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On the controllability of continuous-time Markov jump linear systems ⋆

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Abstract. In this paper we study controllability for continuous-time, linear systems with finite Markov jump parameters. We consider a controllability notion that requires that the expected value of the controllability Gramian is positive definite. We introduce a set of controllability matrices for this class of systems and, based on certain invariance properties, we derive a rank test for controllability. Numerical examples are included.

Keywords: Controllability, stochastic control, hybrid systems modelling and control.

1. INTRODUCTION

The concepts of observability and controlability play an important role in the theory of dynamical systems, as they characterize the strucural relations between state, input and output of the system. Such structures have significant influence on the means for control and filtering, hence are of much relevance for applications. Controllability and observability have deserved a great deal of attention and there are a number of available results, see for instance Bittanti et al. (1984); Ji and Chizeck (1988); Costa et al. (2006); Astolfi and Praly (2006); Davis and Vinter (1984); Gray and Mesko. (1999); Petersen (2002). The situation is somewhat different regarding controllability of continuous-time Markov jump linear systems (MJLS). In fact, MJLS comprise a class of linear stochastic systems that are relevant for applications (see e.g. do Val and Basar (1999); Saridis (1983); Siqueira and Terra (2004)) and present a number of features that parallel the ones of linear deterministic systems, see for instance O. L. V. Costa and Marques (2000) for an operator theory approach for MJLS, Meskin and Khorasani (2010); Zhang et al. (2010); Geromel et al. (2009); Todorov and Fragoso (2008); de Souza and Coutinho (2006); Dragan and Morozan (2008) as an illustration of recent developments and the seminal papers Ji and Chizeck (1988, 1990), however, controllability has been taken into account in a pathwise sense only, see e.g. Ji and Chizeck (1990).

In this paper we consider the continuous-time MJLS defined in a fundamental probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$

$$\dot{x}(t) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t),\ \ x(0) = x_0,\ \theta(0) \sim \pi_0 \qquad (1)$$

where $t \geq 0$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^q$. As usual in MJLS, at each instant of time t, we have $A_{\theta(t)} = A_i$ whenever $\theta(t) = i$, where A_i is a matrix of appropriate dimensions taken from a known collection matrices $A = (A_1, \ldots, A_N)$ and similarly for B. We assume $\Theta = \{\theta(t), t \geq 0\}$ is a Markov chain and $\theta(t)$ takes values on $\mathscr{S} = \{1, 2, \ldots, N\}$. The transition rate matrix is denoted by $\Lambda = [\lambda_{ij}]$ (see details on Markov chains in Bhattacharya and Waymire (1990)), and the probability distribution of θ at time instant t is denoted by $\pi(t) = [P(\theta(t)) = 1, \cdots, P(\theta(t)) = N]$. The state of the system is $(x(t), \theta(t))$ and we refer to $x(t)$ as the continuous state. We consider the controllability Gramian for continuous, linear time-varying systems Davis (2002),

$$\Gamma(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau) B_{\theta(\tau)} B'_{\theta(\tau)} \Phi(t_0, \tau)' d\tau. \qquad (2)$$

where Φ stands for the state transition matrix, and we say that the system is controllable when $E\{\Gamma(0,t)|\theta(0) = i\} > 0, i \in \mathscr{S}$, $t \geq 0$. One interpretation is that, for non-controllable system, there exists initial conditions that can not be driven to zero over the time interval $[0,t]$ almost surely, see Remark 2.

We introduce a set of matrices $\mathscr{C}_i$, $i \in \mathscr{S}$ that play the role of controllability matrices in the sense that the system is controllable if and only if $\mathscr{C}_i$ are of full rank, $i \in \mathscr{S}$. The demonstration of this feature is, however, a somewhat intricate problem, and involves the following tasks. (I) Establish a connection between the null space of $\mathscr{C}_i$ with the null space of $\mathbb{C}_i(k)$, $k \geq 0$, where $\mathbb{C}_i(k)$ is such that $\mathscr{C}_i = [\mathbb{C}_i(0) \quad \mathbb{C}_i(1) \quad \ldots \quad \mathbb{C}_i(n^2N-1)]$; this involves showing that

$$v'\mathbb{C}_i v = 0 \quad \text{ is equivalent to } \quad \mathbb{C}_i v = 0,$$

which is not a trivial issue, since $\mathbb{C}_i(k)$ is symmetric but not necessarily positive semi-definite. We proceed by exploring dual relations with the property P1 in Narváez and Costa (2010) and associated results that are summarized in Section 2. (II)

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Link the quantity $v'\mathbb{C}_i(k)v$ with the cost of an auxiliary problem, allowing to recast a result in (Narváez and Costa (2010)) and to show that

$$B'_jA'^{p_m}_{j_m}\cdots A'^{p_2}_{j_2}A'^{p_1}_{j_1}v=0,, \tag{3}$$

for any sequence $j,j_1,\ldots,j_m,i$ such that $\lambda_{j,j_1}\lambda_{j_1,j_2}\cdots\lambda_{j_m,i}\neq 0$.
(III) Show that (3) is equivalent to $v'E\{\Gamma(0,t)|\theta(0)=i\}v=0$, $i\in\mathscr{S}$. The basic idea is to connect (3) with the continuous state trajectory of a version of the MJLS (1) (with transposes of matrices A_i and initial condition $x(0)=v$), yielding $B'_{\theta(\tau)}\Phi(0,\tau)'v=0$, $\tau\geq 0$, almost surely.

The paper is organized as follows. In Section 2 we present notation and several preliminary results, mainly from Narváez and Costa (2010), that are needed for the main results. Section 3 addresses the aforementioned tasks (I) and (II). The controllability matrices and the rank-test for controllability are presented in Section 4, and we finish with some concluding remarks.

2. NOTATION, DEFINITIONS AND PRELIMINARY RESULTS

We first introduce notation and definitions. Let $\mathscr{R}^{n,q}$ (respectively $\mathscr{R}^n$) be the linear space formed by all matrices of size $n\times q$ (respectively $n\times n$) and $\mathscr{R}^{r0}$ ($\mathscr{R}^{r+}$) the closed convex cone of symmetric semidefinite positive matrices $\{U\in\mathscr{R}^r : U=U'\geq 0\}$, (the open cone of symmetric definite positive matrices $\{U\in\mathscr{R}^r : U=U'>0\}$), U' denoting the transposed of U; $U\geq V (U>V)$ means that $U-V\in R^{r0}$ $(U-V\in R^{r+})$. For $U\in\mathscr{R}^{n,q}$, $\mathscr{N}(U)$ represents the null space of U. The operator $1_{\{.\}}$ is the indicator function (or characteristic function) and $\mathrm{tr}\{.\}$ denotes the trace. Let $\mathscr{M}^{r,n}$ be the linear space formed by a number N of matrices such that $\mathscr{M}^{r,n}=\{U=(U_1,\ldots,U_N) : U_i\in\mathscr{R}^{r,n}, i=1,\ldots,N\}$; also, $\mathscr{M}^r\equiv\mathscr{M}^{r,r}$. We denote by $\mathscr{M}^{r0}$ ($\mathscr{M}^{r+}$) the set $\mathscr{M}^r$ when it is formed by $U_i\in\mathscr{R}^{r0}$ $(U_i\in\mathscr{R}^{r+})$ for all $i=1,\ldots,N$. $\mathscr{M}^{r,n}$ defined as before with the inner product given by

$$\langle U,V\rangle=\sum_{j=1}^N \mathrm{tr}\{U'_jV_j\}$$

is a Hilbert space (Costa and Fragoso (2005)). Furthermore, define the norm $||U||=\langle U,I\rangle$ in $\mathscr{M}^{n0}$. Let the operators $\mathscr{L}:\mathscr{M}^n\longrightarrow\mathscr{M}^n$, $\mathscr{T}:\mathscr{M}^n\longrightarrow\mathscr{M}^n$, and $\mathscr{H}:\mathscr{M}^n\longrightarrow\mathscr{M}^n$ defined as:

$$\begin{aligned}\mathscr{L}_i(U)&=A'_iU_i+U_iA_i+\sum_{j=1}^N\lambda_{ij}U_j,\\ \mathscr{T}_i(U)&=A_iU_i+U_iA'_i+\sum_{j=1}^N\lambda_{ji}U_j,\\ \mathscr{H}_i(U)&=A_iU_i+U_iA'_i+\sum_{j=1}^N\lambda_{ij}U_j,\end{aligned} \tag{4}$$

for $i=1,\ldots,N$. Denote $\mathscr{L}^0(U)=U$ and, for $k\geq 1$, $\mathscr{L}^k(U)=\mathscr{L}(\mathscr{L}^{k-1}(U))$. This is similar for $\mathscr{T}$ and $\mathscr{H}$. We denote $\mathscr{T}_\Lambda$ and $\mathscr{H}_\Lambda$ to emphasize the dependence on the transition matrix Λ. Note that $\mathscr{H}_\Lambda=\mathscr{T}_{(\Lambda')}$.

Consider the MJLS defined as

$$\Psi:\ dx(t)=A_{\theta(t)}x(t)dt+B_{\theta(t)}d\zeta(t),\ x(0)=x_0,\ \theta(0)\sim\pi_0 \tag{5}$$

where $\zeta(t)\in\mathbb{R}^q$ is a Wiener process with incremental covariance operator Idt. The system Ψ allows us to establish a link with the observability notion for MJLS, and to employ the results in Narváez and Costa (2010), leading to the fact that $v'\mathbb{C}_iv=0$ is equivalent to $\mathbb{C}_iv=0$, which is an important element for the proof of the main result, as explained in Section 1. We shall assume that the initial distribution is invariant and is such that $\pi_i(0)>0$, which is restrictive for system Ψ but will not affect the controllability results. We define $X(t)\in\mathscr{M}^{n0}$ by

$$X_i(t)=E\{x(t)x(t)'1_{\{\theta(t)=i\}}|\mathscr{F}_0\},\quad t\geq 0, i\in\mathscr{S},$$

which satisfies the linear differential equations (Costa and Fragoso (2005))

$$\dot{X}_i(t)=\mathscr{T}_i(X(t))+R_i,\qquad X(0)=0, \tag{6}$$

for each $i\in\mathscr{S}$, where $R\in\mathscr{M}^{n0}$ is such that $R_i=B_iB'_i\pi_i$.

We shall employ the linear and invertible operator $\hat{\varphi}$ and φ (O. L. V. Costa and Marques (2000)) defined as follows. For $V\in\mathscr{R}^n$, let us identify the columns of V by

$$V=[v_1\vdots v_2\vdots\cdots\vdots v_n]\quad\text{and define}\quad \varphi(V)=\begin{bmatrix}v_1\\ v_2\\ \vdots\\ v_n\end{bmatrix},$$

and, for $U\in\mathscr{M}^n$, $\hat{\varphi}(U)=[\varphi(U_1)'\ \varphi(U_2)'\ \cdots\ \varphi(U_N)']'$. With this notation we obtain,

$$\hat{\varphi}(\mathscr{T}(U))=\mathscr{A}'\hat{\varphi}(U) \tag{7}$$

where $\mathscr{A}$ is defined as

$$\mathscr{A}=\begin{bmatrix}\hat{A}_1+\lambda_{11}I_{n^2} & \ldots & \lambda_{1N}I_{n^2}\\ \lambda_{21}I_{n^2} & \ddots & \lambda_{11}I_{n^2}\\ \vdots & \vdots & \vdots\\ \lambda_{N1}I_{n^2} & \ldots & \hat{A}_N+\lambda_{NN}I_{n^2}\end{bmatrix}$$

with $\hat{A}_i=I_n\otimes A'_i+A'_i\otimes I_n$, where $\otimes$ is the Kronecker product between matrices, see Costa and do Val (2002).

2.1 Observability matrices

In what follows we gather some results from (Narváez and Costa (2010)) that will be employed in Section 3 to obtain, via dual relations, some invariance results that are needed for the main results. These dualities take into account the relations between system Ψ and the MJLS defined as

$$\Upsilon:\ \begin{aligned}\dot{\vartheta}(t)&=A_{\theta(t)}\vartheta(t), &\vartheta(0)=\vartheta_0,\quad \theta(0)\sim\pi_0,\\ y(t)&=C_{\theta(t)}\vartheta(t), &E\{\vartheta_0\vartheta'_01_{\{\theta(0)=i\}}\}=Q_i,\ i\in\mathscr{S},\end{aligned}$$

where $C\in\mathscr{M}^{r,n}$ and $Q\in\mathscr{M}^{n0}$, and the associated functional

$$W^t(\vartheta,\theta)=E\left\{\int_0^t\vartheta(\tau)'C'_{\theta(\tau)}C_{\theta(\tau)}\vartheta(\tau)d\tau|\mathscr{F}_0\right\} \tag{8}$$

defined whenever $\vartheta(0)=\vartheta$ and $\theta(0)=\theta$.

Let us define $L(t)$, $t\geq 0$, by the linear differential equations:

$$\dot{L}_i(t)=\mathscr{L}_i(L(t))+C'_iC_i,\qquad L(0)=0, \tag{9}$$

for each $i\in\mathscr{S}$.

Definition 1. (Narváez and Costa (2010)). The set of matrices $\mathscr{O} \in \mathscr{M}^{n(n^2N),n}$, defined for each $i = 1,\dots,N$ by

$$\mathscr{O}_i = [O_i(0) \quad O_i(1) \quad \dots \quad O_i(n^2N-1)]' \tag{10}$$

is called the set of observability matrices of the system Υ, where $O_i(k)$ is given as

$$O_i(k) = \frac{d^{k+1}L_i}{dt^{k+1}}(0) \tag{11}$$

with $O_i(0) = C_i'C_i$ for each $i \in \mathscr{S}$.

Proposition 1. (Narváez and Costa (2010)). $\vartheta' O_i(k)\vartheta = 0$ for all $k = 0,\dots,n^2N-1$, if and only if $W^t(\vartheta, i) = 0$, $t \geq 0$.

Proposition 2. (Narváez and Costa (2010)). For each $\vartheta \in \mathbb{R}^n$ and sequence of Markov states $i, i_1, \dots, i_m$ such that $\vartheta' O_i(k)\vartheta = 0$, $k = 0,\dots,n^2N-1$, and $\lambda_{i,i_1}\lambda_{i_1,i_2}\cdots\lambda_{i_{m-1},i_m} \neq 0$, we have

$$C_{i_m}A_{i_{m-1}}^{p_1}A_{i_{m-2}}^{p_2}\cdots A_i^{p_m}\vartheta = 0, \tag{12}$$

for each $p_\ell \geq 0$, $\ell = 1,\dots,m$.

The following example is extracted from (Narváez and Costa (2010)), to illustrate Proposition 2.

Example 1. Consider the system Υ with

$$A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0.01 & 1 & 0 \\ 0.9 & 1 & 0.1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.99 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_3 = 0, \quad \Lambda = \begin{bmatrix} -3 & 1 & 2 \\ 2 & -5 & 3 \\ 0.5 & 0.5 & -1 \end{bmatrix}.$$

Note that $\mathscr{S} = \{1,2,3\}$, $n = 3$, $N = 3$. We consider initial condition $(x(0), \theta(0))$ with $x(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}'$ and $\theta(0) = 1$ (compatible with initial distribution $\mu_0 = [1 \;\; 0 \;\; 0]$), Consider the sequence 1,3,1,2,3,2, for which $\lambda_{1,3}\lambda_{3,1}\lambda_{1,2}\lambda_{2,3}\lambda_{3,2} \neq 0$. We have checked that $x(0)'O_1(k)x(0) = 0$, $k = 0,\dots,26$, thus satisfying the hypotheses of Proposition 2, and

$$C_2A_3^{p_1}A_2^{p_2}A_1^{p_3}A_3^{p_4}A_1^{p_5}x(0) = 0,$$

for each $p_\ell = 0,\dots,20$, $\ell = 1,\dots,5$. This confirms (12).

3. INVARIANCE PROPERTIES

In this section we establish a relation between the quantity $v'\mathbb{C}_i(k)v$ with the cost W defined in (8), allowing us to recast the result in Proposition 2 and to show that

$$B_{j_m}'A_{j_m}'^{p_m}\cdots A_{j_2}'^{p_2}A_{j_1}'^{p_1}v = 0,,$$

for any sequence $i, j_1, \dots, j_m$ such that $\lambda_{i,j_1}\lambda_{j_1,j_2}\cdots\lambda_{j_{m-1},j_m} \neq 0$. Based on this result, and considering $\mathbb{C}(k) \in \mathscr{M}^{n0}$ defined recursively for $k \geq 0$ as

$$\mathbb{C}(k+1) = \mathscr{H}(\mathbb{C}(k)) + BB'\pi, \qquad \mathbb{C}(0) = BB'\pi, \tag{13}$$

we show in Theorem 1 that

$$v'\mathbb{C}_i v = 0 \quad \text{is equivalent to} \quad \mathbb{C}_i v = 0.$$

We proceed as follows. From (6) and applying the operator $\hat{\varphi}$, we obtain:

$$\hat{\varphi}(X(t)) = \mathscr{A}'\hat{\varphi}(X(t)) + \hat{\varphi}(R)$$

and, if we define $z(t) = \hat{\varphi}(X(t))$ and $r = \hat{\varphi}(R)$ we have

$$z(t) = \int_0^t e^{\mathscr{A}'\tau} r d\tau. \tag{14}$$

Next, we present some auxiliary results involving the system Υ. Note that Υ is a version of the system Ψ without additive noise, hence we can obtain results similar to the ones in Section 2 with $R_i = 0$ in a straightforward manner. In particular, in analogy with (6) we have

$$\dot{\mathscr{V}}(t) = \mathscr{T}(\mathscr{V}(t)), \qquad \mathscr{V}(0) = Q, \tag{15}$$

Also, consider

$$W^t = \int_0^t \langle \mathscr{V}(t), C'C \rangle dt.$$

We denote W^t by $W^t_{Q,C}$ to emphasize the dependence on Q and C. Now, let us link systems Ψ and Υ by setting $Q = R$ (the conditional second moment of the initial condition of Υ is equal to the conditional second moment of the additive noise in Ψ).

Lemma 1. $W^t_{R,C} = \hat{\varphi}(C'C)'z(t)$ where $z(t)$ is as in (14).

Proof. Please see the Appendix A.

According to Lemma 1, to calculate the cost functional in the system Υ is equivalent to the inner product of the solution $z(t)$ associated to system Ψ with the vector $\hat{\varphi}(C'C)$. Next, for fixed $v \in \mathbb{R}^n$ and $i \in \mathscr{S}$, we define $C \in \mathscr{M}^{n,m}$ such that

$$C_j = \begin{cases} v', & \text{if} \quad j = i, \\ 0, & \text{if} \quad j \neq i. \end{cases} \tag{16}$$

For the following result, we define for each $i \in \mathscr{S}$, matrices $S_i(k)$, $k = 0,1,\dots,$ as

$$S_i(k) = \frac{d^{k+1}X_i}{dt^{k+1}}(0) = \sum_{\ell=0}^{k} \mathscr{T}_i^\ell(R). \tag{17}$$

Lemma 2. $v \in \mathbb{R}^n$ and $i \in \mathscr{S}$ satisfy $v'S_i(k)v = 0$, $k = 0,1,\dots,$ if and only if $W^t_{R,C} = 0$ for all $t \geq 0$, where C is as defined in (16).

Proof. Please see the appendix B.

Corollary 1. $v \in \mathbb{R}^n$ and $i \in \mathscr{S}$ satisfy $v'S_i(k)v = 0$, $k \geq 0$, if and only if

$$B_i'A_{j_1}'^{p_m}\cdots A_{j_{m-1}}'^{p_2}A_{j_m}'^{p_1}v = 0, \tag{18}$$

for any sequence $i, j_1, \dots, j_m$ such that$\lambda_{i,j_1}\lambda_{j_1,j_2}\cdots\lambda_{j_{m-1},j_m} \neq 0$.

Proof. Please see the Appendix C.

If we replace the transition rate matrix by its transpose Λ' in Corollary 1 and denote $S_{\Lambda'}(k)$ accordingly, it is simple to check that $\mathscr{H}_\Lambda = \mathscr{T}_{(\Lambda')}$ and $\mathbb{C}_\Lambda(k) = S_{\Lambda'}(k)$, leading to the next result.

Corollary 2. $v \in \mathbb{R}^n$ and $i \in \mathscr{S}$ satisfy $v'\mathbb{C}_i(k)v = 0$, $k \geq 0$, if and only if

$$B_{j_m}'A_{j_m}'^{p_m}\cdots A_{j_2}'^{p_2}A_{j_1}'^{p_1}v = 0, \tag{19}$$

for any sequence $i, j_1, \dots, j_m$ such that$\lambda_{i,j_1}\lambda_{j_1,j_2}\cdots\lambda_{j_{m-1},j_m} \neq 0$.

Remark 1. One interpretation for (19) is that there is no sequence in the form $A_{j_m}'^{p_m}\cdots A_{j_2}'^{p_2}A_{j_1}'^{p_1}$ that can take v out of the null space of B_{j_m}'. A_ℓ^p is related to the derivatives of order p of $x(t)$ whenever $\theta(t) = \ell$, which can be employed to show that state trajectories starting with $(x(0) = v, \theta(0) = i)$ remain almost surely in the null space of B'_{j_m}, constituting an invariance property for x similar (dual) to the ones obtained in Narváez and Costa (2010).

One can check, similarly to the case of observability (see Narváez and Costa (2010) for details), that $\mathbb{C}_i(k)$ can be written as a sum involving terms of the form

$$A_{j_1}^{q_1} A_{i_1}^{q_2} \cdots A_{i_{m-1}}^{q_m} B_{j_m} B'_{j_m} A'^{p_m}_{j_m} \cdots A'^{p_2}_{j_2} A'^{p_1}_{j_1}, \qquad (20)$$

where $j_1, \ldots, j_m$ and $j_1, i_1, \ldots, i_{m-1}, j_m$ are sequences of Markov states. Then, assuming $v'\mathbb{C}_i(k)v = 0$ for each $k = 0, \ldots, n^2N-1$, Corollary 2 produces $A_{j_1}^{q_1} \cdots A_{i_{m-1}}^{q_m} B_{j_m} B'_{j_m} A'^{p_m}_{j_m} \cdots A'^{p_1}_{j_1} v = 0$ whenever j_1 reaches j_m, yielding the next results.

Theorem 1. Consider system Ψ and let $i \in \mathscr{S}$. For each $k = 0, 1, \ldots, n^2N-1$ we have $\mathbb{C}_i(k)v = 0$ if and only if $v'\mathbb{C}_i(k)v = 0$.

From Corollary 2 and Theorem 1 we obtain the next result.

Corollary 3. $v \in \mathbb{R}^n$ and $i \in \mathscr{S}$ satisfy $\mathbb{C}_i(k)v = 0$ for $k = 0, 1, \ldots, n^2N-1$ if and only if $v \in \mathscr{N}(B'_{j_m} A'^{p_m}_{j_m} \cdots A'^{p_1}_{j_1})$.

4. CONTROLLABILITY

In this section we present a rank test for controllability. Note that the null space of $\mathbb{C}_i$ is not altered if we replace the forcing term $BB'\pi$ of (13) with BB' (recalling that we have assumed that $\pi_i > 0$); then, in what follows we consider $\mathbb{C}(k) \in \mathscr{M}^{n0}$ defined recursively for $k \geq 0$ as

$$\mathbb{C}(k+1) = \mathscr{H}(\mathbb{C}(k)) + BB', \qquad \mathbb{C}(0) = BB'. \qquad (21)$$

Definition 2. We define $\mathscr{C} \in \mathscr{M}^{n,n^2N}$ as

$$\mathscr{C}_i(A,B,\Lambda) = [\mathbb{C}_i(0) \quad \mathbb{C}_i(1) \quad \ldots \quad \mathbb{C}_i(n^2N-1)], \qquad (22)$$

where $\mathbb{C}(k) \in \mathscr{M}^{n0}$ is as defined in (21). We refer to $\mathscr{C}$ as the set of controllability matrices.

Definition 3. (Controllability). We say that the MJLS in (1) is controllable if $E\{\Gamma(0,t)|\theta(0) = i\} > 0$, $i \in \mathscr{S}$, $t \geq 0$.

Remark 2. If (v,i) is such that $v'E\{\Gamma(0,t)|\theta(0) = i\}v = 0$, then we have that $v'\Gamma(0,t)v = 0$ (a.s.). This yields that the null space of $\Gamma(0,t)$ is not trivial (a.s.), hence the range of the controllability Gramian does not include $\mathbb{R}^n$. This means that there exists a set of initial conditions x_0 such that

$$\int_{t_0}^{t_1} \Phi(t_0,\tau) B_{\theta(\tau)} u(\tau) d\tau \neq x_0 \quad \text{(a.s.)},$$

meaning that the associated continuous state trajectory can not be sent to zero over the finite time interval $[0,t]$ (a.s.).

Theorem 2. The MJLS in (1) is controllable if and only if $\text{rank}(\mathscr{C}_i(A,B,\Lambda)) = n$, $i \in \mathscr{S}$.

Proof. Assume that $i \in \mathscr{S}$ is such that $E\{\Gamma(0,t)|\theta(0) = i\}$ is not a positive definite matrix for some $t > 0$. Then, there exists $v \neq 0$ such that $v'E\{\Gamma(0,t)|\theta(0) = i\}v = 0$. This is equivalent to

$$E\Big\{\int_0^t v'\Phi(0,\tau) B_{\theta(\tau)} B'_{\theta(\tau)} \Phi(0,\tau)' v d\tau | \theta(0) = i\Big\} = 0. \qquad (23)$$

Note that, for a stochastic variable V such that $V \geq c$ and $E\{V\} = c$, we have that $V = c$ (a.s.) and *vice versa*; here, setting $V = v'\Phi(0,\tau)B_{\theta(\tau)}(v'\Phi(0,\tau)B_{\theta(\tau)})'$ for $0 < \tau \leq t$ (note that $V \geq 0$ and (23) means that $E\{V\} = 0$) we get that (23) is equivalent to

$$v'\Phi(0,\tau) B_{\theta(\tau)} B'_{\theta(\tau)} \Phi(0,\tau)' v = 0, \quad \text{(a.s.)}. \qquad (24)$$

Next, using the fact that $\Phi(0,\tau) = \Phi^{-1}(\tau,0)$ and denoting by $i, i_1, \ldots, i_q$ the Markov states visited by θ over the time interval $[0,\tau]$, and the respective jump time instants by $t_1, \ldots, t_q$, (24) can be written as

$$\begin{aligned} & v'\Phi(0,\tau) B_{\theta(\tau)} B'_{\theta(\tau)} \Phi(0,\tau)' v \\ &= v'\Phi^{-1}(\tau,0) B_{\theta(\tau)} B'_{\theta(\tau)} \Phi^{-1}(\tau,0)' v \\ &= \Big[B'_{\theta(\tau)} (\Phi^{-1}(\tau,t_q))' \cdots (\Phi^{-1}(t,0))' v\Big] \\ &\quad \times B'_{\theta(\tau)} (\Phi^{-1}(\tau,t_q))' \cdots (\Phi^{-1}(t,0))' v = 0, \quad \text{(a.s.)}. \end{aligned} \qquad (25)$$

The term on the right-hand side of (25) is equivalent to

$$B'_{i_q} e^{-A'_{i_q}(\tau - t_q)} \cdots e^{-A'_i(t_1 - 0)} v = 0, \quad \text{(a.s.)},$$

and, similarly to the proof in (Narváez and Costa, 2010, Corollary 3) we obtain in a recursive manner

$$B'_{i_q} A'^{p_q}_{i_q} \cdots A'^{p_1}_{i} v = 0,$$

for any exponents $p_1, \ldots, p_q$. Note that $i, i_1, \ldots, i_q$ is any sequence of Markov states satisfying $\lambda_{i,i_1} \cdots \lambda_{i_{q-1},i_q} \neq 0$, and from Corollary 3 we have that $\mathbb{C}_i(k)v = 0$, $k = 0, 1, \ldots, n^2N-1$. This is equivalent to $\text{rank}(\mathscr{C}_i(A,B,\Lambda)) < n$.

Example 2. Consider the continuous-time MJLS (1) with

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = 0, \quad B_2 = I_2,$$

and

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

We consider the initial condition $x_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}'$ and $\theta(0) = 1$. (21) yields

$$\mathbb{C}_1(k) = \begin{bmatrix} c_k & 0 \\ 0 & 0 \end{bmatrix},$$

$k = 0, 1, \ldots, 7$, where $c_0 = 1$, $c_1 = 3$, $c_2 = 7$, $c_3 = 15$, $c_4 = 31$, $c_5 = 63$, $c_6 = 127$, and $c_7 = 255$. This implies that $\text{rank}(\mathscr{C}_1) < 2$ and we conclude from the test of controllability appearing in Theorem 2 that the system is not controllable. Note that $x_0 \in \mathscr{N}(\mathbb{C}_1(k))$, for $k = 0, 1, \ldots, 7$. It is simple to check that the second component of $x(t)$ is given by $x_2(t) = e^t$ (a.s.) for any control $u(t)$, confirming that $x(0)$ can not be driven to zero over a time interval $[0,t]$ (see Remark 2).

Example 3. Consider the continuous-time MJLS (1) with

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = I_2, \quad A_2 = B_2 = 0,$$

$x_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}'$, $\theta(0) = 2$, and Λ as in Example 2. For this system we have checked that $E\{\Gamma(0,t)|\theta(0) = 2\} > 0$ for all $t > 0$, hence the system is controllable. One can check that the controllability matrices are of full rank, confirming Theorem 2. The control $u(t)$ was designed as follows; let T denote the jump time instant when the Markov chain enters $\theta(T) = 1$, and set $u(t) = 0$, $t < T$ and, for $t \geq T$ set $u(t)$ in such a manner that $x(t)$ is sent to zero at time instant $t_f = 0.9$ in a linear fashion.

We illustrate in Fig. 1 three trajectories $x(t,w_\ell)$, $\ell = 1,2,3$, corresponding to three different realizations w_1, w_2 and w_3 of the Markov chain. As we can see from the figure, $x(t,w_3)$ does not reach zero over the interval $[0,t_f]$ as $T(w_3) > t_f$.

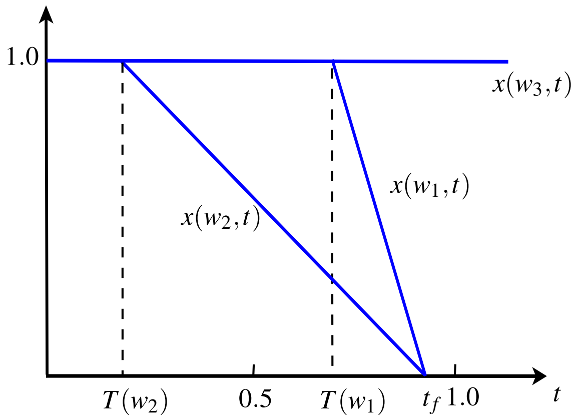

Figure 1. Three realizations of x for the system in Example 3.

5. CONCLUSIONS

We have introduced a controllability notion for continuous-time MJLS, and obtained the invariance property in Corollary 3, see also Remark 1. Based on this result, we have shown in a rather direct manner in Theorem 2 that the MJLS is controllable if and only if the set of controllability matrices are formed of full rank matrices $\mathscr{C}_i$, $i \in \mathscr{S}$, which brings the theory of (continuous-time) Markov jump linear systems to a more complete parallel with the one of deterministic linear systems. The numerical examples illustrate that, for non-controllable MJLS, there exist initial conditions that can not be sent to zero (via the control u) over a finite time interval $[0,t_f]$, almost surely; for controllable systems, $u(t)$ can be selected so $x(t_f) = 0$ with positive probability.

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Appendix A. PROOF OF LEMMA 1

We shall need the following notation. We write $R = \sum_{j=1}^{N} \tilde{R}_j$, where $\tilde{R}_j = (0,\ldots,0,R_j,0,\ldots,0)$ and each R_j is a component of R. We also write $R_j = \sum_{k=1}^{n} r_{jk} r'_{jk}$. Therefore, $\tilde{R}_j = \sum_{k=1}^{n} \tilde{R}_{jk}$ with $\tilde{R}_{jk} = (0,\ldots,0,r_{jk}r'_{jk},0,\ldots,0)$, noting that the non-trivial element is in the j-th position; finally,

$$R = \sum_{j=1}^{N}\sum_{k=1}^{n} \tilde{R}_{jk}. \tag{A.1}$$

Applying the operator $\hat{\varphi}$ in (15) and defining $w(t) = \hat{\varphi}(\mathscr{V}(t))$ we obtain $\dot{w}(t) = \mathscr{A}' w(t)$, whose solution is given by

$$w(t) = e^{\mathscr{A}' t} \sum_{j=1}^{N}\sum_{k=1}^{n} \hat{\varphi}(\tilde{R}_{jk}). \tag{A.2}$$

In addition, for each $j = 1,\ldots,N$ e $k = 1,\ldots,n$, we introduce $\tilde{X}_{jk}(t) \in \mathscr{M}^{n0}$ by

$$\dot{\tilde{X}}_{jk}(t) = \mathscr{T}(\tilde{X}_{jk}(t)), \qquad \tilde{X}_{jk}(0) = \tilde{R}_{jk}. \tag{A.3}$$

In fact, we can define systems $\Upsilon_{j,k}$ such that $\dot{\delta}_{jk}(t) = A_{\theta(t)}\delta_{jk}(t)$, $y(t) = C_{\theta(t)}\delta_{jk}(t)$, $\theta(0) \sim \pi_0$ and $\delta_{jk}(0)$ is such that $X_{jk}(0) = E\{\delta_{jk}(0)\delta_{jk}(0)'\} = \tilde{R}_{jk}$, $j = 1,\ldots,N$. Using again the operator $\hat{\varphi}$ and replacing $z_{jk}(t) = \hat{\varphi}(\tilde{X}_{jk}(t))$ we obtain $\dot{z}_{jk}(t) = \mathscr{A}' z_{jk}(t)$ whose solution is given as $z_{jk}(t) = e^{\mathscr{A}' t}\hat{\varphi}(\tilde{R}_{jk})$. Then, when calculating $W^t_{R,C}$ we have,

$$W^t_{R,C} = \hat{\varphi}(C'C)' \int_0^t e^{\mathscr{A}'\tau} r\, d\tau. \tag{A.4}$$

then, from (14) and (A.4) we have

$$W^t_{R,C} = \hat{\varphi}(C'C)' z(t), \tag{A.5}$$

completing the proof.

Appendix B. PROOF OF LEMMA 2

We need to prove the following proposition before.

Proposition 3. Consider the system Ψ and let $r = \hat{\varphi}(R)$. The following statements are equivalent:

(i) $v' X'_i(s) v = 0$ for some $s \geq 0$
(ii) $\hat{\varphi}(C'C)' \dfrac{d^m z}{dt^m}(0) = 0$ for $m = 1,2,\ldots$
(iii) $\hat{\varphi}(C'C)' \mathscr{A}'^{m-1} r = 0$ for $m = 1,2,\ldots$
(iv) $\hat{\varphi}(C'C)' z(t) = 0$ and $v \in \mathscr{N}(X_i(t))$, for all $t \geq 0$

Proof: $(i) \Rightarrow (ii)$. $X(t) \leq X(s)$, for $0 \leq t \leq s$, then:

$$z(t)'\hat{\varphi}(C'C) = \langle X(t), C'C\rangle \leq \langle X(s), C'C\rangle \leq v' X_i(s) v = 0.$$

Thus,

$$z(t)'\hat{\varphi}(C'C) = 0, \quad \text{for} \quad 0 \leq t \leq s$$

yielding,

$$\hat{\varphi}(C'C)' \frac{d^m z}{dt^m}(0) = 0 \quad \text{for} \quad m = 1,2,\ldots$$

$(ii) \Rightarrow (iii)$. $\hat{\varphi}(C'C)' \dfrac{d^m z}{dt^m}(0) = \mathscr{A}'^{m-1} r$, $m = 1,2,\ldots$, leads to the result.
$(iii) \Rightarrow (iv)$. We have $X(t) = S'S$ for some nonsingular matrix S. Then, for all $t \geq 0$ we have $X_i(t) v = 0$. This is equivalent to $S'Sv = 0$. One can check the following equivalences:

$$S'Sv = 0 \Leftrightarrow v'S'Sv = 0 \Leftrightarrow v'X_i(t)v = 0 \Leftrightarrow \langle X(t), C'C\rangle = 0,$$

and $\langle X(t), C'C\rangle = 0$ is equivalent to

$$\hat{\varphi}(C'C)' z(t) = 0, \ \forall t \geq 0,$$

hence, $\hat{\varphi}(C'C)' z(t) = 0$ is equivalent to $v \in \mathscr{N}(X_i(t))$, for all $t \geq 0$. Now, let us show that $\hat{\varphi}(C'C)' z(t) = 0 \quad \forall t \geq 0$. We have that

$$z(t) = \int_0^t e^{\mathscr{A}'\tau} r\, d\tau = \int_0^t \left(\sum_{k=0}^{\infty} \frac{(\mathscr{A}'\tau)^k}{k!} \right) r\, d\tau = \int_0^t \left(\sum_{k=0}^{\infty} \frac{\tau^k}{k!} \mathscr{A}'^k \right) r\, d\tau,$$

then (iii) leads to

$$\hat{\varphi}(C'C)' z(t) = \sum_{k=0}^{\infty} \left[\int_0^t \left(\frac{\tau^k}{k!} d\tau \right) \hat{\varphi}(C'C)' \mathscr{A}'^k r \right] = 0,$$

concluding the proof. $(iv) \Rightarrow (i)$. Trivial.

B.1 Proof of Lemma 2

Now, for $k = 0,1,\ldots$ we have the equivalences:

$$v' S_i(k) v = 0 \Leftrightarrow v' \frac{d^{k+1} X_i}{dt^{k+1}}(0) v = 0 \Leftrightarrow \hat{\varphi}(C'C)' \frac{d^{k+1} z}{dt^{k+1}}(0) = 0.$$

From the Proposition 3, the last equality is equivalent to $\hat{\varphi}(C'C)' z(t) = 0 \ \forall t \geq 0$ which, in turn, is equivalent to $W^t_{R,C} = 0 \ \forall t \geq 0$ (see (A.4)).

Appendix C. PROOF OF COROLLARY 1

Assume $v' S_i(k) v = 0$, $k \geq 0$, then Lemma 2 leads to $W^t_{R,C} = 0$. Now, we have that

$$W^t(x(0) = r_{ik}, \theta(0) = i) \leq W^t_{\tilde{R}_i,C} \leq W^t_{R,C} = 0,$$

where r_{ik} is as in (A.1), hence,

$$W^t(r_{ik}, i) = 0$$

for $k = 1,\ldots,n$. Then, if j_m is accesible from i, Proposition 2 yields that

$$C_{j_m} A^{p_1}_{j_{m-1}} A^{p_2}_{j_{m-2}} \cdots A^{p_m}_{j_1} r_{ik} = 0,$$

that is equivalent to

$$C_{j_m} A^{p_1}_{j_{m-1}} A^{p_2}_{j_{m-2}} \cdots A^{p_m}_{j_1} \sum_{k=1}^{n} r_{ik} r'_{ik} = 0,$$

and by definition of R_i and transposition we finally have

$$R_i A'^{p_m}_{j_1} \cdots A'^{p_2}_{j_{m-2}} A'^{p_1}_{j_{m-1}} C'_{j_m} = 0,$$

and hence,

$$B'_i A'^{p_m}_{j_1} \cdots A'^{p_2}_{j_{m-2}} A'^{p_1}_{j_{m-1}} C'_{j_m} = 0.$$

Finally, from (16) we have

$$B'_i A'^{p_m}_{j_1} \cdots A'^{p_2}_{j_{m-1}} A'^{p_1}_{j_m} v = 0$$

completing the proof.