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An algorithm for the long run average cost problem for linear systems with indirect observation of Markov jump parameters ⋆

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Abstract: This paper addresses the long run average cost control problem for linear systems with Markov jump parameters and indirect observation of the Markov state. One important issue that arises when employing some standard optimization methods is that the initialization often requires a stabilizing control, which may be difficult to obtain in the partial observation scenario. We propose an algorithm that handles this initialization issue by considering "auxiliary" problems with intermediate levels of observations, starting with complete observation of the Markov chain (allowing to use coupled algebraic Riccati equations to find stabilizing controls) and slowly shifting to the considered indirect observation problem. The effectiveness of the method is illustrated via a numerical example.

Keywords: Stochastic optimal control problems, Linear systems, Evolutionary algorithms.

1. INTRODUCTION

Linear systems with Markov jump parameters (LSMJP) constitute a well known class of system that can be successfully employed in applications featuring random failures, environmental changes and other phenomena that lead to random, abrupt changes of behaviour, which are modeled, in LSMJP, by an underlying Markov chain $\{\theta_k\}$. There exist numerous results for LSMJP, addressing notions of stability (see e.g. Fragoso and Costa (2005); Dragan and Morozan (2008); Todorov and Fragoso (2010)), stabilizability and detectability (e.g. Costa et al. (2005a)), optimal control with H_2, H_∞ norms and other criteria (e.g. Farias et al. (2001); Costa and Tuesta (2003); Todorov and Fragoso (2008); Geromel et al. (2009)), filtering (e.g. Lin et al. (2011); Souza and Fragoso (2003); Souza et al. (2006)), fault detection (e.g. Meskin and Khorasani (2010); Zhang et al. (2010)), and different scenarios of observation, including complete, partial and cluster observations (e.g. do Val et al. (2002)). We also mention the monograph Costa et al. (2005b) and references therein.

The long run average cost (LRAC) control problem for LSMJP arises when additive noise is present in the state, making usual linear quadratic costs to diverge. The problem has been studied in the complete observation scenario and it has been shown that the optimal control in linear state feedback form and can be obtained using coupled algebraic Riccati equations (CARE) (Costa et al., 2005b).

However, in scenarios where θ_k is not observed, or is indirectly observed, there are few results available in literature regarding the LRAC problem for LSMJP. In do Val and Başar (1999), a variational approach have been employed to derive an algorithm for the quadratic, finite horizon T cost, that may provide approximate solutions (assuming that the control in static linear state feedback form) to the LRAC by taking the limit $T \to \infty$. In spite of the fact that convergence of the finite horizon average cost to the LRAC is demonstrated in Vargas et al. (2006), convergence of the static state feedback as $T \to \infty$ is only a conjecture. The problem was also addressed in Silva and Costa (2009), where a genetic algorithm have been proposed with the drawback that the initial solution (initial gain) has to be mean square stabilizing. Obtaining a stabilizing gain can be a complex task in the partial observation scenario, making difficult to use the algorithm in Silva and Costa (2009), as well as other optimization techniques that could be applied or adapted to the LRAC problem. In fact, in Silva and Costa (2009) the gains obtained via the CARE from the complete observation scenario are employed as "guesses" for stabilizing initial solutions, but there is no guarantee that these are good guesses, as the LRAC problems in partial observation and complete observation contexts are quite different.

In this paper we propose an algorithm that makes the aforementioned initialization issue easier to handle and to employ the CARE to initialize the algorithm in a meaningful manner. The basic idea is to consider intermediate problems, with intermediate "observation levels". Actually, we consider indirect observation of the Markov chain via a variable r_k, taking values in the finite set $\mathcal{S} = \{1, \dots, S\}$ and satisfying $P(r_k = \theta_k | \theta_k) = c$ (with the interpretation that the observation of θ may be erroneous, that is, $P(r_k \neq \theta_k | \theta_k) = 1 - c$). We assume that control is in the linear state feedback form $u_k = L_{r_k} x_k$, where u and x stand

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for the control and the so-called continuous state variable, respectively; we say that c is the *level of observation* and that a gain is *stabilizing for the level of observation* c when u_k stabilizes the system. The algorithm starts with $c = c_0 = 1$ (which is equivalent to the complete observation case) and solution L_i, $i \in \mathcal{S}$, given by the CARE, which is meaningful in the sense that it is optimal for the LRAC problem with $c = c_0 = 1$ and provides a good initial solution for $c_1 < c_0$, $c_1 \approx c_0$. Then, at each step i, the algorithm sets $c = c_i < c_{i-1}$ and calculates (using, in this paper, the genetic algorithm of Silva and Costa (2009)) a new MS-stabilizing gain L_i, $i \in \mathcal{S}$. We show that there always exist $c_i < c_{i-1}$ such that the gain obtained in the previous step $i-1$ is stabilizing for the level of observation c_i, ensuring that the method finds new values for c_i until it reaches the desired observation level c_f (in particular, in the scenario of non-observed θ one can set $c_\mathrm{f} = 1/S$, as we explain in Section 3), meaning that the algorithm have obtained a solution for the LRAC problem, or $c_i - c_{i-1}$ satisfies a stop criterion (algorithm failed).

The paper is organized as follows. Section 2 presents the LSMJP, the LRAC and some preliminary results from the context of complete state observation. The structure of observation and the considered class of control are considered in Section 3, as well as a formulation for calculating the LRAC (that can be considered as a relatively simple extension of available results). In Section 4 we propose an algorithm for the LRAC problem, and in Section 5 we present a numerical example illustrating the proposed algorithm.

2. NOTATION AND PROBLEM STATEMENT

Let $\mathcal{M}^{r,s}$ denote the linear space formed by a number S of $r \times s$-dimensional matrices U_i, $i = 1, \ldots, S$, that is, $\mathcal{M}^{r,s} = \{U = (U_1, \ldots, U_S)\}$. Also, $\mathcal{M}^r \equiv \mathcal{M}^{r,r}$. For $U, V \in \mathcal{M}^r$, $U \geq V$ signifies that $U_i - V_i \geq 0$ for each $i \in \mathcal{S}$, and similarly for other mathematical relations. Consider $tr\{\cdot\}$ as the trace operator,

$$\langle U, V \rangle = \sum_{i=1}^{S} tr\{U_i' V_i\}.$$

We consider the discrete-time LSMJP

$$x_{k+1} = A_{\theta_k} x_k + B_{\theta_k} u_k + G_{\theta_k} w_k, \quad k \geq 0 \tag{1}$$

defined in an appropriate probability space, with initial condition $x_0 \in \mathscr{R}^{n,m}$. $x \in \mathscr{R}^{n,m}$ is the state, $u \in \mathscr{R}^m$ is the input and $w_k \in \mathscr{R}^{q,m}$ forms a zero-mean, Gaussian iid random process, satisfying $E\{w_k w_k'\} = \Sigma$. We assume that the sets of matrices $A = (A_1, \ldots, A_S) \in \mathcal{M}^n$, $B = (B_1, \ldots, B_S) \in \mathcal{M}^{n,m}$ and $G = (G_1, \ldots, G_S) \in \mathcal{M}^{n,q}$ are known, and $A_{\theta_k} = A_i$ whenever $\theta_k = i$, and similarly for B_{θ_k} and G_{θ_k}. $\theta \in \mathcal{S} = \{1, \ldots, S\}$ is the state of a finite, homogeneous, ergodic Markov chain having transition probabilities

$$P(\theta_{k+1} = j | \theta_k = i) = p_{ij}, \quad i, j \in \mathcal{S}.$$

We denote $\mu_k = [P(\theta_k = 1), \ldots, P(\theta_k = S)]$, $k \geq 0$, and assume that the initial distribution μ_0 and the transition probability matrix $\mathbb{P} = [p_{ij}]$ are known.

Consider the cost function

$$J^T = \sum_{k=0}^{T} x_k' C_{\theta_k} x_k + u_k' D_{\theta_k} u_k,$$

with $C \in \mathcal{M}^n, C = C' \geq 0$ (meaning that $C_i = C_i' \geq 0$, as stated above) and $D \in \mathcal{M}^m, D = D' > 0$. Since x_k forms a stochastic process, for optimization purposes we consider the expected value of J,

$$\mathcal{J}^T = \mathscr{E}\{J^T\},$$

whenever $T < \infty$, and the LRAC

$$\mathcal{Y} = \lim_{T \to \infty} \sup_{T} \frac{\mathcal{J}^T}{T}.$$

Following the notation of Costa et al. (2005a) we define, for $U \in \mathcal{M}^n$, $V \in \mathcal{M}^n$, the linear operator $\mathcal{T}_U : \mathcal{M}^n \to \mathcal{M}^n$ by

$$\mathcal{T}_{U,i}(V) := \sum_{j=1}^{S} p_{ji} U_j V_j U_j', \quad i \in \mathcal{S}, \tag{2}$$

and we define for convenience $\mathcal{T}^0(V) = V$, and for $t \geq 1$, $\mathcal{T}^t(V) = \mathcal{T}(\mathcal{T}^{t-1}(V))$. We consider the set of conditional second moments $X(k) \in \mathcal{M}^n$ as

$$X_i(k) = E\{x_k x_k' \mathbb{1}_{\{\theta_k = i\}}\}, \quad i \in \mathcal{S}$$

where $\mathbb{1}_\mathcal{C}$ represents the Dirac function of the set $\mathcal{C}$ and we define $\varphi \in \mathcal{M}^n$ as

$$\varphi_i(k) = \sum_{j \in \mathcal{S}} p_{ji} \mu_k(j) G_j \Sigma G_j', \quad \forall i \in \mathcal{S}.$$

The following proposition is an adaptation of the results in Costa et al. (2005b).

Proposition 1. Let $X \in \mathcal{M}^n$ be defined by $X_i = x_0 x_0' \mu_0(i)$, and assume linear state feedback in the form $u_k = K_{\theta_k} x_k$ for some $K \in \mathcal{M}^{m,n}$. Then,

$$X(k+1) = \mathcal{T}_{A+BK}(X(k)) + \varphi(k). \tag{3}$$

Moreover,

$$\mathcal{J}^T = \sum_{k=0}^{T} \Big\langle X(k), C + K'DK \Big\rangle. \tag{4}$$

We say that the control is stabilizing (or the gain K is stabilizing, when appropriate) if, for each initial condition x_0 and μ_0, $X(k)$ is bounded (i.e., exists M such that $X_i(k) \leq M$). In Vargas et al. (2006), it is shown that $|\mathcal{J}^T - T\langle X, C + K'DK\rangle| \leq \alpha \|\mu_0 - \mu_\infty\| + \beta \|X(0) - X\|$, for some non-negative scalars α, β, where $X \in \mathcal{M}^n$ is such that $X = \mathcal{T}_{A+BK}(X) + \varphi(\infty)$, provided that K is stabilizing. Multiplying the above inequality by T^{-1} and taking the limit $T \to \infty$, yields the next result Vargas et al. (2006).

Proposition 2. Consider the LSMJP and assume that the linear state feedback control $u_k = K_{\theta_k} x_k$, for some $K \in \mathcal{M}^{m,n}$, is stabilizing. Then,

$$\mathcal{Y} = \Big\langle X, C + K'DK \Big\rangle \tag{5}$$

where $X \in \mathcal{M}^n$ satisfies

$$X = \mathcal{T}_{A+BK}(X) + \varphi(\infty). \tag{6}$$

3. INDIRECT OBSERVATION OF θ

Assume that θ_k is observed via a variable r_k having conditional distribution

$$P(r_k = \theta_k | \theta_k) = c,$$
$$P(r_k = j | \theta_k) = \frac{1-c}{S-1}, \text{ when } j \neq \theta_k.$$

Note that for $c = 1$, we have $\theta_k = r_k$ almost surely corresponding to the scenario of complete observation and

$c = 1/S$ retrieves the scenario of non-observed θ, as Bayes's rule yields

$$P(\theta = i | r = \tilde{r}) = \frac{P(r = \tilde{r}|\theta = i)P(\theta = i)}{\sum_{j\in\mathcal{S}} P(r = \tilde{r}|\theta = j)P(\theta = j)} = \frac{\frac{1}{S}\mu(i)}{\sum_{j\in\mathcal{S}} \frac{1}{S}\mu_j} = \mu(i) = P(\theta = i),$$

that is, the distribution of θ does not depend on r when $c = 1/S$. Intermediate values of c ($1/S < c < 1$) have an interpretation of different "levels of observation" for θ, and we refer to c as the observation level.

In accordance with this structure of observation, we consider a linear state feedback control in the form,

$$u_k = L_{r_k} x_k, \tag{7}$$

where $L \in \mathcal{M}^{m,n}$. We say that L is stabilizing with the observation level c when the control u given in (7) stabilizes the system (1) and the observation level is c.

Remark 3. One can employ Proposition 2 to calculate the LRAC by considering a modified, augmented Markov chain, having (θ_k, r_k) as the state and transition probabilities $P(\theta_{k+1} = \ell, r_{k+1} = m | \theta_k = i, r_k = j) = p_{i\ell}(c\mathbb{1}_{\{\ell=m\}} + (1-c)(S-1)^{-1}\mathbb{1}_{\{\ell\neq m\}})$, and "cluster" gains in the form $K_{(\theta_k,r_k)} = L_{r_k}$. This leads to a large set of conditional second moments $X_{(i,j)}$, $i, j = 1, \dots, S$, making considerably more difficult to obtain solutions for the system of linear equations (6).

In order to circumvent the difficulty pointed out in Remark 3, in what follows we present a formulation that allows to use the original Markov chain, however with different operators that take into account the indirect observation of θ and the control structure in (7); the results can be considered as a relatively simple extension, and some proofs are omitted.

Consider the closed loop matrix

$$A_{j,\ell} = A_j + B_j L_\ell, \quad j, \ell \in \mathcal{S}.$$

Let $V \in \mathcal{M}^n$; we define the operator $\mathcal{L}_L : \mathcal{M}^n \to \mathcal{M}^n$ by

$$\mathcal{L}_{L,i}(V) := \sum_{\ell\in\mathcal{S}} \sum_{j\in\mathcal{S}} p_{ji} A_{j\ell} V_j A'_{j\ell} \Big(c\, \mathbb{1}_{\{j=\ell\}} + \frac{1-c}{S-1}\, \mathbb{1}_{\{j\neq\ell\}} \Big), \tag{8}$$

and $\bar{\varphi} \in \mathcal{M}^n$ by

$$\bar{\varphi}_i := \sum_{\ell\in\mathcal{S}} \sum_{j\in\mathcal{S}} p_{ji} \mu(j) G_j \Sigma G'_j \Big(c\, \mathbb{1}_{\{j=\ell\}} + \frac{1-c}{S-1}\, \mathbb{1}_{\{j\neq\ell\}} \Big). \tag{9}$$

The next Lemma is a counterpart of Proposition 1, providing a recursive formula for $X(k)$ in terms of the operator in (8) and the forcing term in (9). This is an important formula that allows us to obtain a condition for stability in terms of eigenvalues of a certain matrix $\mathcal{A}$, derived in what follows.

Lemma 4. Consider the LSMJP (1) with indirect observation of the Markov state via the variable r and control in the form (7). Then,

$$X_i(k+1) = \mathcal{L}_{L,i}(X(k)) + \bar{\varphi}_i(k). \tag{10}$$

Proof. Please see the Appendix.

An useful formulation for calculating the LRAC in the context of indirect observation of θ that parallels the one in Proposition 2 is as follows.

Lemma 5. Consider the LSMJP (1) with indirect observation of the Markov state via the variable r and control in the form (7) then,

$$\mathcal{Y} = \Big\langle X, C + L'DL \Big\rangle \tag{11}$$

where $X \in \mathcal{M}^n$ satisfies

$$X = \mathcal{L}_{A+BL}(X) + \bar{\varphi}(\infty). \tag{12}$$

A vector form of (10) can be obtained in a straightforward manner using the Kronecker tensor product (denoted by kron as usual). For $V \in \mathcal{M}^n$, let us identify the columns of V by

$$V = \big[\, v_1 \,\vdots\, v_2 \,\vdots\, \cdots \,\vdots\, v_n \,\big] \text{ and define } \varphi(V) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix},$$

and, for $U \in \mathcal{M}^n$, $\text{vec}(U) = [\varphi(U_1)'\ \varphi(U_2)' \dots \varphi(U_S)']'$.

Lemma 6. Let $X(k) \in \mathcal{M}^n$ defined as in (10), then

$$\text{vec}(X(k+1)) = \mathcal{A}\ \text{vec}(X(k)) + \text{vec}(\bar{\varphi})$$

where $\mathcal{A} \in \mathcal{R}^{nS,1}$ is defined by $\mathcal{A} = [a_{\ell j}]$ with

$$a_{\ell j} = \begin{cases} c p_{\ell i} \text{kron}(A_{\ell j}, A_{\ell j}), & \text{if} \quad \ell = j \\ \frac{1-c}{S-1} p_{\ell i} \text{kron}(A_{\ell j}, A_{\ell j}), & \text{otherwise.} \end{cases}$$

Proof. Consider $X_i(k+1) = \mathcal{L}_{L,i}(X(k)) + \bar{\varphi}_i(k)$. Assume $\bar{\varphi}(k) = 0$ (the proof for general $\bar{\varphi}$ is similar and is not presented). We have

$$X_i(k+1) = \sum_{\ell\in\mathcal{S}} \sum_{j\in\mathcal{S}} p_{ji} \Big(A_{j\ell} X_j(k) A'_{j\ell} \Big) \Big(c\, \mathbb{1}_{\{j=\ell\}} + \frac{1-c}{S-1}\, \mathbb{1}_{\{j\neq\ell\}} \Big).$$

Applying the Kronecker tensor product on both sides we obtain

$$\text{vec}(X_i(k+1)) = \sum_{\ell\in\mathcal{S}} \sum_{j\in\mathcal{S}} p_{ji} \Big(\text{kron}(A_{j\ell}, A_{j\ell})\ \text{vec}(X_j(k)) \Big) \Big(c\, \mathbb{1}_{\{j=\ell\}} + \frac{1-c}{S-1}\, \mathbb{1}_{\{j\neq\ell\}} \Big).$$

This yields

$$\text{vec}(X_i(k+1)) = \sum_{j\in\mathcal{S}} a_{1j}\ \text{vec}(X_1(k)) + \dots + \sum_{j\in\mathcal{S}} a_{Sj}\ \text{vec}(X_S(k)),$$

which leads to the result.

The next result is an adaptation of Costa et al. (2005b), which can be derived by showing that, for each forcing term $\bar{\varphi}$, the solution $X(k)$ of (10) is bounded if and only if the spectral radius of $\mathcal{L}_{L,i}$ is smaller than one, which is equivalent to require that the eigenvalues of $\mathcal{A}$ lie in the open unit disk.

Lemma 7. Consider the LSMJP (1) with indirect observation of the Markov state via the variable r and controls as in (7). The set of gains $L \in \mathcal{M}^{m,n}$ is stabilizing if and only if the eigenvalues of $\mathcal{A}$ lie in the open unit disk.

Lemma 8. If L is stabilizing for c, then there exists $\delta > 0$ such that K is stabilizing for $c - \delta$

Proof. It is a well know fact that the system is stable with control $u_k = L_{r_k} x_k$ if and only if the spectral radius of $\mathcal{L}_{L,c}$, $r_{\mathcal{L}_{L,c}}$ satisfies $r_{\mathcal{L}_{L,c}} < 1$. This is equivalent to require that the eigenvalues of $A_{L,c}$ lie in the open unit disk. The eigenvalues of $A_{L,c}$ depend continuously on c, in suck a manner that there exist $\delta > 0$ suck that the eigenvalues of $A_{L,c}$ also lie in the unit disk. This implies that L is stabilizing for $c - \delta$.

4. ALGORITHM FOR LRAC

In this section we are interested in the LRAC with indirect observation of θ, with observation level $c = c_{\mathrm{f}}$, as presented in Section 3. We consider the Algorithm 1, which generates a sequence of intermediate levels of observation c_ℓ, $\ell = 0, 1, \ldots$, with $c_0 = 1$ and $c_{\mathrm{f}} \leq c_\ell \leq c_{\ell-1}$. The algorithm considers the associated problems in the variables $X \in \mathcal{M}^n$ and $L \in \mathcal{M}^{m,n}$ (see Lemma 5),

$$\mathcal{P}_{c_\ell} : min_L \langle X, C + L'DL \rangle \\ s.t. \begin{array}{l} X = \mathcal{L}_{L,c_\ell}(X) + \varphi(\infty) \\ L \text{ is stabilizing} \end{array} \tag{13}$$

where we denote the operator $\mathcal{L}$ defined in (8) by $\mathcal{L}_{L,c}$ to emphasize the dependence on the set of gains L and the observation level c. The method obtains a sequence of gains $L^\ell \in \mathcal{M}^{m,n}$, $\ell \geq 0$, such that each L^ℓ is stabilizing considering the observation level c_ℓ.

In order to formalize the algorithm, we denote by $\mathcal{A}_{L,c}$ the matrix $\mathcal{A}$ (as in Lemma 6) when it is associated with the set of gains L and observation level c. Stabilizability of L can be checked via the eigenvalues of $\mathcal{A}_{L,c}$, see Lemma 7. We shall also need the following CARE in the variable $P \in \mathcal{M}^n$, $P_i = P_i' \geq 0$,

$$P_i = A_i'\mathcal{E}_i A_i - (A_i'\mathcal{E}_i A_i B_i)(D_i + B_i'\mathcal{E}_i B_i)^{-1}(B_i'\mathcal{E}_i A_i) + C_i \tag{14}$$

where $\mathcal{E}_i = \sum_{j=1}^{S} p_{ij} P_j$ is the coupling term. We can obtain a solution to the CARE via the Algorithm 3 (see Costa and do Val (2002); Costa et al. (2006)) presented in Appendix B for ease of reference.

Algorithm 1

1) Set $c_0 = 1$ and choose a positive scalar quantity ε to serve as a stop criterion.
   If there is no stabilizing solution for the CARE (14), stop – there is no stabilizing solution to $\mathcal{P}_{c_{\mathrm{f}}}$. Else, find the associated set of gains $L \in M^{m,n}$. Set $i = 1$ and $L^0 = L$.
2) Set s_i, $0 < s_i \leq c_{i-1} - c_{\mathrm{f}}$ and $c_i = c_{i-1} - s_i$.
3) If L^{i-1} is stabilizing with the observation level c_i, then find a solution $L^i \in M^{m,n}$ for $\mathcal{P}_{c_i}$.
   Else, set $s_i = s_i/2$, $c_i = c_{i-1} - s_i$ and return to (3).
4) If $c_i = c_{\mathrm{f}}$ or $c_i - c_{i-1} < \varepsilon$, stop. Else, set $i = i+1$, and return to (2).

In order to find a solution L^i for $\mathcal{P}_{c_i}$ at step (3) of Algorithm 1, we use the algorithm developed in Silva and Costa (2009). In what follows we give an overview of that algorithm. We use the set of gains $L^{i-1} = (L_1^{i-1}, \ldots, L_S^{i-1})$ to obtain the initial population as follows:

$$L(\ell) = (L_1^{i-1} + R_1, \ldots, L_S^{i-1} + R_S), \quad \ell = 1, \ldots, q \tag{15}$$

where q is the size of the initial population and R_i, $i \in \mathcal{S}$, are random matrices (typically, the variance of each element of R is small when compared with the modulus of the corresponding element of L_i). The Algorithm 2 describes the Genetic Algorithm for $\mathcal{P}_{c_i}$.

Algorithm 2

1) Set the GA and system parameters and select a stop criterion.
2) Initialize the population $L(\ell)$ as in (15).
3) Perform genetic and evolutionary operators to obtain a new population and the gains $L(\ell)$.
4) For each $L(\ell)$, calculate $\mathcal{Y}_{L(\ell)}$ via (11).
5) If the stop criterion is satisfied, set $L^i = \arg\min_\ell \mathcal{Y}_{L(\ell)}$, otherwise return to (3).

5. NUMERICAL EXAMPLE

The numerical example considered in this paper is an adaptation from (Oliveira et al., 2009, Ex.1). Consider system (1) with

$$A_1 = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}, G_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbb{P} = \begin{bmatrix} 0.6 & 0.4 \\ 1 & 0 \end{bmatrix}, x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$D_1 = D_2 = 1$ and $\mu_k = \begin{bmatrix} 0.514 & 0.486 \end{bmatrix}$. We consider the LRAC with no observation of the Markov state, that is, we set $c_{\mathrm{f}} = 1/S = 1/2$. For the initialization of the Algorithm 1, we employ the CARE (14) (solved using the method presented in the Appendix B), which yields

$$L_1^0 \approx [-1.002 \quad -1.000]; \; L_2^0 = [0 \;\; 0].$$

The Algorithm 1 has produced the sequence $c_0 = 1, c_1 = 0.9, \ldots, c_4 = 0.6, c_5 = 0.5$. The Fig. 1 shows the behaviour of $||X_{L^i}(k)||$ calculated via Lemma 4, for the gains L^1, L^3 and L^5, suggesting that the gains are stabilizing (X is bounded).

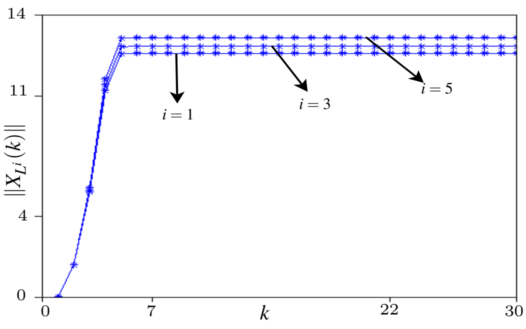

Fig. 1. Norm of X_{L^i} for the gains L^1, L^3 and L^5.

The cost associated with each i, $\mathcal{Y}_{L^i,c^i}$ are show in Fig. 2. We also illustrate in Fig. 3 the cost $\mathcal{Y}_{L^i,c}$ incurred by each gain L^i in the scenario of no observation of θ, that is, with fixed $c = 0.5$; as we can see from the figure, the algorithm generates a sequence of gains whose costs are monotonically decreasing. For comparison purposes, we have employed the variational method of do Val and Başar (1999) presented in Appendix C, which yields

$$K \approx [-1.121 \quad -0.953] \text{ and } \mathcal{Y}_{K,c=0.5} \approx 25.531.$$

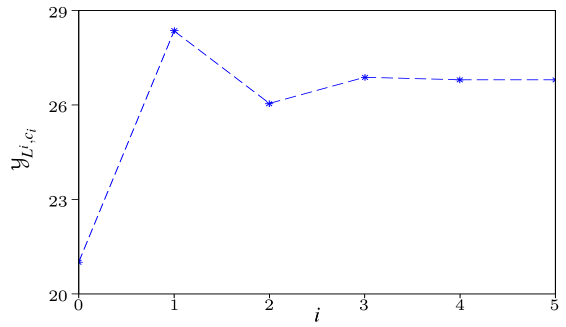

Fig. 2. Values of the LRAC for different levels of observation $c_0,\dots,c_5$ and the associated gains $L^0,\dots,L^5$.

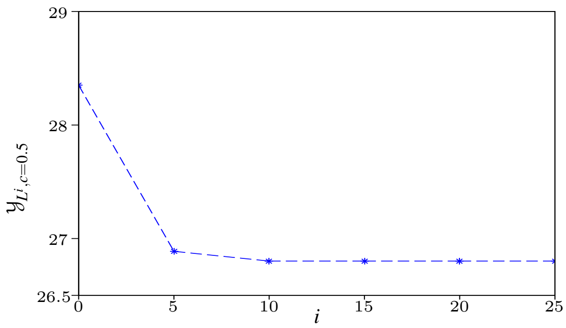

Fig. 3. Values of the LRAC in the non-observed scenario for different gains $L^0,\dots,L^5$.

The Algorithm 1 provides a similar solution

$$L_1^5 \approx [-1.110 \quad -0.975], L_2^5 = 0 \text{ and } \mathcal{Y}_{L,c=0.5} \approx 26.050.$$

6. CONCLUSIONS

Considering the LRAC control problem for linear systems with Markov jump parameters and indirect observation of the Markov state, we have proposed the Algorithm 1 that takes into account a sequence of subproblems $\mathcal{P}_{c_i}$ with intermediate levels of observation c_i. The algorithm makes easier to handle $\mathcal{P}_{c_i}$ as the set of gains L^{i-1} obtained at step $i-1$ (suboptimal solutions for $\mathcal{P}_{c_{i-1}}$) represent a good initial solution for $\mathcal{P}_{c_i}$ when $c_i - c_{i-1}$ is small enough; in particular, we have shown in Lemma 8 that there always exists $c_i < c_{i-1}$ such that L^{i-1} is stabilizing for c_i. Another interesting feature of the algorithm is that it allows to use the CARE solution from the complete observation in a meaningful manner (it is optimal for the observation level c_0) and slowly shifts to the indirect observation problem.

We have derived in Lemma 5 a formulation for the LRAC in the indirect observation scenario, motivated by the fact that, as explained in Remark 3, finding a solution for each intermediate problem $\mathcal{P}_{c_i}$ involves the constraint (6) in the variables $X_{(1,1)},\dots,X_{(S,S)}$ arising from an augmented Markov chain. In the formulation that we have obtained, (6) is substituted with (12), which involves the variables $X_1,\dots,X_S$. A numerical example is included to illustrate the use of Algorithm 1.

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Appendix A. PROOF OF LEMMA 4

Proof.

$$
\begin{aligned}
X_i(k+1) &= \sum_{\ell\in\mathcal{S}} \mathcal{E}\Big\{x_{k+1}x_{k+1}'\ \mathbb{1}_{\{\theta_{k+1}=i,r_k=\ell\}}\Big\} \\
&= \sum_{\ell\in\mathcal{S}}\sum_{j\in\mathcal{S}} \mathcal{E}\Big\{\Big(A_{\theta_k,r_k}x_k+G_{\theta_k}w_k\Big)\Big(A_{\theta_k,r_k}x_k+G_{\theta_k}w_k\Big)' \\
&\quad \mathbb{1}_{\{\theta_{k+1}=i,r_k=\ell,\theta_k=j\}}\Big\} \\
&= \sum_{\ell\in\mathcal{S}}\sum_{j\in\mathcal{S}} \mathcal{E}\Big\{\Big(A_{\theta_k,r_k}x_k+G_{\theta_k}w_k\Big)\Big(A_{\theta_k,r_k}x_k+G_{\theta_k}w_k\Big)' \\
&\quad \mathbb{1}_{\{\theta_{k+1}=i\}}\,|r_k=\ell,\theta_k=j\ \Big\}P(\theta_k=j,r_k=\ell) \\
&= \sum_{\ell\in\mathcal{S}}\sum_{j\in\mathcal{S}} \mathcal{E}\Big\{\Big(A_{j\ell}x_k+G_jw_k\Big)\Big(A_{j\ell}x_k+G_jw_k\Big)'\Big\} \\
&\quad P(r_k=\ell\,|\,\theta_k=j)\,P(\theta_{k+1}=i,\theta_k=j) \\
&= \sum_{\ell\in\mathcal{S}}\sum_{j\in\mathcal{S}} \Big(A_{j\ell}\mathcal{E}\Big\{x_kx_k'\ \mathbb{1}_{\{\theta_k=j\}}\Big\}A_{j\ell}'+G_j \\
&\quad \mathcal{E}\Big\{w_kw_k'\ \mathbb{1}_{\{\theta_k=j\}}\Big\}G_j'\Big)P(r_k=\ell\,|\,\theta_k=j) \\
&\quad P(\theta_{k+1}=i\,|\,\theta_k=j) \\
&= \sum_{\ell\in\mathcal{S}}\sum_{j\in\mathcal{S}} p_{ji}\Big(A_{j\ell}X_j(k)A_{j\ell}'+G_j\Sigma\mu_k(j)G_j'\Big) \\
&\quad \Big(c\ \mathbb{1}_{\{j=\ell\}}+\frac{1-c}{S-1}\ \mathbb{1}_{\{j\neq\ell\}}\Big) \\
&= \sum_{\ell\in\mathcal{S}}\sum_{j\in\mathcal{S}} p_{ji}\Big(A_{j\ell}X_j(k)A_{j\ell}'\Big)\Big(c\ \mathbb{1}_{\{j=\ell\}}+\frac{1-c}{S-1}\ \mathbb{1}_{\{j\neq\ell\}}\Big) \\
&\quad +\sum_{\ell\in\mathcal{S}}\sum_{j\in\mathcal{S}} p_{ji}\Big(G_j\Sigma\mu_k(j)G_j'\Big)\Big(c\ \mathbb{1}_{\{j=\ell\}}+\frac{1-c}{S-1}\ \mathbb{1}_{\{j\neq\ell\}}\Big) \\
&= \mathcal{L}_i(X(k))+\bar{\varphi}_i(k).
\end{aligned}
$$

Appendix B. ALGORITHM FOR SOLVING (14)

Algorithm 3

1) Set $\kappa_i \le 1$ as the largest integer for which $\sqrt{\kappa_i p_{ii}}A_i$ is stable.
2) Set $P^0 = (P_1^0,\dots,P_N^0) \in \mathcal{M}^{n+}$.
3) For $k = 1,2,..$ and $i = 1,2,\dots,N$ solve the standard algebraic Riccati equations:

$$
\begin{aligned}
-P_i^k+\kappa_i p_{ii}A_i'P_i^kA_i+A_i'\tilde{\mathcal{E}}_i^kA_i-(\kappa_i p_{ii}A_i'P_i^kB_i+A_i'\tilde{\mathcal{E}}_i^kB_i) \\
\times(D_i+\kappa_i p_{ii}B_i'P_i^kB_i+B_i'\tilde{\mathcal{E}}_i^kB_i)^{-1} \\
\times(\kappa_i p_{ii}B_i'P_i^kA_i+B_i'\tilde{\mathcal{E}}_i^kA_i)+C_i=0
\end{aligned}
$$

where

$$
\tilde{\mathcal{E}}_i^k=\sum_{j=1}^{i}p_{ij}P_j^k+(1-\kappa_i)p_{ii}P_i^k.
$$

Appendix C. VARIATIONAL METHOD

Algorithm 4

1) Create an sequence of gains $K^{(0)}$.
2) Find $X^{(\eta)}(k)$ as in Proposition 1. Then set $\eta=\eta+1$ and $k=T-1$.
3) Determine $K^{(\eta)}(k)$ define by

$$
\sum_{i=1}^{T}\Big[\Lambda_i^{(\eta)}(k+1)K^{(\eta)}(k)+B_i'E_i\Big(F^{(\eta)}(k+1)\Big)A_i\Big]
$$

$$
X_i^{(\eta-1)}(k)=0
$$

with

$$
\Lambda^{(\eta)}(k)=D+B'E\Big(F^{(\eta)}(k)\Big)B.
$$

Calculate $F^{(\eta)}(k)$ through

$$
\begin{aligned}
F_i^{(\eta)}(k)=C_i+K^{(\eta)}(k)'D_iK^{(\eta)}(k)+\Big(A_i+B_iK^{(\eta)}(k)\Big)' \\
E_i\Big(F^{(\eta)}(k+1)\Big)\Big(A_i+B_iK^{(\eta)}(k)\Big)
\end{aligned}
$$

with $F_i^{(\eta)}(T)=0$ and

$$
E_i\Big(F^{(\eta)}(k)\Big)=\sum_{j\in\mathcal{S}}p_{ji}F_i^{(\eta)}(k).
$$

Do $k=k-1$; if $k\ge 0$, return to (3).

4) If the stop criterion is not satisfied, return to (2).