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# OPTIMAL CONTROLLER TUNING FOR NONLINEAR PROCESSES

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Abstract: This work proposes a systematic methodology for the optimal selection of controller parameters, in the sense of minimizing a performance index which is a quadratic function of the tracking error and the control effort. The performance index is calculated explicitly as an algebraic function of the controller parameters by solving a Zubov-type partial differential equation. Standard nonlinear programming techniques are then employed for the calculation of the optimal values of the controller parameters. The solution of the partial differential equation is also used to estimate the closed-loop stability region for the chosen values of the controller parameters. The proposed approach is illustrated in a chemical reactor control problem. *Copyright © 2002 IFAC*

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## 1.INTRODUCTION

Over the last two decades, significant research efforts and activity have concentrated on the controller synthesis problem for nonlinear processes. The primary objective was to overcome the performance limitations associated with linear controller design methods based on linearized dynamic process models, and instead derive feedback control laws capable of directly coping with process nonlinearities. In this direction, a significant body of research results have been reported on the nonlinear controller synthesis problem that led to explicit, concrete and transparent control schemes and algorithms (Isidori, 1989; Nijmeijer and Van der Schaft, 1990). In all the aforementioned approaches, the tuning of the available controller parameters is based on trial-and-error and heuristic approaches, inevitably resorting to extensive dynamic simulations and/or costly experiments. The proposed approach aims at the development of a systematic way to optimally choose the tuneable parameters of a nonlinear control system, when in addition to the traditional closed-loop performance specifications (stability, fast and smooth set-point tracking, disturbance rejection, etc.) optimality is also requested with respect to a physically meaningful performance index. The formulation of the optimization problem presupposes a fixed-structure controller whose parameters must be optimally selected by minimizing an appropriately defined performance index of quadratic nature, that penalizes both the set-point tracking error as well as excessive input efforts. The optimization problem reduces to a finite-dimensional static optimization problem, since the value of the performance index can be explicitly calculated on the basis of a Lyapunov function which is the solution of a Zubov-type PDE (Margolis and Vogt, 1963; Zubov, 1964; Kalman and Bertram,1960). Moreover, for the optimally chosen controller parameters, an explicit estimate of the size of the closed-loop stability region can be obtained on the basis of the Lyapunov function.

The next section outlines the proposed general methodology for optimal controller tuning. In the following section, numerical results are presented, which evaluate the performance of the proposed approach in a representative chemical engineering example.

## 2.PROPOSED APPROACH

Let's consider a nonlinear system with the following state-space representation.

$$\begin{aligned} \dot{x} &= f(x,u) \\ e &= h(x) \end{aligned} \qquad (1)$$

where $x \in R^n$ is the vector of state variables, $u \in R$ the input variable, $e \in R$ the output variable and $f: R^n \times R \to R^n$, $h: R^n \to R$ real analytic vector and scalar functions respectively. Without loss of generality, it is assumed that the origin $x_0 = 0$ is the reference equilibrium point that corresponds to zero input and that the output map vanishes at the origin: $f(0,0) = 0$ and $h(0) = 0$. The problem of local output regulation involves the design of a feedback controller, which ensures that the resulting closed-loop system is locally asymptotically stable at the origin, and the regulated output e(t) asymptotically decays to 0 as $t \to \infty$.

In order to accomplish the above task, we consider controllers, which are typically modeled by equations of the following form:

$$\begin{aligned} \dot{\xi} &= \eta(\xi, e; p) \\ u &= \theta(\xi, e; p) \end{aligned} \qquad (2)$$

where $\xi \in R^v$ is the controller's state vector, $p \in P$ represents the m-th dimensional vector of controller parameters, P the admissible parameter space which is assumed to be a compact subset of $R^m$, $\eta: R^v \times R \times R^m \to R^v$ a real analytic vector function with $\eta(0,0;p) = 0$ and $\theta: R^v \times R \times R^m \to R$ a real analytic scalar function with $\theta(0,0;p) = 0$. The resulting closed loop system is:

$$\begin{aligned} \dot{x} &= f(x, \theta(\xi, h(x); p)) \\ \dot{\xi} &= \eta(\xi, h(x); p)) \end{aligned} \qquad (3)$$

Consider now the following quadratic performance index defined by:

$$\begin{aligned} J(p) &= \int_0^\infty \{[e(t)]^2 + \rho[u(t)]^2\} dt \\ &= \int_0^\infty \{[h(x(t))]^2 + \rho[\theta(\xi(t), h(x(t)); p)]^2\} dt \end{aligned} \qquad (4)$$

which represents a rather natural choice. It contains a quadratic error term for output regulation and a quadratic input penalty term with a relative weight ρ. Notice that since a fixed-structure controller (2) is always assumed throughout the present study, the performance functional J(p) can be naturally viewed as a function of the parameter vector p, so the problem of optimal tuning involves finding the values of the parameter p, which minimize the performance index J(p). For the closed-loop system (3) and the associated performance index (4), one may define the augmented state vector $\bar{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}$, the vector function

$$F(\bar{x}; p) = \begin{bmatrix} f(x, \theta(\xi, h(x); p)) \\ \eta(\xi, h(x); p) \end{bmatrix}$$

and the positive-definite scalar function

$$Q(\bar{x}; p) = [h(x)]^2 + \rho[\theta(\xi, h(x); p)]^2 .$$

Under this notation, the closed loop system and the associated performance index can be denoted as

$$\dot{\bar{x}} = F(\bar{x}; p)$$

and $$J(p) = \int_0^\infty Q(\bar{x}(t); p) dt$$

respectively.

If we assume that the closed-loop system is asymptotically stable around $\bar{x}=0$ and there is a function $V(\bar{x}; p)$, where $V: R^{n+v} \times R^m \to R$ with $V(0; p) = 0$, which satisfies the following linear first-order non-homogenous PDE:

$$\frac{\partial V}{\partial \bar{x}} F(\bar{x}; p) = -Q(\bar{x}; p) \qquad (5)$$

then:

$$\frac{dV}{dt} = \frac{\partial V(\bar{x}; p)}{\partial \bar{x}} F(\bar{x}; p) = -Q(\bar{x}; p) < 0 \qquad (6)$$

and:

$$\begin{aligned} J(p) &= \int_0^\infty Q(\bar{x}(t); p) dt = -\int_0^\infty \frac{dV}{dt} dt \\ &= V(\bar{x}(0); p) - V(\bar{x}(\infty); p) \\ &= V(\bar{x}(0); p) \end{aligned} \qquad (7)$$

If in addition the above PDE (5) admits a positive-definite solution $V(\bar{x}; p)$ in a neighborhood of $\bar{x}=0$, then $V(\bar{x}; p)$ is a Lyapunov function for the system $\dot{\bar{x}} = F(\bar{x}; p)$ (Khalil, 1991).

The above construction represents Zubov's method for the calculation of Lyapunov functions for nonlinear autonomous dynamical systems (Margolis and Vogt, 1963; Zubov, 1964). With respect to PDE (5), one needs to address the following important issues:

*(i) Existence and Uniqueness of Solutions:*

The origin $\bar{x}=0$ is a characteristic point of Zubov's PDE (5) and therefore (5) is singular at the origin. (Courant and Hilbert, 1962) However, it can be proved that if $\partial F/\partial\bar{x}(0;p)$ is Hurwitz, then equation (5) admits a unique, analytic solution $V(\bar{x};p)$ in a neighborhood of $\bar{x}=0$ (Zubov, 1964).

*(ii) Solution Method:*

Since $F(\bar{x};p)$, $Q(\bar{x};p)$, as well as the solution $V(\bar{x};p)$ are locally analytic, it is possible to calculate the solution $V(\bar{x};p)$ in the form of a multivariate Taylor series around $\bar{x}=0$. The method involves expanding $F(\bar{x};p)$, $Q(\bar{x};p)$, as well as the solution $V(\bar{x};p)$ in Taylor series and equating the Taylor coefficients of the same order of both sides of PDE (5). This procedure leads to linear algebraic equations with respect to the Taylor coefficients of the unknown solution $V(\bar{x};p)$ (See Remark 2 below). Moreover, it should be pointed out that the proposed series solution method can be easily implemented with the aid of a symbolic software package such as MAPLE. In particular, a simple MAPLE code has been written (Tseronis, 2001), which automatically calculates the Taylor coefficients of the unknown solution of Zubov's PDE (5).

*(iii) Local positive-definiteness of the solution $V(\bar{x};p)$:*

Given that the function $Q(\bar{x};p)$ is positive-definite, it can be proved (Zubov, 1964) that, if $\partial F/\partial\bar{x}(0;p)$ is Hurwitz, $V(\bar{x};p)$ is locally positive-definite, in the sense that the second order terms of the series expansion form a positive-definite quadratic form.

*(iv) Stability region estimates:*

Let $V^{(N)}(\bar{x};p)$ be the N-th order truncation of the Taylor series expansion of the solution $V(\bar{x};p)$ of Zubov's PDE (5). Moreover, let:

$$\Omega^{(N)}=\{\bar{x}\in R^{n+v} | \bar{x}\neq 0 \wedge \frac{dV^{(N)}(\bar{x};p)}{dt}=0\}$$

and:

$$C^{(N)}(p)=\min_{\bar{x}\in\Omega^{(N)}} V^{(N)}(\bar{x};p)$$

Then, it can be proven, that the set:

$$S^{(N)}(\bar{x};p)=\{\bar{x}\in R^{n+v} | V^{(N)}(\bar{x};p)\leq C^{(N)}(p)\}$$

is wholly contained in the stability region of the closed-loop system (4), thus providing an estimate of the size of its stability region. (Margolis and Vogt, 1963; Zubov, 1964)

The foregoing properties enable the approximate calculation of the performance index as a function of the controller parameters p using the approximate solution of Zubov's PDE (5), as well as an estimate of the stability region. The final step is the numerical solution of a nonlinear programming problem (Fletcher, 2000; Floudas, 1999) for the minimization of J(p) subject to inequality constraints that guarantee that $\partial F/\partial\bar{x}(0;p)$ is Hurwitz.

*Remark 1*: In the linear quadratic case, where the closed-loop dynamics is linear, of the form

$$\dot{\bar{x}}=A_{(p)}\bar{x}$$

and the performance index is quadratic of the form

$$J(p)=\int_0^\infty [\bar{x}(t)]^T Q_{(p)} [\bar{x}(t)]dt$$

the solution to the Zubov PDE is a quadratic function:

$$V(\bar{x};p)=\bar{x}^T V_{(p)}\bar{x}$$

where the matrix $V_{(p)}$ satisfies the Lyapunov matrix equation

$$A_{(p)}^T V_{(p)}+V_{(p)}A_{(p)}=-Q_{(p)}$$

It is well known (see e.g. Chen, 1984) that, if $A_{(p)}$ is Hurwitz and $Q_{(p)}$ is positive definite, the above matrix equation admits a unique solution $V_{(p)}$ which is positive definite. Moreover, that the performance index J(p) can be calculated from

$$J(p)=[\bar{x}(0)]^T V_{(p)} [\bar{x}(0)]$$

We see therefore, that the proposed method for calculating J(p) in nonlinear systems, is a direct generalization of standard results on linear-quadratic problems.

*Remark 2*: It is possible to derive recursion formulas for the application of the series solution method for Zubov's PDE (5). In this direction, it is convenient to use the following tensorial notation:

a) A lower index denotes the component of a vector function, whereas an upper index denotes the index of a variable with respect to which differentiation is performed. Thus, for the μ-th component $f_\mu(x)$ of a vector function f(x), we denote:

$$f_\mu^i=\frac{\partial f_\mu}{\partial x_i}(0),\ f_\mu^{ij}=\frac{\partial^2 f_\mu}{\partial x_i \partial x_j}(0)\text{, etc.}$$

b) The standard summation convention, where repeated upper and lower indices are summed up.

Using the above notation, the series solution method for the PDE (5) gives rise to the following recursion relations among the Taylor coefficients of the unknown and known functions:

- From matching of quadratic terms:

$$V^{\mu i_1}F_\mu^{i_2}+V^{\mu i_2}F_\mu^{i_1}=-Q^{i_1 i_2}\ ,\quad i_1,i_2=1,2,...,n$$

- From matching of third order terms:

$$\begin{aligned}&\left(V^{\mu i_1 i_2}F_\mu^{i_3}+V^{\mu i_1 i_3}F_\mu^{i_2}+V^{\mu i_2 i_3}F_\mu^{i_1}\right)\\&+\left(V^{\mu i_1}F_\mu^{i_2 i_3}+V^{\mu i_2}F_\mu^{i_1 i_3}+V^{\mu i_3}F_\mu^{i_1 i_2}\right)\\&=-Q^{i_1 i_2 i_3}\ ,\quad i_1,i_2,i_3=1,2,...,n\end{aligned}$$

- From matching of N-th order terms:

$$\sum_{L=1}^{N-1}\sum_{\binom{N}{L}} V^{\mu i_1\ldots i_L}F_\mu^{i_{L+1}\ldots i_N}=-Q^{i_1\ldots i_N}\ ,\quad i_1,...,i_N=1,...,n$$

where the second summation is over the $\binom{N}{L}$ possible combinations of the indices $i_1, i_2, \ldots, i_N$. Also, it should be noted that in all the above recursion relations, the dependence on the parameter p is not explicitly indicated, in order to simplify notation. Finally, it is important to observe that the recursion relations are all linear in the unknown coefficients and this enables the symbolic calculation using MAPLE.

## 3.ILLUSTRATIVE EXAMPLE

To illustrate the main aspects of the proposed parametric optimization approach, a representative chemical engineering example is considered next. In particular, an isothermal continuous stirred tank reactor (CSTR) is considered, where the series/parallel Van de Vusse reaction is taking place (Van de Vusse, 1964; Wright and Kravaris, 1992).

$$A \rightarrow B \rightarrow C$$
$$2A \rightarrow D$$

where the rates of formation of species A and B are given by:

$$r_A = -k_1 C_A - k_3 C_A{}^2$$
$$r_B = k_1 C_A - k_2 C_B$$

The reaction rate constants are considered to be: $k_1=50\ \mathrm{h}^{-1}, k_2=100\ \mathrm{h}^{-1}, k_3=10\ \mathrm{l/mol \cdot h}$. Under the assumption that the feed stream consists of pure A, the mass balance equations for species A and B lead to the following nonlinear dynamic process model:

$$\dot{C}_A = \frac{F}{V}(C_{A_0} - C_A) - k_1 C_A - k_3 C_A{}^2$$

$$\dot{C}_B = -\frac{F}{V} C_B + k_1 C_A - k_2 C_B \qquad (8)$$

where F is the inlet flow rate of A, V is the volume of the reactor that is considered to be constant during the operation, $C_A$ and $C_B$ are the concentrations of species A and B, respsctively, and $C_{A0}=10$ mol/l is the concentration of A in the feed stream. Our goal is to control $C_B$ at a certain set-point value by manipulating the dilution rate (F/V). The system is initially at steady state with $C_B$ at a constant set-point and then it is subjected to a step change in the set point. The final steady state values are:

$C_{A_s}=2.697$ mol/l, $C_{B_s}=1.05$ mol/l and $(F/V)_s=28.428$ l/h. Furthermore, defining deviation variables with respect to the final steady state values:

$x_1=C_A-2.697,\ x_2=C_B-1.05,\ u=F/V-28.428,$

the CSTR model (8) is put in the form of equation (1):

$$\dot{x}_1=f_1(x_1,x_2,u)$$
$$\dot{x}_2=f_2(x_1,x_2,u)$$
$$e=x_2$$

A simple linear static state-feedback control law is applied to the system:

$$u=-p_1x_1-p_2x_2$$

where $p_1$ and $p_2$ are the controller parameters (gains) which, according to the proposed method, must be optimally selected by minimizing the following performance index:

$$J(p_1,p_2) = \int_0^\infty \{[x_2(t)]^2 + \rho[u(t)]^2\}dt$$
$$= \int_0^\infty \{[x_2(t)]^2 + \rho[p_1x_1(t)+p_2x_2(t)]^2\}dt \qquad (9)$$

The above performance index can be calculated as follows:

$$J(p_1,p_2)=V(x_1(0),x_2(0);p_1,p_2) \qquad (10)$$

where $V(x_1,x_2;p_1,p_2)$ is the solution of Zubov's PDE:

$$\frac{\partial V}{\partial x_1} f_1(x_1,x_2,-p_1x_1-p_2x_2)+\frac{\partial V}{\partial x_2} f_2(x_1,x_2,-p_1x_1-p_2x_2)=$$
$$=-x_2{}^2-\rho(p_1x_1+p_2x_2)^2 \qquad (11)$$

The above PDE was solved symbolically using MAPLE, applying the series solution method outlined in the previous section, up to a certain truncation order N. The result was evaluated at the initial condition (steady state corresponding to the initial set-point value) and the function $V^{[N]}(x_1(0),x_2(0);p_1,p_2)$ was minimized using the nonlinear programming library of MAPLE (see Tseronis, 2001).

Figures 1 and 2 provide the optimal values of $p_1$ and $p_2$ as a function of the size of the step change in the set point, for various truncation orders N. The value of $\rho=10^{-5}$ was used for the weight coefficient and the final set point value was $C_{Bsp}=1.05$ in all calculations.

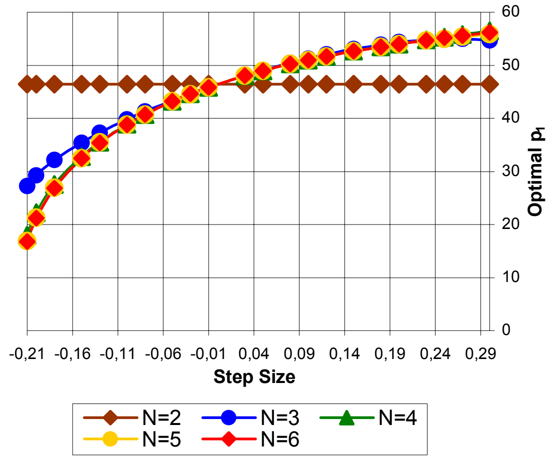


Fig. 1. Optimal values of $p_1$ as a function of the size of the step change in the set point. ($\rho=10^{-5}$)

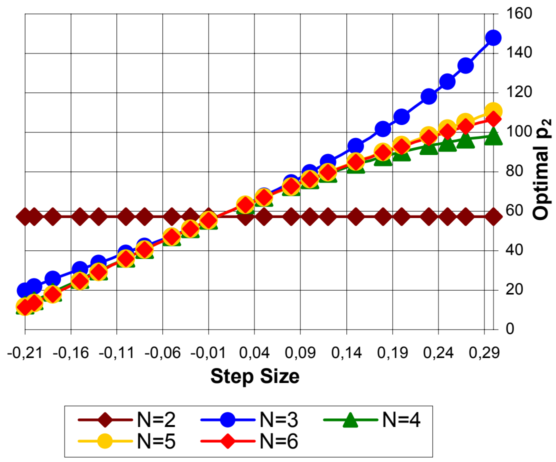


Fig. 2. Optimal values of $p_2$ as a function of the size of the step change in the set point. ($\rho=10^{-5}$)

The results for truncation order N=2 are exactly what would have been obtained from a linear approximation of the dynamics. The optimal values of $p_1$ and $p_2$ are, as expected, independent of the step size. This is not the case for higher truncation order N>2: the optimal $p_1$ and $p_2$ are strongly dependent upon the step size, as a result of the nonlinearity of the system.

The results also indicate numerical convergence for the optimal $p_1$ and $p_2$ values with increasing N. For the value of weight coefficient ρ and the range of step sizes considered, a truncation order N>4 provides a good approximation.

Figures 3 and 4 show the effect of the weight coefficient ρ on the optimal $p_1$ and $p_2$ values, for a step change in the set point from $C_{Bsp}$=1.2 to $C_{Bsp}$=1.05 (step size of –0.15). Calculations were performed with truncation order N=5.

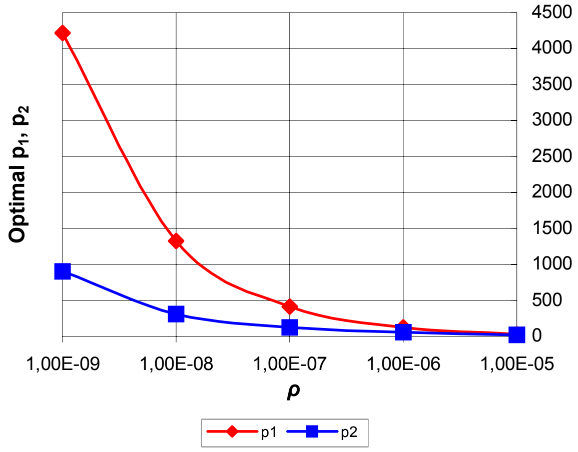


Fig. 3. Optimal $p_1$, $p_2$ as a function of ρ, in the range $10^{-9}$-$10^{-5}$ for a step change in the set point from 1.2 to 1.05 (step size=-0.15).

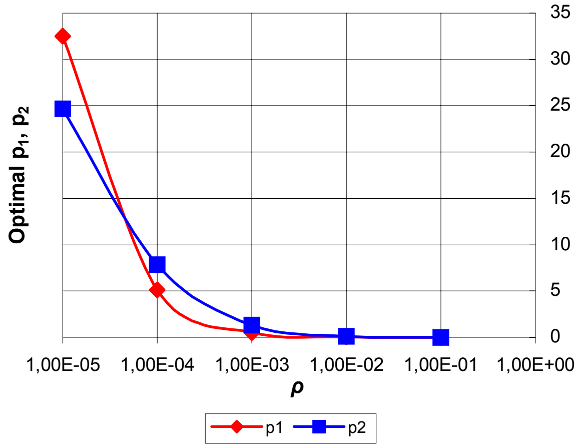


Fig. 4. Optimal $p_1$, $p_2$ as a function of ρ, in the range $10^{-5}$-$10^{0}$ for a step change in the set point from 1.2 to 1.05 (step size=-0.15).

As expected, optimal $p_1$ and $p_2$ are strongly dependent on the weight coefficient ρ. In fact, the numerical results seem to indicate that the optimal $p_1$ and $p_2$ tend to infinity as $\rho \rightarrow 0$ and to zero as $\rho \rightarrow \infty$.

Figures 5 and 6 depict the optimal closed-loop responses for three representative values of the weight coefficient, $\rho=10^{-7}$, $\rho=10^{-5}$, $\rho=10^{-3}$. As expected, the small value of ρ gives very fast output response but physically unrealistic values of the dilution rate. On the other hand, the large value of ρ gives unnecessarily slow response.

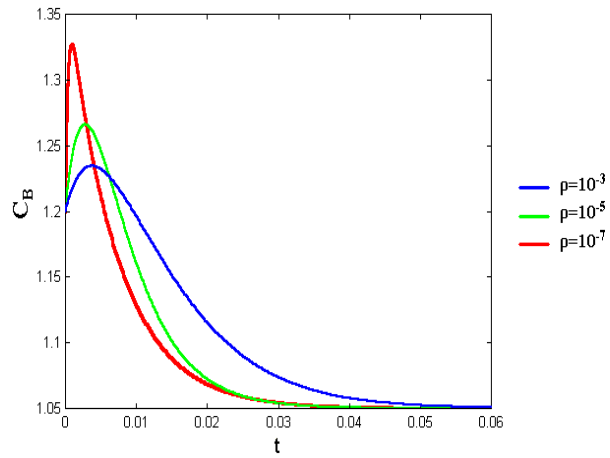


Fig. 5. Optimal output responses to a step change in the set point from 1.2 to 1.05 ($\rho=10^{-7}$,$10^{-5}$, $10^{-3}$).

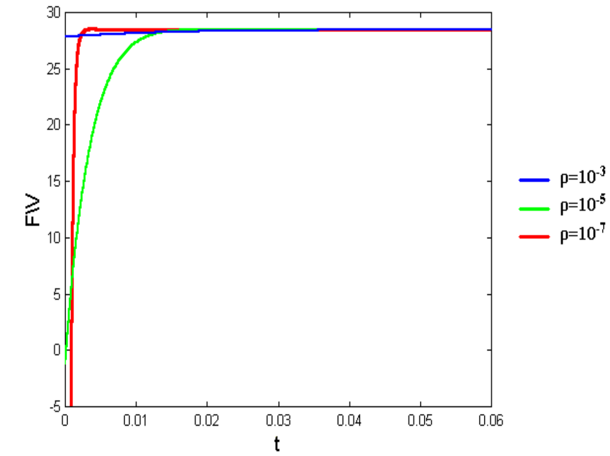


Fig. 6. Optimal input responses to a step change in the set point from 1.2 to 1.05 ($\rho=10^{-7}$,$10^{-5}$, $10^{-3}$).

To complete the simulation study, estimates of the stability region have been obtained using the method outlined in the previous section. For a step change in the set-point from 1.2 to 1.05, weight coefficient $\rho=10^{-5}$ and truncation order N=6, the optimal gains are $p_1$=50.99 and $p_2$=76.27. Figure 7 illustrates the method of estimating the stability region in this case.

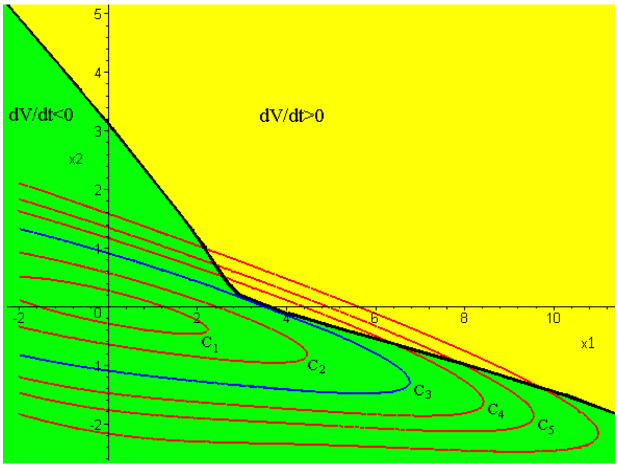


Fig. 7. Geometric interpretation of the method for estimating the stability region.

The figure shows the curve $\frac{dV}{dt}=0$, which separates the $x_1$-$x_2$ plane into two regions, one with $\frac{dV}{dt}<0$ and another with $\frac{dV}{dt}>0$. It also shows contours of the function $V=V^{[6]}(x_1,x_2;p_1=50.99,p_2=76.27)$. The estimate of the stability region is exactly the interior of the contour of $V^{[6]}(x_1,x_2)=C_3$, which is tangent to the curve $\frac{dV}{dt}=0$ and is wholly contained in the region with $\frac{dV}{dt}<0$. Because $C_1<C_2<C_3<C_4<C_5$, the value $C_3$ is exactly the smallest C-value on the curve $\frac{dV}{dt}=0$, as requested by the method.

Figure 8 depicts the estimates of the stability region for $p_1$=50.99 and $p_2$=76.27 and for different truncation orders, N=2, 4, 6, 8, 10, 12.

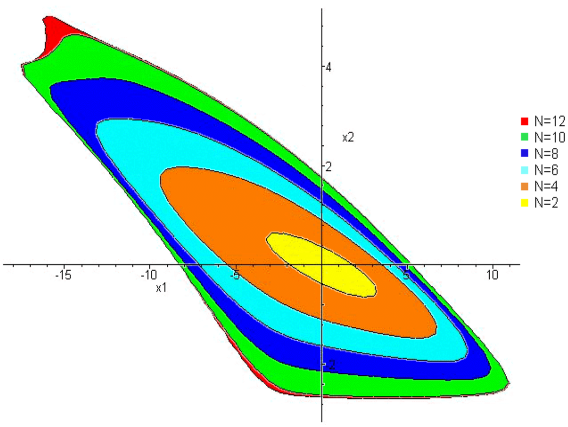


Fig. 8. Stability region estimates for different truncation orders ($p_1$=50.99, $p_2$=76.27)

For N=2, one obtains the standard quadratic estimate of the stability region (Khalil, 1991), which is a rather conservative estimate. The estimate for N=4 is a superset of the one for N=2, the estimate for N=6 is a superset of the one for N=4, etc. With increasing N, the results seem to indicate numerical convergence to a limiting region, which corresponds to the stability region estimate that would have been obtained if an exact solution of Zubov's PDE were available.

## REFERENCES


Chen, C.T. (1984), *Linear System Theory and Design*, Holt, Rineheart and Winston, New York.

Courant, R. and Hilbert, D. (1962), *Methods of Mathematical Physics*, *Volume II*, John Wiley & Sons, New York.

Fletcher, R. (2000), *Practical Methods of Optimization*, John Wiley & Sons, New York.

Floudas, C.A. (1999), *Deterministic Global Optimization*, Kluwer, New York.

Isidori, A. (1989), *Nonlinear Control Systems: An Introduction*, Springer-Verlag, New York.

Kalman, R.E. and Bertram, J.E. (1960), Control system analysis and design via the second method of Lyapunov, *J. Basic Eng. Trans., ASME*, **82**, 371.

Khalil, H.K. (1991), *Nonlinear Systems*, Prentice Hall, Englewood Cliffs, New Jersey.

Margolis, S.G. and Vogt, W.G. (1963), Control engineering applications of V.I. Zubov's construction procedure for Lyapunov functions, *IEEE Trans. Autom. Contr.*, **8**, 104.

Nijmeijer, H. and Van der Schaft, A.J. (1990), *Nonlinear Dynamical Control Systems*, Springer Verlag, Berlin, Germany.

Tseronis, C. (2001), Optimization of Quadratic Performance Indexes for Nonlinear Control Systems, Master's Thesis, University of Patras, Greece. (in greek)

Van de Vusse, J.G. (1964), Plug-flow reactor versus tank reactor, *Chem. Eng. Sci.*, **19**, 994.

Wright, R.A. and Kravaris, C. (1992), Nonminimum-phase compensation for nonlinear processes, *AICHE J.*, **38**, 26.

Zubov, V.I. (1964), *Methods of A.M. Lyapunov and Their Applications*, Noordhoff LTD, Groningen.