15th Triennial World Congress, Barcelona, Spain

NONLINEAR H-INFINITY CONTROL: AN LMI APPROACH [1]

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Abstract: In this paper, we consider the problem of robust H-infinity Control for a class of uncertain nonlinear systems. We derive LMI conditions for analyzing regional robust stability and performance based on Lyapunov functions which are polynomial functions of the state and uncertain parameters. More specifically, we provide an energy bound on the input disturbance which guarantees that the state of the system stays inside a given region. For the given bound on the input disturbance, we also minimize the L2-gain of the input/output operator. Through an iterative algorithm, the proposed technique is applied to control synthesis. Numerical examples are presented to illustrate our results.

Keywords: H-infinity control, convex-optimization, uncertainty, Lyapunov methods.

1. INTRODUCTION

During the past ten years many researchers have generalized the linear robust control theory to deal with the $\mathcal{L}_2$-gain of nonlinear systems (Lu and Doyle, 1995). Unfortunately, the nonlinear $\mathcal{H}_\infty$ problem needs the solution of Hamilton-Jacob Inequalities (HJI, or equations - HJE) which are difficult to solve. Some alternative approaches have been developed (Huang and Lu, 1996) to solve HJI (or HJE) indirectly by reducing the problem to algebraic inequalities (or equations), but these methods are only applicable to problems with low dimension.

On the other hand, the so-called linear matrix inequality (LMI) approach has been used widely to solve problems in linear robust control, gain-scheduling and multi-objective control (Boyd *et al.*, 1994). Since the work (El Ghaoui and Scorletti, 1996) that showed a solution to the nonlinear problem using LMIs, researchers have proposed different solutions to nonlinear robust $\mathcal{H}_\infty$ control problems, e.g. (Sasaki and Uchida, 1997).

In this paper, we also address the nonlinear $\mathcal{H}_\infty$ control problem using the LMI framework. We consider a class of nonlinear systems which are subject to both parameter uncertainties in the system model and an affine input disturbance. The system matrices are allowed to be rational functions of the state and uncertain parameters. We first study conditions for analyzing regional robust stability and performance of the system. More precisely, we consider a given polytopic region in the state space and study two problems: 1) Determine an energy bound on the input disturbance which guarantees that the state trajectory stays inside the given polytopic region, assuming zero initial conditions; 2) Minimize the $\mathcal{L}_2$-gain of the system, assuming that the state trajectory stays in the given polytopic region. Both problems are solved in terms of LMIs. We then extend these results to a synthesis problem, i.e., we want to design a state feedback control law that minimizes the $\mathcal{L}_2$-gain of the system while guarantee-

[1] This work was partially supported by 'CAPES', Brazil, under grant BEX 0784/00-1.

ing that the state stays in a given polytopic region. The synthesis is done using an iterative scheme. We point out that our approach is based on Lyapunov functions which are polynomial functions of the state and uncertain parameters. We will also show via examples that this type of Lyapunov functions yield much less conservative results compared to functions with a Lyapunov matrix which is constant or affinely dependent on the state and uncertain parameters.

The rest of the paper is structured as follows. Sections 2 and 3 study the analysis problems. Section 4 discusses the synthesis problem and proposes an iterative design scheme. Numerical examples are given in section 5. Some conclusions are drawn in section 6. The notation used in this paper is standard. For a real matrix S, S' denotes its transpose, $S > 0$ means that S is symmetric and positive-definite, and $\mathrm{He}(S) = S + S'$. Matrix and vector dimensions are omitted whenever they can be inferred from the context. In this paper, the proofs of theorems and some references are omitted from the original work because of space limitation. For further details, the reader is referred to the full version of this paper (Coutinho *et al.*, 2001) (available for download at ftp://warhol.newcastle.edu.au/pub/Reports/EE01045.ps.gz).

2. REGIONAL STABILITY ANALYSIS

Consider the uncertain nonlinear system described as follows:

$$\dot{x} = A(x,\delta)x + B_w(x,\delta)w\ , \quad x(0) = 0 \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state, $\delta \in \mathbb{R}^l$ the uncertain parameters, and $w \in \mathbb{R}^m$ the disturbance input. We assume that:

A1. The uncertain parameters represented by the vector δ and its time-derivative $\dot{\delta}$ lie in a given polytope $\mathcal{B}_\delta$, i.e., $(\delta,\dot{\delta}) \in \mathcal{B}_\delta$;

A2. The system's matrices $A(x,\delta)$ and $B_w(x,\delta)$ are bounded for all $(\delta,\dot{\delta}) \in \mathcal{B}_\delta$ and $x \in \mathcal{B}_x$, where $\mathcal{B}_x$ represents a given polytopic region of state containing the origin.

The problem of concern in this section is to analyze the regional stability of the system (1) for a given set of input disturbances. To this end, we will use the following definition of regional stability.

Definition 1. Consider the nonlinear uncertain system in (1), satisfying the assumptions **A1** and **A2**, and a given set of input disturbances $\mathcal{W}$. The system is called regionally stable (with respect to $\mathcal{W}$ and $\mathcal{B}_x$) if $x(t) \in \mathcal{B}_x$ for all $t \geq 0$ and all $w \in \mathcal{W}$. The corresponding set $\mathcal{W}$ is called a *set of admissible input disturbances*.

Hereafter, we describe the class of admissible input disturbances as follows:

$$\mathcal{W} \triangleq \left\{ w(t)\ :\ \mu^{-1}\int_0^\infty w(t)'w(t)\,dt \leq 1 \right\} \tag{2}$$

where $\mu > 0$ controls the "size" of $\mathcal{W}$.

In this paper, we will represent the polytope $\mathcal{B}_x$ by its vertices or by using a set of inequalities, i.e.,

$$\mathcal{B}_x\ =\ \left\{ x\ :\ a_k'\,x\ \leq\ 1\ ,\ \ k = 1,\ldots,n_e \right\} \tag{3}$$

where a_k are given vectors associated with the n_e edges of the polytope $\mathcal{B}_x$.

The key idea involved in the study of admissible input disturbances is to overbound the state trajectory generated by an input disturbance using a level set of a Lyapunov function, which in turn is overbounded by $\mathcal{B}_x$. More precisely, we consider the following Lyapunov matrix candidate:

$$\mathcal{P}_r(x,\delta) = \begin{bmatrix} I_n \\ \Theta(x,\delta) \end{bmatrix}' P_r \begin{bmatrix} I_n \\ \Theta(x,\delta) \end{bmatrix} \tag{4}$$

where P_r is a fixed symmetric matrix to be determined and $\Theta(x,\delta) \in \mathbb{R}^{n\times n}$ is an affine matrix function of (x,δ) that we will specify later. The corresponding Lyapunov function candidate is given by

$$v_r(x,\delta)\ =\ x'\mathcal{P}_r(x,\delta)x.$$

The overbounding level set is given by

$$\mathcal{R}_c = \{x : v_r(x,\delta) \leq 1, \forall\ (\delta,\dot{\delta}) \in \mathcal{B}_\delta\}$$

Suppose $v_r(x,\delta)$ satisfies the following conditions for all $x \in \mathcal{B}_x$, $(\delta,\dot{\delta}) \in \mathcal{B}_\delta$ and $w \in \mathcal{W}$:

$$v_r(x,\delta) > 0 \tag{5}$$

$$\dot{v}_r(x,\delta) \leq \mu^{-1}w'(t)w(t),\ \forall t \geq 0 \tag{6}$$

Integrating both sides of (6) from 0 to T for any $T \geq 0$ yields

$$v_r(x(T),\delta) \leq \mu^{-1}\int_0^T w'(t)w(t)dt \leq 1, \quad \forall\,(\delta,\dot{\delta}) \in \mathcal{B}_\delta,\ w(t) \in \mathcal{W} \tag{7}$$

Hence, the trajectory $x(t)$ driven by $w(t) \in \mathcal{W}$ belongs to $\mathcal{B}_x$ if the condition $\mathcal{R}_c \subset \mathcal{B}_x$ is satisfied.

In order to make the above conditions testable via LMIs, we rewrite the system (1) as follows:

$$\begin{aligned}\dot{x} &= \sum_{i=0}^{q} A_i(x,\delta)\pi_i + \sum_{j=0}^{\tilde{q}} B_j(x,\delta)\xi_j \\ &= \mathbf{A}(x,\delta)\pi + \mathbf{B}(x,\delta)\xi\end{aligned} \tag{8}$$

with $\pi = [\,\pi_0' \ \cdots\ \pi_q'\,]'$; $\xi = [\,\xi_0' \ \cdots\ \xi_{\tilde{q}}'\,]'$; and

$$\mathbf{A}(x,\delta) = [\,A_0(x,\delta)\ \cdots\ A_q(x,\delta)\,]$$

$$\mathbf{B}(x,\delta) = [\,B_0(x,\delta)\ \cdots\ B_{\tilde{q}}(x,\delta)\,]$$

In addition, the auxiliary vectors π and ξ are nonlinear functions of (x,δ) satisfying

$$\Omega(x,\delta)\pi = 0;\quad \Lambda(x,\delta)\xi = 0$$

for some non-zero matrix functions $\Omega(x,\delta)$ and $\Lambda(x,\delta)$. Also, π_0, π_1 and ξ_0 are chosen as $\pi_0 = x$, $\pi_1 = \Theta(x,\delta)x$ and $\xi_0 = w(t)$, where $\Theta(x,\delta)$ is the same matrix used to define the Lyapunov matrix in (4). Moreover, $\mathbf{A}(x,\delta) \in \mathbb{R}^{n\times(q+1)n}$, $\mathbf{B}(x,\delta) \in \mathbb{R}^{n\times(\tilde{q}+1)m}$, $\Omega(x,\delta)$ and $\Lambda(x,\delta)$ are affine functions of x and δ. To simplify the notation, we may use the auxiliary matrices and vectors without explicitly mentioning their respective dependence on x,δ,w and t.

With the above discussion, we modify Assumption **A2** to the following:

A2'. The matrices $\mathbf{A}(x,\delta)$ and $\mathbf{B}(x,\delta)$ in (8) are affine functions of x and δ and are bounded for $x \in \mathcal{B}_x$ and $(\delta,\dot{\delta}) \in \mathcal{B}_\delta$.

By definition of the Lyapunov matrix in (4), it should be noted that the matrix $\Theta(x,\delta)$ is an affine function of x and δ. Hence, we can represent it by

$$\Theta(x,\delta) \;=\; \sum_{j=1}^{n} T_j\, x_j \;+\; \sum_{j=1}^{l} U_j\, \delta_j \;+\; V \qquad (9)$$

where x_j, δ_j are j-th entries of the vectors x and δ, respectively, and T_j, U_j and V are constant matrices with the same dimensions of $\Theta(x,\delta)$. The analysis above leads to the following result.

Theorem 2. Let $\mathcal{B}_x$, $\mathcal{B}_\delta$ and $\mathcal{W}$ in (2) be given. Consider the system (1), the associated system (8), the assumptions **A1** and **A2'** and the following notation:

$$\tilde{\Theta}(x) = \sum_{j=1}^{n} T_j\, x\, s_j;\ \hat{\Theta}(\dot{\delta}) = \sum_{j=1}^{l} U_j\, \dot{\delta};\ N = [\, I_{(2n)}\ \ 0\,];$$

$$\tilde{N} = [\, I_m\ \ 0\,];\ E = \begin{bmatrix} I_n & 0 \\ -(\Theta(x,\delta)+\tilde{\Theta}(x)) & I_n \end{bmatrix};$$

$$F = \begin{bmatrix} \mathbf{A}(x,\delta) \\ [\,\hat{\Theta}(\dot{\delta})\ \ 0\,] \end{bmatrix};\ \Psi_a \;=\; \begin{bmatrix} M_x & 0 \\ \Theta(x,\delta) & -I_n \end{bmatrix};$$

$$M_x = \begin{bmatrix} x_2 & -x_1 & 0 & \cdots & 0 \\ 0 & x_3 & -x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_n & -x_{n-1} \end{bmatrix};$$

$$\Psi_b = \begin{bmatrix} 0 & \Omega(x,\delta) & 0 \\ 0 & [\,\Psi_a\ \ 0\,] & 0 \\ -E & F & \begin{bmatrix} \mathbf{B}(x,\delta) \\ 0 \end{bmatrix} \\ 0 & 0 & \Lambda(x,\delta) \end{bmatrix};$$

where s_j is the *j-th* row of the identity matrix I_n, $N \in \mathbb{R}^{(2n)\times(q+1)n}$ and $\tilde{N} \in \mathbb{R}^{m\times(\tilde{q}+1)m}$.

Suppose there exist matrices P_r, L_{ar} and L_{br} that solve the following LMIs constructed at all vertices of $\mathcal{B} = \mathcal{B}_x \times \mathcal{B}_\delta$:

$$\begin{array}{l} P_r + L_{ar}\Psi_a + \Psi_a' L_{ar}' > 0\ ,\ \ P_r' = P_r, \\ \Upsilon(P_r,\mu^{-1}) + L_{br}\Psi_b + \Psi_b' L_{br}' < 0 \\ \begin{bmatrix} 1 & \begin{bmatrix} a_k' & 0 \end{bmatrix} \\ \begin{bmatrix} a_k \\ 0 \end{bmatrix} & \begin{pmatrix} P_r + L_{ar}\Psi_a \\ +\Psi_a' L_{ar}' \end{pmatrix} \end{bmatrix} \geq 0\ ,\ \ \forall k \end{array} \qquad (10)$$

where

$$\Upsilon(P_r,\mu^{-1}) = \begin{bmatrix} 0 & P_r N & 0 \\ N' P_r & 0 & 0 \\ 0 & 0 & -\mu^{-1}\tilde{N}'\tilde{N} \end{bmatrix} \qquad (11)$$

Then, the disturbed system (1) is regionally stable and the set $\mathcal{W}$ is an admissible set of input disturbances (with respect to $\mathcal{B}_x$). □

The proof of above theorem was omitted because of space limitation and can be obtained in the full version of this paper (Coutinho *et al.*, 2001).

3. $\mathcal{L}_2$-GAIN PERFORMANCE

Consider that the uncertain nonlinear system in (1) has the following performance output:

$$z = C_z(x,\delta)x + D_{zw}(x,\delta)w \qquad (12)$$

where $z \in \mathbb{R}^r$.

For the above output vector, we assume that:

A3. The matrix functions $C_z(x,\delta)$ and $D_{zw}(x,\delta)$ are bounded for all $(\delta,\dot{\delta}) \in \mathcal{B}_\delta$ and $x \in \mathcal{B}_x$.

In this section, we are interested in the performance of the nonlinear system (1) and (12). To this end, we will use the following definition of $\mathcal{L}_2$-gain.

Definition 3. Consider the nonlinear uncertain system (1) and (12), satisfying assumptions **A1-A3**, and a given set of input disturbances $\mathcal{W}$. The (worst-case) $\mathcal{L}_2$-gain of the input/output operator, denoted by G_{wz}, of system (1) and (12) is given by

$$\|G_{wz}\|_\infty = \begin{cases} \infty & \text{if the system is not regionally stable} \\ \displaystyle\sup_{\substack{0\neq w\in\mathcal{W} \\ \forall(\delta,\dot{\delta})\in\mathcal{B}_\delta}} \frac{\|z\|_2}{\|w\|_2} & \text{otherwise} \end{cases}$$

With the above assumptions and definitions, we can state the problem of concern in this section.

Problem 4. Given two polytopes $\mathcal{B}_x$ and $\mathcal{B}_\delta$ and the set $\mathcal{W}$, the problem of concern in this section is to find an upper-bound on the $\mathcal{L}_2$-gain of the system (1) and (12).

To solve the above problem in terms of LMIs, we will consider the following Lyapunov function candidate $v(x,\delta) = x'\mathcal{P}(x,\delta)x$, where $\mathcal{P}(x,\delta)$ is given by

$$\mathcal{P}(x,\delta) = \begin{bmatrix} I_n \\ \Theta(x,\delta) \end{bmatrix}' P \begin{bmatrix} I_n \\ \Theta(x,\delta) \end{bmatrix} \quad (13)$$

and $P = P'$ is a constant matrix to be determined.

To test if the $\mathcal{L}_2$-gain is bounded by a given $\gamma > 0$, we require that the following inequalities are satisfied for all $(x, \delta, \dot{\delta}) \in \mathcal{B}$, $w \in \mathcal{W}$ and $t \geq 0$:

$$v(x,\delta) > 0 \quad (14)$$

$$\dot{v}(x,\delta) + z'z - \gamma w'(t)w(t) \leq 0 \quad (15)$$

which is a well-known result in the literature, see for instance (Boyd *et al.*, 1994).

In the same way of Section 2, we rewrite the performance output (12) as follows:

$$z = \sum_{i=0}^{q} C_i(x,\delta)\pi_i + \sum_{j=0}^{\tilde{q}} D_j(x,\delta)\xi_j \quad (16)$$

or, equivalently, in the concise form $z = \mathbf{C}(x,\delta)\pi + \mathbf{D}(x,\delta)\xi$, with $\mathbf{C}(x,\delta) = [\, C_0(x,\delta) \ \cdots \ C_q(x,\delta) \,]$ and $\mathbf{D}(x,\delta) = [\, D_0(x,\delta) \ \cdots \ D_{\tilde{q}}(x,\delta) \,]$.

Hereafter, we modify Assumption **A3** to the following:

A3'. The matrices $\mathbf{C}(x,\delta) \in \mathbb{R}^{r\times(q+1)n}$ and $\mathbf{D}(x,\delta) \in \mathbb{R}^{r\times(\tilde{q}+1)m}$ in (16) are affine functions of x and δ and are bounded for $x \in \mathcal{B}_x$ and $\delta \in \mathcal{B}_\delta$.

With the above analysis, we can state the main result of this paper as follows.

Theorem 5. Let $\mathcal{B}_x$ and $\mathcal{B}_\delta$ and $\mathcal{W}$ be given. Consider the system (1) and (12), the associated system (8) and (16), the assumptions **A1**, **A2'** and **A3'** and the notation of Theorem 2. Suppose the given $\mathcal{W}$ is admissible. Then, the $\mathcal{L}_2$-gain of the system is bounded by γ, where γ is the solution of the following optimization problem, with the decision variables P, L_a, L_b and γ, and the LMIs constructed at all vertices of $\mathcal{B} = \mathcal{B}_x \times \mathcal{B}_\delta$:

$\min \gamma$ subject to:

$$P + L_a\Psi_a + \Psi_a' L_a' > 0, \ P = P'$$

$$\begin{bmatrix} \begin{pmatrix} \Upsilon(P,\gamma) + L_b\Psi_b \\ +\Psi_b' L_b' \end{pmatrix} & \begin{bmatrix} 0 \\ \mathbf{C}(x,\delta)' \\ \mathbf{D}(x,\delta)' \end{bmatrix} \\ [\, 0 \ \mathbf{C}(x,\delta) \ \mathbf{D}(x,\delta) \,] & -I_r \end{bmatrix} < 0 \quad (17)$$

where $\Upsilon(P,\gamma)$ is given by (11).

Moreover, the origin of the unforced system ($w \equiv 0$) is locally exponentially stable. □

The proof of above theorem was omitted because of space limitation. See (Coutinho *et al.*, 2001) for further details.

REMARK: For linear time-invariant (LTI) systems with no uncertain parameters, the optimization problem in Theorem 5 leads to a necessary and sufficient condition. To illustrate this point, let us consider the following LTI system:

$$\dot{x} = Ax + B_w w \ \text{ and } \ z = C_z x + D_{zw} w.$$

Define the multipliers $L_a = 0$ and $L_b = [\, F' \ G' \ H' \,]'$. From (17), we get the following: $\min \gamma$ subject to: $P > 0$ ($P = P'$) and

$$\begin{bmatrix} \Delta(P,F,G,H,\gamma) & \begin{bmatrix} 0 \\ C_z' \\ D_{zw}' \end{bmatrix} \\ [\, 0 \ C_z \ D_{zw} \,] & -I \end{bmatrix} < 0 \quad (18)$$

where the matrix $\Delta(P,F,G,H,\gamma)$ is given by

$$\begin{bmatrix} -(F+F') & P+FA-G' & FB_w - H' \\ P+A'F'-G & GA+A'G' & GB_w + A'H' \\ B_w'F' - H & HA + B_w'G' & HB_w + B_w'H' - \gamma I \end{bmatrix}$$

Now, applying the Schur complement to (18) and defining $F = 0$, $G = P$ and $H = 0$, we get

$$\begin{bmatrix} A'P + PA + C_z'C_z & PB_w + C_z'D_{zw} \\ B_w'P + D_{zw}'C_z & D_{zw}'D_{zw} - \gamma I \end{bmatrix} \leq 0 \quad (19)$$

recovering the classical result for LTI systems (Boyd *et al.*, 1994). Note that the second LMI of (18) has multipliers which the classical result (19) does not have. The use of multipliers allows us to deal with nonlinearities and uncertain parameters. The same idea has been used to solve analysis and synthesis problems of uncertain (continuous and discrete time) linear systems see, e.g. (de Oliveira *et al.*, 1999; Apkarian *et al.*, 2000).

4. CONTROL DESIGN

Consider the uncertain nonlinear system as follows:

$$\begin{aligned} \dot{x} &= A(x,\delta)x + B_u(x,\delta)u + B_w(x,\delta)w \\ z &= C_z(x,\delta)x + D_{uz}(x,\delta)u + D_{wz}(x,\delta)w \end{aligned} \quad (20)$$

where $x(0) = 0$, $(\delta, \dot{\delta}) \in \mathcal{B}_\delta$, $u \in \mathbb{R}^p$ denotes the control input, and $B_u(x,\delta)$ and $D_{uz}(x,\delta)$ are affine matrix functions of x and δ with appropriate dimensions.

In this section we are concerned with the problem of determining a control law to improve the performance of the closed-loop system. In particular, we use a control law $u = \mathbf{K}(x,\delta)x$, where the control matrix is given by $\mathbf{K}(x,\delta) = \sum_{i=0}^{q} K_i\pi_i$, the auxiliary vectors π_i are as defined in section 3 and $K_i \in \mathbb{R}^{p\times n}$ are fixed matrix gains to be determined. Note that this control law can represent a gain scheduler or a non-fragile controller. For simplicity, we assume that the state information is available for feedback and the parameters δ_i are known on-line to the controller.

Theorems 2 and 5 provide the foundation for solving our synthesis problem. To analyze the closed-loop regional stability for given admissible input disturbance $\mathcal{W}$ and control-gains K_i ($i = 0, \ldots, q$), we can replace the matrices $\mathbf{A}(x,\delta)$ and $\mathbf{C}(x,\delta)$ that are used in (10) and (17) with the following:

$$\begin{array}{l}\mathbf{A} = [\, A_0 + B_u K_0 \ \cdots \ A_q + B_u K_q \,] \\ \mathbf{C} = [\, C_0 + D_{uz} K_0 \ \cdots \ C_q + D_{uz} K_q \,]\end{array} \tag{21}$$

and apply Theorem 2 to verify the regional stability and Theorem 5 to determine an upper bound on the $\mathcal{L}_2$-gain.

To extend this result to the design case, we observe that the matrix inequalities in (10) and (17) will be bilinear matrix inequalities (BMIs). BMI problems appear commonly in robust control design. In order to avoid this technical difficulty, we use an iterative algorithm, similar to an algorithm proposed by (Feron *et al.*, 1996), in which the BMI problem is solved via two LMI sub-problems. It is well-known that iterative algorithms may provide a locally optimal solution for BMI problems. Nevertheless, each iteration tends to improve the closed-loop performance, so the technique is often effective.

Algorithm 1. Consider the system (20) with given $\mathcal{B}_x$, $\mathcal{B}_\delta$, $\mathcal{W}$, and Theorems 2 and 5 with the matrices $\mathbf{A}(x,\delta)$ and $\mathbf{C}(x,\delta)$ as defined in (21).

STEP 1 Determine a stabilizing controller such that $\mathcal{W}$ is admissible;
STEP 2 For a given controller and taking into account (21), solve the LMIs in (10) and the optimization problem (17) to obtain the matrices L_{br} and L_b respectively.
STEP 3 For given matrices L_{br} and L_b, solve respectively (10) and (17) to obtain the new control parameters $K_0,\cdots,K_q$. Note that the inequalities in (10) and (17) are affine in all variables including $K_0,\cdots,K_q$.
STEP 4 Iterate over steps 2 and 3 until convergence or satisfaction of a pre-defined $\mathcal{L}_2$-gain.

At each iteration i, note that Algorithm 1 guarantees the regional stability of the closed-loop system and $\gamma_i \leq \gamma_{(i-1)}$. As a result, this algorithm converges on a local minimum. To overcome the problem of finding an initial stabilizing controller (STEP 1), we propose the following result.

4.1 *Stabilizing Controller*

Consider the differential equation of system (20), i.e.

$$\dot{x} = A(x,\delta)x + B_u(x,\delta)u + B_w(x,\delta)w \tag{22}$$

We assume for above system that:

A4. The system matrix $A(x,\delta)$ can be rewritten as $A(x,\delta) = \Pi(x,\delta)\tilde{\mathbf{A}}(x,\delta)$ where the matrix $\tilde{\mathbf{A}}(x,\delta)$ is an affine function of (x,δ) and $\Pi(x,\delta)$ is a nonlinear function of (x,δ) that satisfy the following: (1) the matrix $\tilde{\mathbf{A}}(x,\delta)$ is bounded for $x \in \mathcal{B}_x$ and $\delta \in \mathcal{B}_\delta$; (2) there exists a non-zero matrix $\Omega_\Pi(x,\delta)$ affine function of (x,δ) such that $\Pi(x,\delta)\ \Omega_\Pi(x,\delta) = 0$; and (3) there exists a constant matrix N_Π such that $\Pi(x,\delta)N_\Pi = I_n$.

A5. The system matrix $B_w(x,\delta)$ can be rewritten as $B_w(x,\delta) = \Phi(x,\delta)\tilde{\mathbf{B}}(x,\delta)$, where the matrix $\tilde{\mathbf{B}}(x,\delta)$ is an affine function of (x,δ) and $\Phi(x,\delta)$ is a nonlinear function of (x,δ) that satisfy the following: (1) the matrix $\tilde{\mathbf{B}}(x,\delta)$ is bounded for $x \in \mathcal{B}_x$ and $\delta \in \mathcal{B}_\delta$; (2) there exists a non-zero matrix $\Lambda_\Phi(x,\delta)$ affine function of (x,δ) such that $\Phi(x,\delta)\ \Lambda_\Phi(x,\delta) = 0$; and (3) there exists a constant matrix N_Φ such that $\Phi(x,\delta)N_\Phi = I_m$.

Then, we can propose the following state feedback control law that assures the regional stability of system (22) for a given constant μ.

Theorem 6. Let $\mathcal{B}_x$, $\mathcal{B}_\delta$ and μ be given. Consider the system (22) and Assumptions **A1**, **A4** and **A5**. Suppose there exist matrices X, Y and L that solve the following LMIs constructed at all vertices of $\mathcal{B} = \mathcal{B}_x \times \mathcal{B}_\delta$.

$$\begin{array}{l} X > 0,\ X = X' \\ 1 - a_k' X a_k \geq 0,\ \forall k \\ \mathsf{He}\left(\Gamma(X,Y,\mu^{-1}) + \begin{bmatrix} \Omega_\Pi & 0 \\ 0 & \Lambda_\Phi \end{bmatrix} L\right) < 0 \end{array} \tag{23}$$

where $\Gamma(X,Y,\mu^{-1})$ is given by

$$\begin{bmatrix} N_\Pi X \tilde{\mathbf{A}}' + N_\Pi B_u Y & 0 \\ N_\Phi \tilde{\mathbf{B}}' & -0.5\mu^{-1} N_\Phi N_\Phi' \end{bmatrix}$$

Then, the system (22) with $u = Y\Pi(x,\delta)' X^{-1} x$ is regionally stable and the set $\mathcal{W}$ defined by μ is an admissible set of input disturbances. □

The proof of above theorem was omitted because of space limitation. See (Coutinho *et al.*, 2001) for further details.

5. NUMERICAL RESULTS

To illustrate the potential of the proposed technique, we analyze two numerical examples. In the first one, we compute an upper bound on the $\mathcal{L}_2$-gain of the system. In the second example, we design a nonlinear controller that minimizes the $\mathcal{L}_2$-gain. In both examples, we assume that the sets of admissible input disturbances are given.

Example 1: consider the following uncertain system which is based on the Van der Pol equation (El Ghaoui and Scorletti, 1996):

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & \varepsilon(x_1^2 - 1) \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \tag{24}$$

with $z = [\,1\ \ 0\,]x + w$, $w \in \mathcal{W}$ (for some $\mu > 0$), and the nonlinear dumping factor ε is constant and approximately known, i.e. $\varepsilon \in [\varepsilon_0 - \Delta\varepsilon, \varepsilon_0 + \Delta\varepsilon]$, where $\varepsilon_0 = 0.8$ and $\Delta\varepsilon = 0.2$.

Our objective in this example is to determine a bound on the output energy as well as estimate the size of set

$\mathcal{W}$ using different Lyapunov matrices (constant, affine and quadratic). To this end, let $\mathcal{B}_x$ be the polytope defined by the following set $\{x_1, x_2 : |x_i| \leq \alpha,\ i = 1, 2\}$, where α is a given scalar.

Now, consider that the matrix P is partitioned as follows $P = \begin{bmatrix} P_0 & P_1 \\ P_1' & P_2 \end{bmatrix}$. With this partition, we obtained the following Lyapunov matrices: (i) P_0, P_1 and P_2 are free, i.e the matrix $\mathcal{P}(x,\delta)$ is quadratic in (x,δ); (ii) P_0 and P_1 are free and $P_2 = 0$, i.e the matrix $\mathcal{P}(x,\delta)$ is affine in (x,δ); and (iii) $P_1 = 0$, $P_2 = 0$ and P_0 is free, i.e. the matrix $\mathcal{P}(x,\delta)$ is constant.

Table 1 shows the estimated upper-bounds on the $\mathcal{L}_2$-gain of the input/output operator and sizes of the input disturbances for the proposed approach using Theorems 2 and 5 with different Lyapunov matrices. For all the solutions, $\alpha = 0.7$ is used. As expected, the

Upper-bounds	Lyapunov Matrix		
	Constant	Affine	Quadratic
γ	84.5	84.4	10.7
μ	0.10	0.11	0.30

Table 1. $\mathcal{L}_2$-gain and size of $\mathcal{W}$

polynomial Lyapunov function (quadratic Lyapunov matrix) achieved less conservative results, thus justifying the required extra computation and showing the potential of our approach.

Example 2: consider the following nonlinear system which is based on the benchmark example proposed in (Kokotović *et al.*, 1991).

$$\dot{x} = \begin{bmatrix} 0 & 1 & x_3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w \qquad (25)$$

with $z = x_1$ and $w \in \{w : \int_{\infty}^{\infty} w' w dt \leq 0.1\}$.

The objective in this example is to stabilize the system in a $\mathcal{H}_\infty$ sense using a static state-feedback ($u = K_0 x$).To this end, let $\mathcal{B}_x$ be the polytope defined by the following bounds on the state-space variables: $|x_1| \leq 1$, $|x_2| \leq 1$ and $|x_3| \leq 1$.

Note that system (25) is open-loop unstable. Then, using theorem 6 we obtained $\tilde{K}_0 = [\,-2.0 \ \ -3.8 \ \ -2.2\,] \times 10^5$ that stabilizes regionally the closed-loop system with $\gamma = 58$.

After 2 iterations of algorithm 1, we obtained $\gamma = 0.1$ for the control gain $K_0 = [\,-3.8 \ \ -6.4 \ \ -3.8\,] \times 10^5$.

6. CONCLUDING REMARKS

In this paper, we have proposed a new technique to analyze the regional stability and compute an upper-bound on the $\mathcal{L}_2$-gain for a class of uncertain nonlinear systems using the LMI framework. To ascertain the robustness and performance of the nonlinear system, we use Lyapunov functions which are polynomial functions of the state and uncertain parameters. Through an iterative algorithm, this technique is extended to the synthesis problem. The numerical examples show the potential of our approach when compared with techniques using quadratic and affine quadratic Lyapunov functions. Future research may concentrate on devising a better algorithm for the synthesis problem.

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