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ON MODEL ORDER REDUCTION VIA PROJECTION

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Abstract: The paper investigates the properties of reduced order models obtained by projection of a high order system. It answers questions such as are any two models of different orders related by a projection? Is it possible to obtain the same reduced order model using different projections? Etc. It is shown that in cases where not all models of a certain reduced order can be obtained by a projection, the optimal L2 reduced order model is obtained by a unique projection, and it resides on the boundary of the set of attainable models. *Copyright © 2002 IFAC*

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1. INTRODUCTION

The problem of model order reduction for continuous time systems has received a considerable amount of interest over the years, and many methods for obtaining the reduced order model have been suggested. Most of them include the following two steps. First a state transformation into a state space realization in which the state variables can be ranked according to some measure of importance. The second step is truncation of the least important state variables. The two operations together constitute a projection into a lower dimension. We will therefore call such operation Projection Order Reduction (POR).

The various POR methods differ in the criterion which is used for ranking the state variables. In partial fraction expansion, known for lightly damped mechanical systems as 'modal truncation', the state space model is transformed into a diagonal realization. Another POR method is truncated balanced realization (Moore, 1981; Kabamba, 1985; Halevi, 1999), where in the new realization the controllability and observability gramians are diagonal and equal. State variables that correspond to larger diagonal elements are more controllable and observable, and are therefore retained in the reduced order model. Yet another POR method is component cost analysis (Skelton and Yousuff, 1983) where the contribution of each state variable to a certain cost is investigated.

In the methods that have been described so far the projection is an intentional part of a heuristic algorithm. Wilson (1970) used the same structure and derived the optimal L_2 reduced order model. Later, Hyland and Bernstein (1985) have solved the problem by direct optimization, without imposing any structure on the reduced order model. It turned out that it is given in terms of a projection into a lower order subspace and therefore is sometimes referred to as the 'optimal projection'.

The first part of this paper is concerned with problems, which apply to POR in its general form. For example, given two models of different orders, is it always possible to obtain the one with the lower dimension by POR of the other? Following this line of investigation, we present unique properties of the optimal L_2 reduced order model.

2. ORDER REDUCTION VIA PROJECTION

The model order reduction problem for linear systems is usually defined as follows. Given the n-th order, linear, time invariant, system G(s), find an r-th order (r<n) system $G_r(s)$, with the same number of inputs and outputs, which is an approximation of it. POR methods start with a state space realization

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (1a)$$

$$y(t) = Cx(t) + Du(t) \qquad (1b)$$

where $x\in R^n$, $u\in R^m$, and $y\in R^p$. Let the nonsingular matrix T, and its inverse, be partitioned as

$$T = \begin{bmatrix} R & \overline{R} \end{bmatrix} \quad , \quad T^{-1} = \begin{bmatrix} L \\ \overline{L} \end{bmatrix} \qquad (2)$$

with $R\in R^{n\times r}$, $L\in R^{r\times n}$. The state transformation x=Tx' leads to the following realization.

$$\begin{bmatrix} \dot{x}'_1(t) \\ \dot{x}'_2(t) \end{bmatrix} = \begin{bmatrix} LAR & LA\overline{R} \\ \overline{L}AR & \overline{L}A\overline{R} \end{bmatrix} \begin{bmatrix} x'_1(t) \\ x'_2(t) \end{bmatrix} + \begin{bmatrix} LB \\ \overline{L}B \end{bmatrix} u(t) \qquad (3a)$$

$$y(t) = \begin{bmatrix} CR & C\overline{R} \end{bmatrix} x'(t) + Du(t) \qquad (3b)$$

Suppose that, using any criterion, x'_1 is more important than x'_2. It is assumed that $x'_2 \approx 0$ and the reduced order approximation of the system (1) is

$$\dot{x}_r(t) = LARx_r(t) + LBu(t) \qquad (4a)$$

$$y(t) = CRx_r(t) + Du(t) \quad (4b)$$

It is evident from (4) that the direct transmission term Du(t) plays no role in this order reduction procedure. For convenience, it will therefore be assumed from now on that D=0. Since L and R are sub-blocks of T and its inverse they satisfy $LR=I_r$. A model (4), with any L and R satisfying $LR=I_r$ is a Projection Reduced Order Model (PROM). To see the origin of this name, we define the matrix

$$P = RL \quad (5)$$

It follows immediately that $P^2=P$, hence P is a projection matrix. We also define the pseudo full order state vector

$$\begin{aligned}\hat{x}(t) &= Rx_r(t) \\ &= Px(t)\end{aligned} \quad (6)$$

which is x_r expressed in the coordinates of the n-th order space of x. Multiplying eq. (4a) by R, the reduced order model can be written as

$$\dot{\hat{x}}(t) = P(A\hat{x}(t) + Bu(t)) \quad (7a)$$

$$y(t) = C\hat{x}(t) \quad (7b)$$

Hence P projects the time derivative of the state vector into its image, and the 'angle' is determined by its null space. The PROM is therefore a minimal realization of the system (PA, PB, C). Assuming zero initial conditions, $\hat{x}(t)$ is confined to the image of P, hence $P\hat{x}(t) = \hat{x}(t)$ and the reduced order model is also a minimal realization of (PAP, PB, C). This later form is sometimes preferred since it resembles the familiar similarity transformation.

Despite their wide use, PROM's have very few generic properties. They do not preserve stability or instability, relative degree, and even minimality or non-minimality. The following results discuss the invariance properties under state transformations.

Property 1: The projection P that relates (A,B,C) and (A_r,B_r,C_r) is invariant under state transformation of the reduced order realization.
Proof: Under a state transformation $x_r=T_r x_r'$, $(A_r,B_r,C_r) \rightarrow (T_r^{-1}A_rT_r, T_r^{-1}B_r, C_rT_r)$. This is equivalent to $(L,R) \rightarrow (T_r^{-1}L, RT_r)$, hence R' L'= R L=P.

Property 2: The projection P that relates (A, B, C) and (A_r, B_r, C_r) changes under state transformation of the full order realization into TPT^{-1}.
Proof: Under a state transformation x=Tx', $(A,B,C) \rightarrow (T^{-1}AT, T^{-1}B, CT)$. For the same (A_r,B_r,C_r) This is equivalent to $(L,R) \rightarrow (LT^{-1}, TR)$, hence $R'L' = TR\,LT^{-1} = TP\,T^{-1}$.

Assuming that $G_r(s)=(A_r,B_r,C_r)$ is minimal, Property 1 means that P is a projection into all of its realizations. Property 2 means that if there exists a projection relation between certain (A,B,C) and (A_r,B_r,C_r), there exists a projection relation between any pair of realizations of G(s) and $G_r(s)$. However the specific projection is not preserved. Hence for systems, rather then realizations, the only relevant question is whether a projective relation exists.

3. EXISTENCE PROPERTIES

We begin this section by considering the following questions: Given the system G(s), is any r-th order $G_r(s)$, with the same dimensions, a PROM of it? In other words, is it always possible to find a projection that relates the two models. As was shown in the previous section, the existence of a projection is independent of a specific realization. Let (A, B, C) and (A_r, B_r, C_r) be any realizations of G(s) and $G_r(s)$ respectively, then for $G_r(s)$ to be a PROM of G(s) the following relationships must hold.

$$LAR = A_r \quad (8a)$$

$$LB = B_r \quad (8b)$$

$$CR = C_r \quad (8c)$$

$$LR = I_{r\times r} \quad (8d)$$

Considering L and R as the unknowns, equations (8) are a set of *(2r+m+p)r* equations with *2nr* unknowns. There are three possible cases.

1. *r < n-(m+p)/2* - There are more unknowns than equations. In general any $G_r(s)$ is a PROM of a given G(s), and can be obtained via infinitely many projections.
2. *r = n-(m+p)/2* – The number of equations and unknowns is the same. Not every $G_r(s)$ is a PROM of a given G(s). Those who are, can be obtained by a finite number of projections.
3. *r > n - (m+p)/2* - There are more equations than unknowns. For a given G(s), the class of models $G_r(s)$ that are PROM has measure zero.

The first two cases suggest that the same reduced order model can be obtained from a specific realization of G(s), using different projections. Case 3 is impossible in SISO systems. However models with a large number of inputs and outputs are used in some cases. For example, in mechanical systems it is customary to assume external forces at all degrees of freedom, and the output is often defined as the entire displacement or velocity profile. Case 3 implies that in some cases the class of reduced order models that can be obtained by projection is very narrow. In heuristic methods the question is whether it is justified to look only at that narrow class.

4. THE INVERSE PROBLEM – PROJECTION CALCULATION

As was explained in section 3, eqs (8) are a set of *(2r+m+p)r* equations with *2nr* unknowns. Focusing on the case where *r=n-(m+ p)/2*, the first question is how many solutions exist, which is not obvious. Eqs

(8b-c) are linear, while (8a) and (8d) are quadratic but with a distinct structure, i.e. only cross terms between two groups of unknowns (L and R) appear. Another question is how to solve these equations. Using numerical methods, such as Newton-Raphson methods, one is not guaranteed to find all the solutions. Generic solvers for polynomial equations do get all the solutions, but the required computational effort increases rapidly. Instead, we develop in this section an analytical solution for the problem. The linear equations can be replaced by

$$L = B_r B^+ + X B_\perp \tag{9}$$

$$R = C^+ C_r + C_\perp Y \tag{10}$$

where B^+(m×n) is a left inverse of B, and $B_\perp$((n-m)×n) is a basis for the left null-space of B. Similarly, C^+(n×p) is a right inverse of C, and $C_\perp$(n×(n-p)) is a basis for the left null-space of C. X(r×(n-m)) and Y ((n-p)×r) are the new unknown matrices. Substituting them into (8a,d) and rearranging, the equivalent equations are

$$\begin{bmatrix} X & I_r \end{bmatrix} H_i \begin{bmatrix} Y \\ I_r \end{bmatrix} = 0 \qquad i = 1,2 \tag{11}$$

where

$$H_1 = \begin{bmatrix} B_\perp A C_\perp & B_\perp A C^+ C_r \\ B_r B^+ A C_\perp & B_r B^+ A C^+ C_r - A_r \end{bmatrix} \tag{12}$$

$$H_2 = \begin{bmatrix} B_\perp C_\perp & B_\perp C^+ C_r \\ B_r B^+ C_\perp & B_r B^+ C^+ C_r - I_r \end{bmatrix} \tag{13}$$

At this point we make two assumptions.

A4.1: $m=p$, i.e. H_1, H_2 are square.
A4.2: The matrix pencil λH_1-H_2 is regular (Kailath, 1980), i.e. there exist a scalar λ_0 such that $\lambda_0 H_1$-H is nonsingular.

From A4.2 we can assume, without loss of generality that H_1 is nonsingular, because if it is not, it can be replaced by $\lambda_0 H_1$-H_2 , which is then labeled as H_1. S and V are square real matrices given as

$$S = \begin{bmatrix} S_1 \\ \vdots \\ S_N \end{bmatrix} \qquad V = \begin{bmatrix} V_1 & \cdots & V_N \end{bmatrix} \tag{14}$$

such that

$$S H_1 V = I_{n-m+r} \tag{15}$$

$$S H_2 V = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_N \end{bmatrix} \tag{16}$$

and J_k is a real Jordan block, corresponding to the eigenvalue λ_k of the generalized eigenvalue problem

$$(\lambda H_1 - H_2) v = 0 \tag{17}$$

This result is a degenerate form of the Kronecker canonical form (Kailath, 1980), for the case of nonsingular H_1.

Lemma 4.1: Let si and vj be a row of S and a column of V respectively. Then $s_i H_1 v_j = s_i H_2 v_j = 0$ if

1) $s_i \in S_k$, $v_j \in V_l$, $k \neq l$, or
2) s_i is the i_k row of S_k, v_j is the j_k column of V_k, $J_k(i_k, j_k)$=0.

Proof: Immediate from (15)-(16).

Corollary: Let

$$S_r = \begin{bmatrix} S_{I\, r\times(n-m)} & S_{II\, r\times r} \end{bmatrix} \quad , \quad V_r = \begin{bmatrix} V_{I\,(n-m)\times r} \\ V_{II\, r\times r} \end{bmatrix} \tag{18}$$

consist of rows of S and columns of V that mutually satisfy the conditions of Lemma 1. If S_{II} and V_{II} are nonsingular, then

$$X = S_{II}^{-1} S_I \quad , \quad Y = V_I V_{II}^{-1} \tag{19}$$

is a solution of (8).

Lemma 4.2: If *r=n-m*, then
1) All the solutions of (11) are of the form (18)-(19).
2) The maximum number of real solutions to (8) is

$$\binom{n-m}{2(n-m)} = \frac{(2(n-m))!}{(n-m)!(n-m)!}$$

Proof: S and V are nonsingular, therefore any solution can be written as [X I_r]=X_1S, $[Y^T\ I_r]^T$=VY_1. Substituting into (11) we have $X_1 Y_1$=0 and $X_1 J Y_1$=0. Since in this case the dimension of H_1 and H_2 is *2(m-n)*, and since X_1, Y_1 have full rank, it follows that the columns of Y_1 form a basis for the nullspace of X_1, and therefore there exists a matrix T such that.

$$J Y_1 = Y_1 T \tag{20}$$

Hence Y_1 is a linear combination of n-m vectors which are generalized eigenvectors of J. For an eigenvalue λ_k with multiplicity N_k, those vectors includes only strings of successive generalized eigenvectors starting from the true eigenvector. From the structure of J it follows that V_r consists of n-m columns of V. The result for S_r follows similarly.

Maximum freedom in selecting the rows for S_r and the columns for V_r is obtained where all the eigenvalues of (H_1, H_2) are real and distinct. In that case the problem is selecting n-m numbers out of 2(n-m), hence part 2.

Suppose now that one solution to (8) is known. It satisfies (9)-(10), with certain X_0, Y_0.

$$L_0 = B_r B^+ + X_0 B_\perp \ , \ R_0 = C^+ C_r + C_\perp Y_0 \tag{21}$$

Any other solution can be written as

$$L = L_0 + \overline{X}B_\perp \;,\; R = R_0 + C_\perp \overline{Y} \qquad (22)$$

where $\overline{X} = X - X_0$, $\overline{Y} = Y - Y_0$. Substituting these expressions into (12)-(13) and rearranging, we obtain

$$\begin{bmatrix} \overline{X} & I_r \end{bmatrix} \begin{bmatrix} B_\perp AC_\perp & B_\perp AR_0 \\ L_0 AC_\perp & 0 \end{bmatrix} \begin{bmatrix} \overline{Y} \\ I_r \end{bmatrix} = 0 \qquad (23a)$$

$$\begin{bmatrix} \overline{X} & I_r \end{bmatrix} \begin{bmatrix} B_\perp C_\perp & B_\perp R_0 \\ L_0 C_\perp & 0 \end{bmatrix} \begin{bmatrix} \overline{Y} \\ I_r \end{bmatrix} = 0 \qquad (23b)$$

Notice that $\overline{X} = 0$, $\overline{Y} = 0$ is always a solution, which corresponds to the basis solution in (22). To see the relationship between (23) and (11), notice that

$$\begin{bmatrix} I_{n-m} & 0 \\ X_0 & I_r \end{bmatrix} H_1 \begin{bmatrix} I_{n-p} & Y_0 \\ 0 & I_r \end{bmatrix} = \begin{bmatrix} B_\perp AC_\perp & B_\perp AR_0 \\ L_0 AC_\perp & 0 \end{bmatrix}$$

$$\begin{bmatrix} I_{n-m} & 0 \\ X_0 & I_r \end{bmatrix} H_2 \begin{bmatrix} I_{n-p} & Y_0 \\ 0 & I_r \end{bmatrix} = \begin{bmatrix} B_\perp C_\perp & B_\perp R_0 \\ L_0 C_\perp & 0 \end{bmatrix}$$

Since the eigenvalues of (H_1, H_2) and of $(T_1H_1T_2, T_1H_2T_2)$ are the same, (23) and (11) are equivalent in case a solution exists. The simpler structure enables further insight into the problem. Assuming again that *m=p*, it follows that

$$\lambda(H_1, H_2) = \lambda(B_\perp R_0, B_\perp AR_0) \oplus \lambda(L_0 C_\perp, L_0 AC_\perp) \qquad (24)$$

Furthermore, the right eigenvectors corresponding to the *n-m* eigenvalues of the first group are of the form $[0\ v_i^T]^T$, and the left eigenvectors of the second group are $[0\ m_j]$. Dividing the eigenvalues along those groups, in the solutions described in Lemma 4.2, yields the basis solution $\overline{X} = 0$, $\overline{Y} = 0$.

5. GEOMETRICAL STUDY OF ORDER REDUCTION FROM SECOND TO FIRST

In this section we consider the simplest case of order reduction, i.e. second order to first order.

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -a_0 & -a_1 & 1 \\ \hline b_0 & b_1 & 0 \end{array}\right] \xrightarrow{P} \left[\begin{array}{c|c} -\alpha & \beta \\ \hline 1 & 0 \end{array}\right] \qquad (25)$$

The characteristic polynomial of the eigenvalue equation is of second order. From its coefficients it can be shown that a real solution exists if and only if

$$\begin{aligned} & b_1{}^2\alpha^2 + (a_1{}^2 - 4a_0)\beta^2 + (4b_0 - 2a_1b_1)\alpha\beta \\ & + (-2b_0b_1)\alpha + (4a_0b_1 - 2a_1b_0)\beta + b_0{}^2 \ge 0 \end{aligned} \qquad (26)$$

Eq. (26), with equality sign, is a conic sector in the α-β plane. Generically it can be written as

$$c_{\alpha\alpha}\alpha^2 + c_{\beta\beta}\beta^2 + 2c_{\alpha\beta}\alpha\beta + 2c_\alpha\alpha + 2c_\beta\beta + c_0 \ge 0 \qquad (27)$$

Its shape depends on the two parameters, δ and Δ, given by

$$\delta = \begin{vmatrix} c_{\alpha\alpha} & c_{\alpha\beta} \\ c_{\alpha\beta} & c_{\beta\beta} \end{vmatrix} \qquad \Delta = \begin{vmatrix} c_{\alpha\alpha} & c_{\alpha\beta} & c_\alpha \\ c_{\alpha\beta} & c_{\beta\beta} & c_\beta \\ c_\alpha & c_\beta & c_0 \end{vmatrix} \qquad (28)$$

Direct calculation and some algebraic manipulations lead to the following expressions

$$\delta = 4(b_0b_1a_1 - b_0{}^2 - a_0b_1{}^2) \quad , \quad \Delta = -\delta^2/4 \qquad (29)$$

Out of the six possible cases in general, only three are possible in our case.

Case 1: δ>0, Δ<0. α and β that satisfy the inequality (27), hence representing attainable PROM's, reside in an area on and outside an ellipse in the α-β plane.
Case 2: δ<0, Δ<0. α and β that satisfy the inequality (27), are located in the α-β plane between the two branches of a hyperbola.
Case 3: δ=0, Δ=0. The conic sector reduces in that case to a single straight line. Outside that line there exists a single solution while on it no solution exists, except for one point that has infinitely many solutions. An interesting observations is that δ=0 if and only if G(s) is non-minimal, and that the point with infinitely many solutions corresponds to a minimal realization of G(s).

The three cases are summarized in figures 1a-c. The circled point in each plot represents the optimal L_2 reduced order model. Its position on the borderline is not coincidental, and its meaning is discussed in section 6.

6. PROPERTIES OF THE OPTIMAL L_2 REDUCED ORDER MODEL

The L_2 optimal reduced order model (OROM) is the r-th order $G_r(s)$ which minimizes

$$J = \left\| G(s) - G_r(s) \right\|_2 \qquad (30)$$

In (Wilson, 1970), it was assumed that the OROM is a PROM, and equations defining the optimal L and R were given. Later, in (Hyland and Bernstein, 1985), no assumptions regarding $G_r(s)$ were made, and the matrices (A_r, B_r, C_r) were sought using direct optimization. The starting point is the augmented system

$$\dot{\tilde{x}}(t) = \begin{bmatrix} A & 0_{n\times n_r} \\ 0_{n_r\times n} & A_r \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} B \\ B_r \end{bmatrix} u(t) \qquad (31a)$$

$$\tilde{y}(t) = \begin{bmatrix} C & -C_r \end{bmatrix} \tilde{x}(t) \qquad (31b)$$

whose output is the error between the two systems. Its controllability and observability gramians are given by

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{B}\tilde{B}^T = 0 \qquad (32)$$

$$\tilde{P}\tilde{A} + \tilde{A}^T\tilde{P} + \tilde{C}^T\tilde{C} = 0 \qquad (33)$$

Where $(\tilde{A}, \tilde{B}, \tilde{C})$ denote the augmented matrices. The gramians are partitioned as

$$\tilde{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \tilde{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, Q_{11}, P_{11} \in R^{n \times n}$$

The following Theorem is the main result in (Hyland and Bernstein, 1985), given in a slightly different form.

Theorem 6.1: The reduced order model

$$\dot{x}_r = -P_2^{-1} P_{12}^T A Q_{12} Q_2^{-1} x_r - P_2^{-1} P_{12}^T B u \qquad (34a)$$

$$y_r(t) = C Q_{12} Q_2^{-1} x_r(t) \qquad (34b)$$

minimizes J in (30).

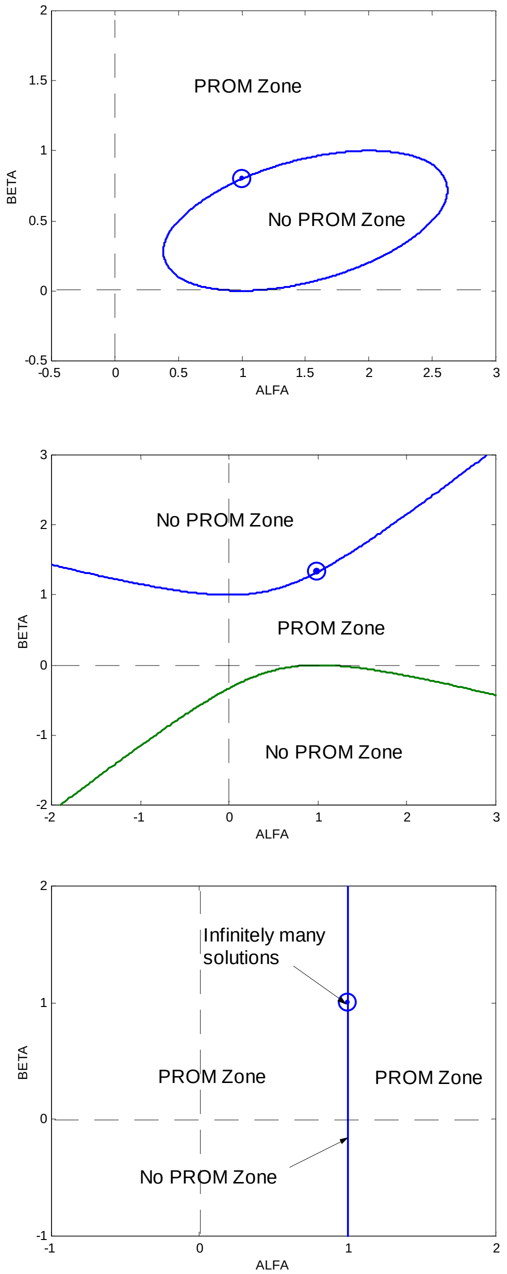

Figure 1: Zones of achievable 1st order PROM's for G(s)=(s+1)/(s^2+3s+1) (top), (s+1)/(s^2+s+1) (middle) and (s+1)/(s^2+2s+1) (bottom).

It can be shown that if (A_r, B_r, C_r) are as in (34), it follows that

$$-P_2^{-1} P_{12} Q_{12} Q_2^{-1} = I_r \qquad (35)$$

Hence the OROM is a PROM with L= $-P_2^{-1}P_{12}^T$, R=$Q_{12}Q_2^{-1}$. This fact was never emphasized in (Hyland and Bernstein, 1985), or in other works on optimal order reduction, and was considered as 'natural'. However the analysis in the section 3 indicates that in certain cases being a PROM is the exception rather than the obvious possibility. The definition of the OROM in Theorem 1 is not constructive since the calculation of the gramians requires (A_r, B_r, C_r). Furthermore, as was explained in section 2, L and R are realization dependent. A more explicit form of the result appears in (Hyland and Bernstein, 1985), where the generalized Lyapunov equations that have to be solved consist of realization invariant quantities. However (34) is very convenient from the PROM analysis point of view. In the remainder of this section an interesting PROM related result of OROM is discussed. As for notation, superscript * will be used to denote all quantities of the OROM. Before stating the main result of this section, we make two technical assumptions.

A6.1: $B_\perp R^*$ and $L^* C_\perp$ are nonsingular.

A6.2: $(\lambda B_\perp C_\perp - B_\perp A C_\perp) v_1 \notin Im(\lambda B_\perp R^* + B_\perp R^* Q_2 A_r^T Q_2^{-1})$ where λ and v_1 are an eigenvalue and an eigenvector of $(L^* C_\perp, -P_2^{-1} A_r^T P_2 L^* C_\perp)$ respectively.

Both assumptions are satisfied generically. A6.1 does not hold when (A, B, C) is non-minimal, and indeed in those cases there are infinitely many projections leading to the same lower order realization. A6.2 will be used in the proof of the following Theorem.

Theorem 6.2: For *r=n-m*, the system (A,B,C) and the OROM (Ar, Br, Cr) are related by a unique projection .

Proof: the upper left sub-block of eq.(32) is given by

$$A Q_{12} + Q_{12} A_r^T + B B_r = 0 \qquad (36)$$

Pre-multiplying eq. (36) by $B_\perp$, and post-multiplying it by Q_2^{-1}, recalling that $B_\perp B=0$ and the definition of R*, lead to

$$B_\perp A R^* = -B_\perp R^* Q_2 A_r^T Q_2^{-1} \qquad (37)$$

Similarly, the lower right sub-block of (33), together with the definition of L*, lead to

$$L^* A C_\perp = -P_2^{-1} A_r^T P_2 L^* C_\perp \qquad (38)$$

Substituting these relationships into eq. (24) yields

$$\lambda(H_1^*, H_2^*) = \lambda(B_\perp R^*, B_\perp A R^*) \oplus \lambda(L^* C_\perp, L^* A C_\perp)$$
$$= \lambda(B_\perp R^*, -B_\perp R^* Q_2 A_r^T Q_2^{-1}) \oplus \lambda(L^* C_\perp, -P_2^{-1} A_r^T P_2 L^* C_\perp)$$
$$= \lambda(I, -Q_2 A_r^T Q_2^{-1}) \oplus \lambda(I, -P_2^{-1} A_r^T P_2)$$
$$= \lambda(-A_r) \oplus \lambda(-A_r) \quad (39)$$

Hence every eigenvalue has multiplicity two. Notice that assumption A6.1 enables the move from the second line to the third one. Assumption A6.2 rules out the possibility of having two eigenvectors for each eigenvalue (this technical part of the proof is omitted due to space limitations). Hence all the eigenvalues have geometric multiplicity one. From Lemma 4.2 It follows that there is only one choice of $\tilde{Y}$, and it includes all the eigenvectors. Therefore $\overline{X} = 0$, $\overline{Y} = 0$ is the only solution in that case.

The same phenomenon has some other implications. Denoting the LHS of eqs. (23a-b), which are nominally zero, by F_i, define

$$f = \begin{bmatrix} \mathrm{Vec}(F_1) \\ \mathrm{Vec}(F_2) \end{bmatrix} \quad , \quad w = \begin{bmatrix} \mathrm{Vec}(\overline{X}) \\ \mathrm{Vec}(\overline{Y}) \end{bmatrix}$$

where the operator Vec stacks the columns of a matrix into a single vector. Then we have the following result.

Theorem 6.3: Let ρ be the number of eigenvalues of A_r, counted by geometrical multiplicity. Then the Jacobean matrix $\partial f / \partial w$, evaluated at w=w*=0, has row rank deficiency ρ.

Proof: The differentials of eqs. (23a-b), evaluated at the solution $\overline{X} = 0$, $\overline{Y} = 0$ are given as

$$dF_1 = d\overline{X}(-B_\perp R^* Q_2 A_r^T Q_2^{-1}) + (-P_2^{-1} A_r^T P_2 L^* C_\perp) d\overline{Y} \quad (40)$$

$$dF_2 = d\overline{X}(B_\perp R^*) + (L^* C_\perp) d\overline{Y} \quad (41)$$

Let *s* and *v* be a left eigenvector of $P_2^{-1} A_r^T P_2$ and a right eigenvector of $Q_2 A_r^T Q_2^{-1}$ respectively. Then

$$s dF_1 v = s d\overline{X}(-B_\perp R^* Q_2 A_r^T Q_2^{-1}) v + s(-P_2^{-1} A_r^T P_2 L^* C_\perp) d\overline{Y} v$$
$$= -s d\overline{X} B_\perp R^* v \lambda - \lambda s L^* C_\perp d\overline{Y} v$$
$$= -\lambda s dF_2 v \quad (42)$$

The Jacobian matrix is given by

$$\frac{\partial f}{\partial w} = \begin{bmatrix} (-B_\perp R^* Q_2 A_r^T Q_2^{-1})^T \otimes I_r & I_r \otimes (-P_2^{-1} A_r^T P_2 L^* C_\perp) \\ (B_\perp R^*)^T \otimes I_r & I_r \otimes (L^* C_\perp) \end{bmatrix} \quad (43)$$

The translation the result in (42) to this matrix is that

$$\begin{bmatrix} v^T \otimes s & \lambda v^T \otimes s \end{bmatrix} \frac{\partial f}{\partial w} = 0 \quad (44)$$

Since there are ρ eigenvalues, there are ρ linear combinations of the rows which are zero. Hence the rank deficiency is ρ.

Remark 6.1 : Since eqs. (11) and (23) are related by a constant linear transformation, the Jacobian of (11) has the same rank.

Rank deficiency of the Jacobian matrix is a necessary condition for G_r*(s) to be on the boundary of the set of reduced order systems which are attainable OROM's of a given G(s). In the second order to first order case, it can be shown that the singularity of the Jacobian is also a sufficient condition. Therefore the OROM is always on the boundary, as is shown in Figures 1a-c. To avoid any confusion, we stress that if the OROM were just the optimal PROM, its location on the boundary of the set to which it is confined would not have been surprising. However the OROM is the optimum of all models, PROM's or not, yet it is located on the boundary between the two sets.

7. CONCLUTIONS

The properties of the projection relationship between two models of different orders have been investigated. An algorithm to calculate all the projections that relate the two models, as well as conditions for the existence and uniqueness, have been presented. It was shown that in case there is a finite number of projections that lead to the same reduced order model, the optimal L_2 model is obtained by a single projection. Furthermore, in such cases the solution has a singular Jacobian matrix, indicating that the model is on the boundary of the attainable models zone.

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