ \dot{\xi} &= F\xi + Gx\,, \end{aligned} \tag{5}$$

with F and G to be determined, and the problem of studying the stability property of the origin using the function $V(x,\xi)$, defined in (4), as a *candidate* Lyapunov function. To begin with note that the partial derivatives of the function V are given by

$$\begin{aligned} V_x &= x^T P + (x-\xi)^T(R-P)\,, \\ V_\xi &= x^T P - (x-\xi)^T R\,. \end{aligned} \tag{6}$$

Therefore, the time derivative of the function V along the trajectories of the augmented system (5) is $\dot{V} = V_x Ax + V_\xi(F\xi + Gx)$. Setting

$$F = -kR\,, \tag{7}$$

$$G = k(R-P)\,, \tag{8}$$

$k > 0$, yields $\dot{\xi} = -kV_\xi^T$. Consequently,

$$\begin{aligned} \dot{V}(x,\xi) &= x^T PAx + x^T A^T(R-P)(x-\xi) \\ &\quad - k(x^T P - (x-\xi)^T R)(Px - R(x-\xi)) \\ &= -x^T Qx + x^T A^T(R-P)(x-\xi) \\ &\quad - k(x^T P - (x-\xi)^T R)(Px - R(x-\xi)) \\ &= -x^T Qx + x^T A^T(R-P)(x-\xi) \\ &\quad - k[x^T\ (x-\xi)^T]C^T C[x^T\ (x-\xi)^T]^T\,, \end{aligned} \tag{9}$$

with $C = [P\ \ -R]$, where the second equality is obtained using the condition (2). Note that the time derivative (9) can be rewritten as a quadratic form in x and $(x-\xi)$, *i.e.*

$$\dot{V}(x,\xi) = -[x^T\ (x-\xi)^T][M + kC^T C][x^T\ (x-\xi)^T]^T\,,$$

where the matrix M is defined as

$$M = \begin{bmatrix} Q & -\frac{1}{2}A^T(R-P) \\ -\frac{1}{2}(R-P)A & 0_n \end{bmatrix}.$$

Before stating the main result of this section – providing conditions on the choice of the matrix R such that the function V defined in (4) is indeed a Lyapunov function for the system (5) – we recall the following preliminary lemma.

Lemma 1. Anstreicher and Wright (2000) Let M be an $n \times n$ symmetric matrix and C an $m \times n$ matrix of rank m, where $m < n$. Let Z denote a basis for the null space of C.

(i) If $Z^T MZ$ is positive semidefinite and singular, then there exists a finite $\bar{k} \geq 0$ such that $M + kC^T C$ is positive semidefinite for all $k \geq \bar{k}$, if and only if $Ker(Z^T MZ) = Ker(MZ)$. In this case, $M + kC^T C$ is singular for all k.
(ii) $Z^T MZ$ is positive definite if and only if there exists a finite $\bar{k} \geq 0$ such that $M + kC^T C$ is positive definite for all $k \geq \bar{k}$.

Proposition 1. There exists a $\bar{k}$ such that the function $V(x,\xi)$, defined in (4), is positive definite and its time derivative along the trajectories of the system (5) is negative definite for all $k \geq \bar{k}$ if

$$\underline{\sigma}(R) > \frac{1}{2}\bar{\sigma}(P)\left[\frac{\bar{\sigma}(PA)}{\underline{\sigma}(Q)}\right]\,, \tag{10}$$

where $\underline{\sigma}(B)$ ($\bar{\sigma}(B)$, resp.) denotes the minimum (maximum, resp.) singular value of the matrix B. ◇

In other words, the positive definite scalar function $V(x,\xi)$ is indeed a Lyapunov function for the system (5), with F and G as in (7) and (8) respectively, *i.e.*

$$\begin{aligned} \dot{x} &= Ax\,, \\ \dot{\xi} &= -kR\xi + k(R-P)x\,, \end{aligned} \tag{11}$$

provided $k \geq \bar{k}$, proving asymptotic stability of the origin in the *extended* state-space.

2.1 Invariant Subspace

The result in Proposition 1 can be exploited to *construct* a Lyapunov function for the linear system (1). Suppose that there exists a linear subspace $\mathcal{L}$ parameterized in x, *i.e.* $\mathcal{L} = \{(x,\xi) \in \mathbb{R}^{2n} : \xi = Yx\}$ where $Y \in \mathbb{R}^{n\times n}$ is a non-singular matrix, which is invariant with respect to the dynamics of the augmented system (5) and such that the flow of the system (5) restricted to $\mathcal{L}$ is a copy of the flow of the system (1). The previous conditions hold if and only if there exists a matrix $Y \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} A & 0_n \\ k(R-P) & -kR \end{bmatrix}\begin{bmatrix} I_n \\ Y \end{bmatrix} = \begin{bmatrix} I_n \\ Y \end{bmatrix} A\,,$$

or equivalently such that

$$k(R-P) - kRY = YA\,, \tag{12}$$

for some $k \geq \bar{k}$.

Remark 1. The condition $\sigma(A) \cap \sigma(-kR) = \emptyset$ guarantees existence and unicity of the matrix Y and therefore the existence of the invariant subspace $\mathcal{L}$. ▲

Proposition 2. Suppose that the condition (10) is satisfied and let $k \geq \bar{k}$ and Y be the solution of (12). Then the restriction of the function $V(x,\xi)$ to the invariant subspace $\mathcal{L}$, defined as

$$V_{\mathcal{L}}(x) = V(x, Yx) = \frac{1}{2}x^T[Y^TP + PY + (I-Y)^TR(I-Y)]x\,, \tag{13}$$

depends only on the variable x, it is positive definite and its time derivative is negative definite along the trajectories of the system (1), hence $V_{\mathcal{L}}(x)$ is a Lyapunov function for the system (1). ◇

Remark 2. $V_{\mathcal{L}}(x)$ describes a family of Lyapunov functions for the system parameterized by the matrix $R \geq \bar{R}$ and $k \geq \bar{k}$. ▲

Proposition 3. $V_{\mathcal{L}}(x)$ coincides with the function (3) if $\hat{Y}$ is a simultaneous solution of the Sylvester equation (12) and of the algebraic Riccati equation

$$Y^TD + DY + Y^TRY - D = 0\,, \tag{14}$$

where $D = P - R$, $D < 0$ with $R > P$. Then, $V(x, \hat{Y}x) = 1/2x^TPx$ and the *original* quadratic Lyapunov function (3) is recovered. ◇

It is reasonable to expect the additional condition (14), since the Lyapunov function $V(x)$ defined as in (3) does not necessarily belong to the family parameterized by $V_{\mathcal{L}}$.

To have a better insight about this comment it is worth recalling that the matrix P is defined together with the matrix Q, *i.e.* the pair (P,Q) is such that $V(x)$ in (3) is a quadratic positive definite function and $\dot{V} = -x^TQx$ along the trajectories of the linear system (1). Therefore the function $V(x)$ belongs to the family of Lyapunov functions $V_{\mathcal{L}}$ if and only if it is possible to obtain at the same time $\dot{V}(x,Yx) = -x^TQx$ with a proper choice of the matrices R and Y and the parameter k. To be more precise, it can be easily noticed from (9) that, provided (10) is satisfied, larger values of the constant k yield a *more negative* $\dot{V}(x,\xi)$. Thus, if we let k be such that $\dot{V}(x,Yx) < -x^TQx$ for all $x \in \mathbb{R}^n$ then it is obvious that the quadratic function $V(x)$ defined in (3) can not belong to the family $V_{\mathcal{L}}$. On the contrary, if the parameter k is selected such that the solution Y of the Sylvester equation is in the set of solutions of the additional algebraic Riccati equation (14), then we can guarantee that the function (3) belongs to the family of Lyapunov functions.

Remark 3. With respect to the previous comment, the case $\mathcal{L} = Ker(C)$ must be dealt with separately, since $\dot{V}(x,Yx)$ is not affected by the choice of the parameter k. To begin with, write $Ker(C)$ as the set of (x,ξ) such that

$$\begin{bmatrix} P & -R \end{bmatrix} \begin{bmatrix} x \\ x-\xi \end{bmatrix} = 0$$

that is the set $\{(x,\xi) : \xi = -R^{-1}(P-R)x\}$. Obviously, if $Ker(C)$ coincides with $\mathcal{L}$, then the matrix

$$K = -R^{-1}(P-R)$$

must satisfy the Sylvester equation that defines $\mathcal{L}$. By substitution, it can be easily seen that $Ker(C) = \mathcal{L}$ if and only if $R = P$ and in this case not only the family of Lyapunov functions automatically contains the *original* Lyapunov function but actually the family *reduces* to the function $V(x)$ defined in (3). In fact, since the solution of the Sylvester equation is $Y = 0$, the invariant subspace is defined by $\xi = 0$ hence, with $R = P$, $V(x,0) = 1/2x^TPx$ and moreover the time derivative $\dot{V}(x,\xi)$ in (9) is equal to $-x^TQx$ for any value of the parameter k. ▲

3. NONLINEAR SYSTEMS

Consider a nonlinear, autonomous, system described by equations of the form

$$\dot{x} = f(x)\,, \tag{15}$$

with $x(t) \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable and suppose that the origin of the state-space is an equilibrium point for the system (15), *i.e.* $f(0) = 0$. Hence, there exists a continuous mapping $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $f(x) = F(x)x$.

Assumption 1. The equilibrium point $x = 0$ of the system (15) is locally exponentially stable, hence there exists a matrix $\bar{P} = \bar{P}^T > 0$ such that

$$\frac{1}{2}\bar{P}A + \frac{1}{2}A^T\bar{P} = -Q\,, \tag{16}$$

for some given $Q = Q^T > 0$, with $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$.

In this section - mimicking the results presented in Section 2 for linear systems - it is shown how to exploit the knowledge of a Lyapunov function for the linearized system $\dot{x} = Ax$, namely

$$V_l(x) = \frac{1}{2}x^T\bar{P}x\,, \tag{17}$$

to construct a Lyapunov function for the nonlinear system (15). To begin with, consider the following notion of solution of the Lyapunov partial differential inequality

$$V_x f(x) < 0\,, \tag{18}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$.

Definition 1. Consider the system (15). An *algebraic $\bar{P}$ solution* of the inequality (18) is a $\mathcal{C}^1$ mapping $P(x) : \mathbb{R}^n \to \mathbb{R}^{1\times n}$, $P(0) = 0$, such that

$$P(x)f(x) \leq -x^T\Gamma(x)x \triangleq -\alpha(x)\,, \tag{19}$$

for all $x \in \mathbb{R}^n$, with $\Gamma = \Gamma^T > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Moreover $P(x)$ is tangent at the origin to $\bar{P}$, namely

$$\left.\frac{\partial P}{\partial x}\right|_{x=0} = \bar{P}\,.$$

Note that $P(x)$ may not be a *gradient vector* of any scalar (positive definite) function. Define now similarly to (4) the function

$$V(x,\xi) = P(\xi)x + \frac{1}{2}\|x-\xi\|_R^2\,, \tag{20}$$

with $\xi(t) \in \mathbb{R}^n$ and $R = R^T \in \mathbb{R}^{n\times n}$ positive definite.

Remark 4. Consider V as in (20) and note that there exist a non-empty compact set $\Omega_1 \subseteq \mathbb{R}^{2n}$ containing the origin and a positive definite matrix $\bar{R}$ such that for all $R \geq \bar{R}$ the function $V(x,\xi)$ in (20) is positive definite for all $(x,\xi) \in \Omega_1 \subseteq \mathbb{R}^{2n}$. In fact, since $P(x)$ is tangent at $x = 0$ to the solution of the Lyapunov equation (16), the function $P(x)x : \mathbb{R}^n \to \mathbb{R}$ is, locally around the origin, quadratic and moreover has a local minimum for $x = 0$. Hence the function $P(\xi)x$ is (locally) quadratic in (x,ξ) and, restricted to the manifold $\mathcal{E} = \{\xi \in \mathbb{R}^n : \xi = x\}$, is positive definite in Ω_1. ▲

The partial derivatives of the function V defined as in (20) are given by

$$\begin{aligned} V_x &= P(x) + (x-\xi)^T(R - \Phi(x,\xi))^T \,, \\ V_\xi &= x^T\Psi(\xi) - (x-\xi)^T R \,, \end{aligned} \tag{21}$$

where $\Phi(x,\xi)$ is a smooth mapping such that

$$P(x) - P(\xi) = (x-\xi)^T\Phi(x,\xi)^T$$

and $\Psi(\xi)$ denotes the Jacobian matrix of the mapping $P(\xi)$. As in the linear setting, define an *augmented* nonlinear system described by equations of the form

$$\begin{aligned} \dot{x} &= f(x) \,, \\ \dot{\xi} &= -k(\Psi(\xi) - R)^T x - kR\xi \triangleq g(\xi)x - kR\xi \,, \end{aligned} \tag{22}$$

and let the function (20) be a *candidate* Lyapunov function to investigate the stability properties of the equilibrium point $(x,\xi) = (0,0)$ of the system (22). Finally, to streamline the presentation of the following result - providing conditions on the choice of the parameter k such that the function V in (20) is indeed a Lyapunov function for the augmented system (22) - let

$$\Delta(x,\xi) = (R - \Phi(x,\xi))\Lambda(\xi)^T \,, \tag{23}$$

with $\Lambda(\xi) = \Psi(\xi)R^{-1}$.

Proposition 4. There exist a set $\Omega \subset \mathbb{R}^{2n}$ and $\bar{k} > 0$ such that $V(x,\xi)$, defined in (20), is positive definite in Ω and its time derivative along the trajectories of the system (22) is negative definite for all $k \geq \bar{k}$ if and only if

$$\frac{1}{2}F(x)^T\Delta(x,\xi) + \frac{1}{2}\Delta(x,\xi)^T F(x) < \Gamma(x) \,, \tag{24}$$

for all $(x,\xi) \in \Omega \subset \mathbb{R}^{2n} \setminus \{0\}$. Therefore, $V(x,\xi)$ is a Lyapunov function for the *augmented* system (22) that guarantees asymptotic stability of the equilibrium point $(x,\xi) = (0,0)$. ⋄

Remark 5. If $P(x) = x^T\bar{P}$ is an *algebraic* $\bar{P}$ *solution*, then the choice $R = \bar{P}$ guarantees that $g(\xi) \equiv 0$ and that the condition (24) is satisfied for all $(x,\xi) \in \mathbb{R}^{2n} \setminus \{0\}$. Moreover, the *gain* k in (22) might be defined as a function of x, *i.e.* $k(x)$. ▲

Remark 6. The function $V(x,\xi)$ in (20) is a Lyapunov function for the *extended* system (22), hence there exists a $\mathcal{KL}$-class function β such that

$$\|[x(t),\, \xi(t)]\| \leq \beta(t, \|[x(0),\, \xi(0)]\|) \,,$$

for all $t \geq 0$ and for any $x(0)$, $\xi(0)$. Therefore

$$\|x(t)\| \leq \beta(t, \|[x(0),\, 0]\|) \triangleq \bar{\beta}(t, \|x(0)\|) \,,$$

proving asymptotic stability of the origin for the system (15). ▲

3.1 Invariant Submanifold

As in the linear case the proposed approach consists in exploiting the Lyapunov function for the *augmented* system (22) to construct a Lyapunov function for the system (15).

Proposition 5. Suppose that there exists a smooth mapping $h \in \mathbb{R}^{n\times 1}$ such that the manifold

$$\mathcal{M} = \{(x,\xi) \in \mathbb{R}^{2n} : \xi = h(x)\}$$

is an invariant submanifold with respect to the dynamics of the *augmented* system (22), *i.e.* such that

$$g(h(x))x - kRh(x) = \frac{\partial h}{\partial x}f(x) \,. \tag{25}$$

Then, the restriction of the function $V(x,\xi)$ as in (20) to the manifold $\mathcal{M}$, namely

$$V_{\mathcal{M}}(x) = P(h(x))x + \frac{1}{2}\|x - h(x)\|_R^2 \,, \tag{26}$$

yields a family of Lyapunov functions for the nonlinear system (15) parameterized by R and $k \geq \bar{k}$. ⋄

Note that, by (25), $\mathcal{M}$ is invariant under the flow of the system (22) and moreover the restriction of the flow of the *augmented* system (22) to the manifold $\mathcal{M}$ is a *copy* of the flow of the nonlinear system (15).

Remark 7. The equation (25) is a partial differential equation without constraints on the sign of the solution, *i.e.* the mapping $h(x)$ is not required to be positive definite. ▲

3.2 Approximate Solution of the Invariance pde

An explicit solution of the partial differential equation (25) may still be difficult to determine even without the sign constraint. Therefore, consider the following *algebraic* condition which allows to uniformly approximate, with an arbitrary degree of accuracy, the closed-form solution of the partial differential equation (25), which may be hard or impossible to find. Suppose that there exists a matrix $H \in \mathbb{R}^{n\times n}$ such that

$$H(x)f(x) + kRH(x)x - g(H(x)x)x = 0 \,. \tag{27}$$

Note that the solution of the condition (27) is parameterized by k, namely $H_k(x)$. Let now $\hat{h}(x) = H_k(x)x$ and consider the submanifold $\mathcal{M}_\eta \triangleq \{(x,\xi) \in \mathbb{R}^{2n} : \xi = \hat{h}(x)\}$.

Proposition 6. Suppose that the condition (27) is satisfied and that there exists a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|H_k(x)\| < \phi(\|x\|) \,, \tag{28}$$

uniformly in k. Then there exist a matrix $R \geq \bar{R}$ and $k \in (\bar{k}, \infty)$ such that the submanifold $\mathcal{M}_\eta$, on which the flow of the system (22) is a copy of the nonlinear system (15), is *almost*-invariant. [1] ⋄

[1] A submanifold $\mathcal{F}$ is said to be almost-invariant with respect to the system (15) if, given $\varepsilon > 0$, $x_0 \in \mathcal{F}$ yields $\text{dist}(x(t),\mathcal{F}) \leq \varepsilon$ for all $t \geq 0$, where $\text{dist}(x(t),\mathcal{F})$ denotes the distance of $x(t)$, solution of the system (15) with $x(0) = x_0$, from the submanifold $\mathcal{F}$.

Proposition 7. Suppose that the conditions (27) and (28) are satisfied. Then there exist a matrix $R \geq \bar{R}$ and $k \in (\bar{k}, \infty)$ such that the functions, parameterized by R and k

$$V_{\mathcal{M}_\eta}(x) = P(H_k(x)x)x + \frac{1}{2}\|x - H_k(x)x\|_R^2 \,, \qquad (29)$$

yield a family of Lyapunov functions for the nonlinear system (15). ◇

4. NUMERICAL EXAMPLE

Consider the nonlinear system described by equations of the form

$$\begin{aligned} \dot{x}_1 &= -x_1 \,, \\ \dot{x}_2 &= x_1^2 - x_2 \,, \end{aligned} \qquad (30)$$

with $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$. Note that the zero equilibrium of the system (30) is globally asymptotically stable and locally exponentially stable. A natural choice of a Lyapunov function for the linearization around the origin of the system (30) is provided by $V_l = \frac{1}{2}(x_1^2 + x_2^2)$, *i.e.* with $\bar{P} = I$ which is a solution of the equation (16) associated to the choice of the matrix $Q = I$. The quadratic function V_l is then employed to estimate the region of attraction, $\mathcal{R}_0$, of the zero equilibrium of the nonlinear system (30). Specifically, the estimate is given by the largest connected component, containing the origin of the state-space, of the level set of the considered Lyapunov function entirely contained in the set $\mathcal{N} \triangleq \{x \in \mathbb{R}^2 : \dot{V} < 0\}$. Figure 1

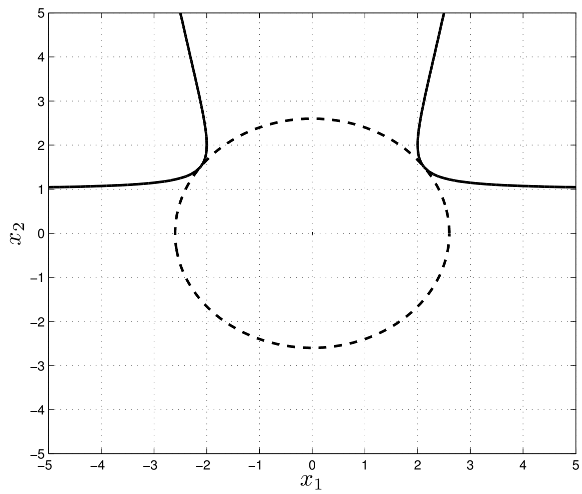

Fig. 1. Estimate of the basin of attraction of the zero equilibrium of system (30) given by the quadratic function V_l, dashed line.

displays the zero level line of the time derivative of V_l along the trajectories of system (30), solid line, together with the largest level set of V_l entirely contained in the set $\mathcal{N}$. Note that $\mathcal{N} \subset \mathbb{R}^2$ and consequently $\mathcal{R}_0 \subset \mathbb{R}^2$. It can be shown that a quadratic function $V_q = \frac{1}{2}x^T\bar{P}x$, with

$$\bar{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} ,$$

$p_2 \neq 0$, does not allow to obtain $\mathcal{N} = \mathbb{R}^2$. In fact, the time derivative of V_q along the trajectories of the system (30) yields

$$\dot{V}_q = -p_1x_1^2 - 2p_2x_1x_2 - p_3x_2^2 + p_2x_1^3 + p_3x_2x_1^2 \,,$$

which is equal to

$$\dot{V}_q\Big|_{x_2=0} = -x_1^2(p_1 - p_2x_1) \,,$$

if evaluated along $x_2 = 0$. Therefore, $\dot{V}_q > 0$ for $x_1 > \frac{p_1}{p_2}$.

Suppose now that the mapping $P(x) : \mathbb{R}^{1\times 2}$ defined as the gradient vector of the quadratic function V_l is actually an *algebraic* $\bar{P}$ *solution* for the nonlinear system (30), as detailed in the Definition 1. The proposed approach consists in constructing the family of Lyapunov functions defined in Proposition 5, exploiting the knowledge of the local strict Lyapunov function V_l, such that the robustness property is preserved and such that the region in which the time derivative is negative definite can be enlarged. To begin with note that the choice $R = \bar{P}$ guarantees that $g(\xi)$ is identically equal to zero for all $\xi(t) \in \mathbb{R}^2$ and that the condition (24) is trivially satisfied for all $(x, \xi) \in \mathbb{R}^4 \setminus \{0\}$. To construct the Lyapunov function V_d defined in Proposition 5 it is required to determine $h_1 : \mathbb{R}^2 \to \mathbb{R}$ and $h_2 : \mathbb{R}^2 \to \mathbb{R}$ such that the manifold

$$\{(x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4 : \xi_1 = h_1(x_1, x_2), \xi_2 = h_2(x_1, x_2)\}$$

is invariant for the dynamics of the *augmented* system (22). Note that the system of partial differential equations (25) reduces to two identical (decoupled) partial differential equations given by

$$-\frac{\partial h_i}{\partial x_1}(x_1, x_2)x_1 + \frac{\partial h_i}{\partial x_2}(x_1, x_2)(x_1^2 - x_2) + kh_i(x_1, x_2) = 0 \,, \qquad (31)$$

for $i = 1, 2$. The solutions $h_1(x_1, x_2)$ and $h_2(x_1, x_2)$ are defined as

$$h_1(x) = h_2(x) = L\left(\frac{x_2 + x_1^2}{x_1}\right)x_1^k \,,$$

$k \geq 1$, where $L(a)$ can be any function of a. In what follows let $L(a) = a$ and construct the family of Lyapunov functions

$$\begin{aligned} V_d(x) &= h(x)^T x + \frac{1}{2}\|x - h(x)\|^2 \\ &= \frac{1}{2}(x_1^2 + x_2^2) + (x_2 + x_1^2)^2 \left(x_1^{k-1}\right)^2 \,, \end{aligned} \qquad (32)$$

with $h(x) = [h_1(x),\, h_2(x)]^T$. For instance, letting $k = 1$, we obtain the function

$$V_d^1 = \frac{1}{2}(x_1^2 + x_2^2) + (x_2 + x_1^2)^2$$

the time derivative of which along the trajectories of the system (30), namely

$$\dot{V}_d^1 = -x_1^2 - 3x_2^2 - 3x_2x_1^2 - 2x_1^4 \,,$$

is negative definite for all $(x_1, x_2) \in \mathbb{R}^2$, hence $\mathcal{N} = \mathbb{R}^2$ and the estimate of the basin of attraction coincides with the entire plane. Finally, note that $\dot{V}_d^k$ is negative definite for all $(x_1, x_2) \in \mathbb{R}^2$ and for all $k \geq 1$. Interestingly, the pde (32) has a structure similar to the equation (18) but the solution obtained for (32) is clearly not positive definite.

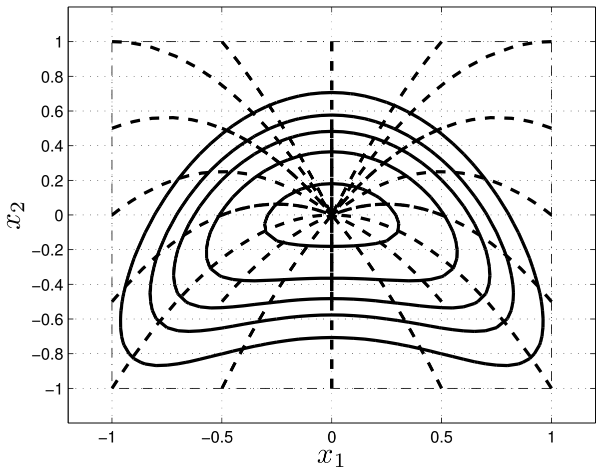

Fig. 2. Phase portraits of the trajectories of the system (30) together with the level lines of the lyapunov function V_d.

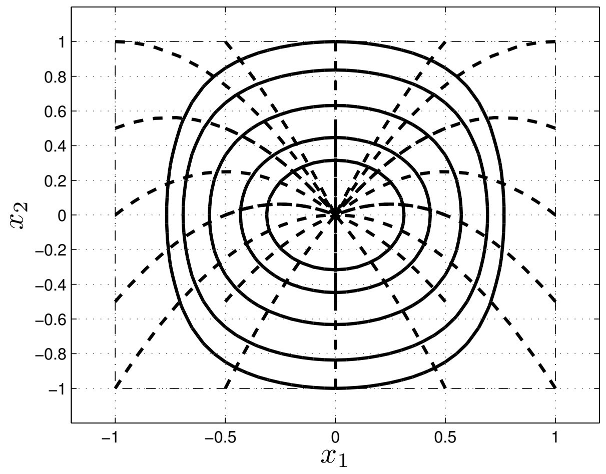

Fig. 3. Phase portraits of the trajectories of the system (30) together with the level lines of the lyapunov function V_{d_a}.

Even if the partial differential equation (31) admits a closed-form solution, suppose that we are interested in determining the approximate solution proposed in Proposition 7. Letting

$$H(x) = \begin{bmatrix} \bar{h}_1(x) & \bar{h}_2(x) \\ \bar{h}_3(x) & \bar{h}_4(x) \end{bmatrix},$$

and considering the condition (27), note that it reduces to two identical conditions on $\bar{h}_1$, $\bar{h}_2$ and $\bar{h}_3$, $\bar{h}_4$, namely

$$-x_1\bar{h}_1 + \bar{h}_2(x_1^2 - x_2) + k\bar{h}_1 x_1 + k\bar{h}_2 x_2 = 0\,, \qquad (33)$$

and the same condition obtained substituting $\bar{h}_1$ and $\bar{h}_2$ with $\bar{h}_3$ and $\bar{h}_4$, respectively. A possible solution of (33) is given by $\bar{h}_1(x) = -x_2 - x_1^2(k-1)^{-1}$ and $\bar{h}_2(x) = x_1$ and consequently, since the condition (28) holds, the manifold $\{(x,\xi) \in \mathbb{R}^4 : \xi_1 = \xi_2 = -x_1^3(k-1)^{-1}\}$ is *almost invariant* for $k \in (1,\infty)$. Moreover, it is interesting to note that the solution $\delta(x) \triangleq [\bar{h}_1(x), \bar{h}_2(x)]$ is not the gradient of any scalar function since the Jacobian $\nabla\delta(x)$ is not a symmetric matrix. Letting $k = 2$, the Lyapunov function presented in Proposition 7 is then defined as

$$V_{d_a} = \frac{1}{2}(x_1^2 + x_2^2) + x_1^6\,,$$

and the corresponding time derivative along the trajectories of the system (30) is $\dot{V}_{d_a} = x_2x_1^2 - x_1^2 - x_2^2 - 6x_1^6$ which is negative definite for all $(x_1, x_2) \in \mathbb{R}^2$.

5. CONCLUSIONS

A family of Lyapunov functions for linear and nonlinear systems can be obtained by means of a dynamic extension, *i.e.* considering the immersion of the system into an *augmented* system. A Lyapunov function, proving asymptotic stability of the origin of the extended state-space, can be *constructed*. In particular, techniques initially introduced for linear systems are then extended to the nonlinear case. Specifically, the family of Lyapunov functions is obtained considering the restriction of the extended Lyapunov function to an invariant submanifold - parameterized by the state variable x - on which the flow of the augmented system is a *copy* of the original system.

REFERENCES

Anstreicher, K.M. and Wright, M.H. (2000). A note on the augmented Hessian when the reduced Hessian is semidefinite. *SIAM J. on Optimization*, 11(1), 243–253.

Bacciotti, A. (1992). *Local Stabilizability of Nonlinear Control Systems*. World Scientific.

Bacciotti, A. and Rosier, L. (2005). *Lyapunov Functions and Stability in Control Theory*. Berlin: Springer-Verlag.

Chiang, H.D., Hirsch, M., and Wu, F. (1988). Stability regions of nonlinear autonomous dynamical systems. *IEEE Transactions on Automatic Control*, 33(1), 16–27.

Isidori, A. (1995). *Nonlinear Control Systems*. Springer-Verlag New York, Inc., Secaucus, NJ, USA.

Khalil, H.K. (2001). *Nonlinear Systems (3rd Edition)*. Prentice Hall.

La Salle, J.P. (1976). *The Stability of Dynamical Systems*. Philadelphia, PA : SIAM.

Lin, Y., Sontag, E.D., and Wang, Y. (1996). A smooth converse Lyapunov theorem for robust stability. *SIAM Journal on Control and Optimization*, 34, 124–160.

Lyapunov, A.M. (1992). *The general Problem of Stability of Motion*. London: Taylor and Francis.

Sassano, M. and Astolfi, A. (2010a). Dynamic solution of the Hamilton-Jacobi inequality in the L_2-disturbance attenuation problem. *Proc. of the 8th IFAC Symposium on Nonlinear Control Systems, Bologna*, 3081–3086.

Sassano, M. and Astolfi, A. (2010b). Dynamic solution of the HJB equation and the optimal control of nonlinear systems. *Proc. of the 49th Conference on Decision and Control , Atlanta, GA*.

Sontag, E. (1989). A 'universal' construction of Artstein's theorem on nonlinear stabilization. *Systems & Control Letters*, 13(2), 117–123.

Vannelli, A. and Vidyasagar, M. (1985). Maximal Lyapunov functions and domain of attraction for autonomous nonlinear systems. *Automatica*, 21(1), 69–80.