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ABOUT POLY-QUADRATIC STABILITY AND SWITCHED SYSTEMS

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Abstract: In this paper, a link between Poly-Quadratic stability and stability of a class of Hybrid systems is established. This class concerns the class of linear switched systems. Poly-Quadratic stability aims to check asymptotic stability of a polytopic system by mean of polytopic quadratic Lyapunov functions. The necessary and sufficient condition of Poly-Quadratic stability proposed in Daafouz and Bernussou (2001) is shown to be immediately applicable to a class of switched control and observer design problems. Chaos synchronization for which the transmitter is described by a piecewise linear map is presented as an application.

Keywords: Switched Discrete Time Systems, Polytopic Lyapunov functions, Switched control, Switched observer, Chaos synchronization, $\mathcal{L}$inear $\mathcal{M}$atrix $\mathcal{I}$nequalities ($\mathcal{LMI}$).

1. INTRODUCTION

In recent years, the study of hybrid dynamical systems has received a growing attention in control theory and practice. Switched systems are an important class of hybrid systems consisting of a family of continuous (or discrete) time subsystems and a rule that orchestrates the switching between them. A survey of basic problems, and recent results, in stability and design of switched systems is given in Liberzon and Morse (1999). Among these recent results, many contributions proposed to analyze stability of arbitrary switched systems use some conservative arguments to conclude. The most pessimistic assume that there exists a single Lyapunov function, or that the dynamical matrices commute. Even if these conditions are easily tractable, they can be used only in a very few applications.

More recently, less conservative conditions have been proposed. Piecewise Lyapunov function based result are proposed in Johansson and Rantzer (1998), Mignone et al. (2000). Discrete-time state description is used in Rubensson et al. (2000), to derive a set of LMI based stability conditions for a continuous-time hybrid system provided a bound on duty times is satisfied.

In this paper, we look for less conservative LMI based conditions for stability analysis and control (or observer) design for linear switched systems. Compared with Mignone et al. (2000), Johansson and Rantzer (1998), Rubensson et al. (2000), this new approach involves a Poly-Quadratic stability concept. This paper is organized as follows. In the next section, stability analysis of arbitrary switched systems is addressed. The necessary and sufficient condition of Poly-Quadratic stability proposed in Daafouz and Bernussou (2001) is shown to be immediately applicable. Section 3 is dedicated to state feedback and static output feedback switched control design. In Section 4, switched observer design is investigated and chaos synchronization is presented as an application. We end the paper by some illustrative examples and a conclusion. Due to space limitations, all the proofs are ommitted.

Notations: We use standard notations throughout the paper. M' is the transpose of the matrix M. $M > \mathbf{0}$ ($M < \mathbf{0}$) means that M is positive definite (negative definite). $\mathbf{0}$ and $\mathbf{I}$ denote the null matrix and the identity matrix with appropriate dimensions.

2. STABILITY ANALYSIS

Consider the autonomous switched system given by

$$x_{k+1} = A_\alpha x_k \tag{1}$$

where $\{A_i : i \in \mathcal{E}\}$ is a family of matrices parameterized by an index set $\mathcal{E} = \{1, 2, ..., N\}$ and $\alpha : \mathbb{N} \to \mathcal{E}$ is a piecewise constant function of time k called a switching signal. At a given time k, the value of $\alpha(.)$, denoted α for simplicity reasons, might depend on k or x_k, or both, or may be generated by any other hybrid scheme. We assume that the value of $\alpha(.)$ is unknown a priori. This class of systems is one of the most commonly treated in the literature. One among the basic problems reported in Liberzon and Morse (1999) concerns finding conditions which guarantee that the switched system (1) is asymptotically stable for any switching signal. In this section, we use the concept of Poly-Quadratic stability to solve this problem. This reduces to check stability by mean of particular quadratic Lyapunov functions taking into account the switching nature of this system. To recall this concept, consider a dynamical discrete time system

$$x_{k+1} = \mathcal{A}(\xi_k) x_k \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $\xi_k \in \Xi \subset \mathbb{R}^N$ is a unknown but bounded time-varying parameter. The structure of the dynamical matrix $\mathcal{A}$ is assumed to depend in a polytopic way on the parameter ξ_k:

$$\mathcal{A}(\xi_k) = \sum_{l=1}^{N} \xi_k^l \tilde{A}_l, \qquad \xi_k^l \geq 0, \qquad \sum_{l=1}^{N} \xi_k^l = 1 \tag{3}$$

where $\tilde{A}_l$ are given constant matrices called vertices. To check asymptotic stability of system (2), Poly-Quadratic stability uses Lyapunov function with a polytopic structure similar to that of the system description:

$$V(x_k, \xi_k) = x_k' \mathcal{P}(\xi_k) x_k, \quad \text{with} \quad \mathcal{P}(\xi_k) = \sum_{l=1}^{N} \xi_k^l P_l \tag{4}$$

where P_l are symmetric positive definite constant matrices of appropriate dimension. If such a positive definite Lyapunov function $V(x_k, \xi_k)$ exists and its difference:

$$\Delta V(x_k, \xi_k) = V(x_{k+1}, \xi_{k+1}) - V(x_k, \xi_k)$$

is negative definite along the solutions of (2), then the origin of the system given by (2) is globally asymptotically stable.

To check asymptotic stability of the switched system (1) under arbitrary switching laws, notice that the switched system (1) can be viewed as a polytopic system with the particularity that the allowable values for the dynamical matrix are those corresponding to the vertices of the polytope. The components of the parameter vector ξ_k of the polytopic representation (2) appear as indicator functions given by:

$$\xi_k^l = \begin{cases} 1 & \text{when the switched system is described} \\ & \text{by the matrix } A_l \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

with $l \in \mathcal{E}$ and

$$\xi_k = \left[\, \xi_k^1, \; ..., \; \xi_k^N \,\right]'$$

Then the autonomous switched system (1) can also be written as

$$x_{k+1} = \sum_{l=1}^{N} \xi_k^l A_l x_k \tag{6}$$

Using this fact, the stability condition given in Daafouz and Bernussou (2001) is adapted to the case of switched systems.

Proposition 1. There exists a Lyapunov function of the form (4) whose difference is negative definite proving asymptotic stability of the system (1) if and only if there exist N symmetric matrices $S_1, ..., S_N$ and N matrices $G_1, ..., G_N$ satisfying

$$\begin{bmatrix} G_i + G_i' - S_i & G_i' A_i' \\ A_i G_i & S_j \end{bmatrix} > \mathbf{0} \qquad \forall (i,j) \in \mathcal{E} \times \mathcal{E} \tag{7}$$

The Lyapunov function is then given by:

$$V(x_k, \xi_k) = x_k' \big(\sum_{l=1}^{N} \xi_k^l P_l \big) x_k \quad \text{with} \quad P_l = S_l^{-1}$$

3. SWITCHED CONTROL DESIGN

3.1 *Switched State feedback*

In this section, we consider switched discrete time systems given by

$$x_{k+1} = A_\alpha x_k + B_\alpha u_k \tag{8}$$

where $x_k \in \mathbb{R}^n$ is the state, u_k is the control. α is the switching signal defined as previously. Moreover, we assume that the switching signal is unknown a priori but available in real-time.

The stabilizing state feedback control problem is to find

$$u_k = K_\alpha x_k, \tag{9}$$

ensuring stability of the corresponding closed loop switched system

$$x_{k+1} = \big(A_\alpha + B_\alpha K_\alpha\big)x_k \tag{10}$$

The following Theorem gives sufficient conditions to build a stabilizing switched state feedback controller.

Theorem 1. If there exist symmetric matrices S_i, matrices G_i, and R_i, $\forall i \in \mathcal{E}$, such that $\forall (i,j) \in \mathcal{E} \times \mathcal{E}$:

$$\begin{pmatrix} G_i + G_i' - S_i & (A_i G_i + B_i R_i)' \\ A_i G_i + B_i R_i & S_j \end{pmatrix} > \mathbf{0} \tag{11}$$

then the state feedback control given by (9) with

$$K_i = R_i G_i^{-1} \quad \forall i \in \mathcal{E} \tag{12}$$

stabilizes asymptotically the system (8).

3.2 *Switched Static output feedback*

We consider the class of switched systems given by

$$\begin{aligned} x_{k+1} &= A_\alpha x_k + B_\alpha u_k \\ y_k &= C_\alpha x_k \end{aligned} \tag{13}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input and $y_k \in \mathbb{R}^p$ is the output vector. α is the usual switching signal assumed unknown a priori but real time available.

The following assumption is made:

H_1: matrix C_i, $\forall i \in \mathcal{E}$ is of full row-rank,

Now, the control synthesis problem is related to the design of a switched output feedback control

$$u_k = K_\alpha y_k, \tag{14}$$

ensuring stability of the corresponding closed loop switched system:

$$x_{k+1} = \big(A_\alpha + B_\alpha K_\alpha C_\alpha\big)x_k \tag{15}$$

A sufficient condition to build such a control is proposed in the following Theorem.

Theorem 2. If there exist symmetric matrices S_i, matrices G_i, U_i and V_i, $\forall i \in \mathcal{E}$, such that $\forall (i,j) \in \mathcal{E} \times \mathcal{E}$:

$$\begin{pmatrix} G_i + G_i' - S_i & (A_i G_i + B_i U_i C_i)' \\ A_i G_i + B_i U_i C_i & S_j \end{pmatrix} > \mathbf{0} \tag{16}$$

and

$$V_i C_i = C_i G_i \quad \forall i \in \mathcal{E} \tag{17}$$

then the output feedback control given by (14) with

$$K_i = U_i V_i^{-1} \quad \forall i \in \mathcal{E} \tag{18}$$

stabilizes asymptotically the system (13).

In Theorem 2, the equality constraint (17) has to be satisfied by the slack variables G_i ($\forall i \in \mathcal{E}$). The Lyapunov matrices $P_i = S_i^{-1}$ ($\forall i \in \mathcal{E}$) are let free and have to satisfy the stability condition only. This means that reporting the constraint (17) on the slack variables G_i makes the conditions of Theorem 2 less conservative. Notice that conditions given in Theorem 2 can be adapted to the switched state feedback control design. This can be achieved by replacing in the corresponding condition (16) the matrices C_i by $C_i = \mathbf{I}$ $\forall i \in \mathcal{E}$. In this case, the constraint (17) is not required and we recover the result in Theorem 1.

4. SWITCHED OBSERVER DESIGN

Consider the switched hybrid system defined by (13). The aim in this section is to design a switched observer for this system. For this purpose, we look for a full order switched observer with the following structure:

$$\begin{aligned} \hat{x}_{k+1} &= A_\alpha \hat{x}_k + B_\alpha u_k + L_\alpha (y_k - \hat{y}_k) \\ \hat{y}_k &= C_\alpha \hat{x} \end{aligned} \tag{19}$$

where $\hat{x}_k \in \mathbb{R}^n$ is the estimated state and $\hat{y}_k \in \mathbb{R}^p$ is the output estimation. The switching signal α is assumed unknown a priori but available in real-time. The gain matrices L_i, $i \in \mathcal{E}$, have to be calculated such that the estimation error between the state of the switched system (13) and the state of the observer (19) is asymptotically stable. The convergence to the origin of the estimation error has to be independent of the initial conditions x_0 and $\hat{x}_0$, the input u_k and the switching signal α.

Let the reconstruction error be $\varepsilon_k = x_k - \hat{x}_k$, the error dynamic is given by:

$$\varepsilon_{k+1} = (A_\alpha - L_\alpha C_\alpha)\varepsilon_k. \tag{20}$$

The switched observer design reduces to the computation of the gain matrices L_i, $i \in \mathcal{E}$, ensuring the asymptotic stability for the switched system (20) under arbitrary switching signal. The following theorem, gives sufficient conditions to build such a switched observer.

Theorem 3. If there exist symmetric matrices S_i, matrices F_i and G_i solutions of:

$$\begin{bmatrix} G_i + G_i' - S_i & G_i' A_i - F_i' C_i \\ A_i' G_i - C_i' F_i & S_j \end{bmatrix} > 0, \quad \forall (i,j) \in \mathcal{E} \times \mathcal{E}, \tag{21}$$

then a switched observer (19) for system (13) exists and the resulting gains L_i are given by $L_i = G_i'^{-1} F_i'$.

A direct application of Theorem 3 concerns the chaos synchronization problem toward which a growing interest has been directed since the pioneering works outlined in Carroll and Pecora (1991). This is due to the fact that reproductibility of chaotic behaviors through synchronization has been found interesting for secure communications and has spurred numerous advancements in encoding, masking or encrypting information. Chaos synchronization consists in conveying a signal produced by a chaotic system called *drive* which forces a second system called *response* to produce identical chaotic motion in the sense that the difference between both state vectors decays to zero. Hence, synchronization can be formulated as a reconstruction and an observer design problem. One of the important survey on chaos synchronization dealing with an observer approach is Nijmeijer and Mareels (1997) and the related problem, in its general formulation, is an open problem.

Piecewise linear maps might exhibit chaotic motions and are frequently encountered to describe the dynamics of the *drive*. The synchronization scheme is given by:

$$\begin{aligned} &\begin{cases} x_{k+1} &= A_i x_k + B_i \\ y_k &= C_i x_k \end{cases} \\ &\begin{cases} \hat{x}_{k+1} &= A_i \hat{x}_k + B_i + L_i(y_k - \hat{y}_k) \\ \hat{y}_k &= C_i \hat{x}_k \end{cases} \end{aligned} \tag{22}$$

with

- x_k and x_{k+1} (*resp.* $\hat{x}_k$ and $\hat{x}_{k+1}$) are the n-dimensional state vectors of the *drive* (*resp. response*) at the discrete times k and $k+1$.
- the output vector $y_k = C_i x_k$ of dimension m constitutes the signal transmitted to the *response* for obtaining the synchronization.
- the state space $\mathbb{R}^n$ is partitioned into N regions R_i with $\bigcup_{i=1}^{i=N} R_i = \mathbb{R}^n$. A_i is the $n \times n$ dynamical matrix governing the dynamics of the system in region R_i. B_i is a vector of dimension n representing the constant part of the local characteristic of the map. All matrices depend on a scalar i. This scalar is related, with a one-to-one correspondence, to the region R_i visited at the discrete time k by x_k and constitutes the switching signal value unknown a priori but available in real time.
- L_i is a $n \times m$ matrix referred as *gain matrix*. The *response* equation is that of an observer and the synchronization scheme is called "observer configuration".

The results of Theorem 3 can be applied directly. In fact, define an indictor function

$$\xi_k = \left(\xi_k^1 \ \xi_k^2 ... \xi_k^N\right)'$$

as:

$$\xi_k^i = \begin{cases} 1 & \text{when the state } x_k \text{ is in the region } R_i \\ 0 & \text{otherwise} \end{cases}$$

for all $i = 1, ..., N$. From (22), the synchronization error equation verified by $\varepsilon_k = x_k - \hat{x}_k$ is:

$$\varepsilon_{k+1} = \mathcal{A}(\xi_k)\varepsilon_k = \sum_{l=1}^{N} \xi_k^l (A_l - L_l C_l)\varepsilon_k \tag{23}$$

The chaos synchronization problem reduces to find the gain matrices L_l, $l \in \mathcal{E}$, such that the switched system (23) is asymptotically stable. This problem is the same as the one discussed previously for the switched observer design and for which a solution is proposed in Theorem 3. Hence, conditions of this Theorem are sufficient for chaos synchronization and less conservative than the single Lyapunov function approach used in Millerioux and Daafouz (2001).

5. ILLUSTRATIVE EXAMPLES

5.1 *Example 1*

This first example is an application of the switched state feedback conditions to stabilize a switched system formed by two stable subsystems but which, under arbitrary switching signal, may be unstable.

Consider a switched system given by

$$x_{k+1} = A_\alpha x_k + B_\alpha u_k \tag{24}$$

where $\{A_i : i \in \mathcal{E}\}$ and $\{B_i : i \in \mathcal{E}\}$ are a family of matrices parameterized by an index set $\mathcal{E} = \{1, 2\}$ and

$$A_1 = \begin{bmatrix} 0.0094 & 0.3010 \\ -3.0098 & 0.0094 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0.0094 & 3.0098 \\ -0.3010 & 0.0094 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}'$$

The switching signal $\alpha : \mathbb{N} \to \mathcal{E}$ is an arbitrary piecewise constant function of time k. In the autonomous case ($u_k = 0$), clearly both subsystems are stable and the matrices A_1 and A_2 have the same eigen values: $0.0094 \pm 0.9518i$.

However, if the switching signal is characterized by

$$\alpha = \begin{cases} 1 & \text{if } x_k^1 x_k^2 \geq 0, \quad \text{with} \quad x_k = [x_k^1 \ \ x_k^2]' \\ 2 & \text{otherwise} \end{cases} \tag{25}$$

then the result is unstable as can be verified in this case by the trajectories of the switched system (Thick line) depicted in Fig. 1 with an initial state vector $x_0 = \begin{bmatrix} 0 & 0.1 \end{bmatrix}'$. Dashed and solid lines represent the trajectories of the LTI subsystems $x_{k+1} = A_1 x_k$ and $x_{k+1} = A_2 x_k$.

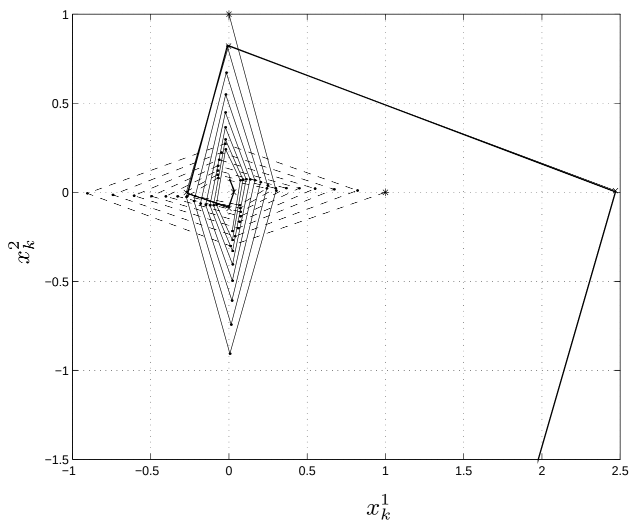

Fig. 1. Trajectory of the switched system (24) with the switching signal (25) (Thick line). Initial state vector $x_0 = \begin{bmatrix} 0 & 0.1 \end{bmatrix}'$.

The aim is to build a switched state feedback control

$$u_k = K_\alpha x_k \tag{26}$$

which stabilizes the closed loop system for arbitrary switching signal α. This cannot be achieved using a single quadratic Lyapunov based approach ($x_k' P x_k$, with P a constant matrix). In fact the corresponding LMI based conditions are found to be unfeasible. Using conditions of Theorem 1, we find a feasible solution and the switched state feedback is then given by (26) with

$$K_1 = \begin{bmatrix} -0.0172 & -0.3010 \end{bmatrix},\ K_2 = \begin{bmatrix} -0.0102 & -3.0098 \end{bmatrix}$$

The behaviour of the closed loop switched system with a switching signal given by (25) is shown in Fig. 2.

5.2 *Example 2*

This second example illustrates the conservatism improvement of the Poly-Quadratic stability based condition.

Consider a switched system given by

$$\begin{aligned} x_{k+1} &= A_\alpha x_k + B_\alpha u_k \\ y_k &= C_\alpha x_k \end{aligned} \tag{27}$$

where $\{A_i : i \in \mathcal{E}\}$, $\{B_i : i \in \mathcal{E}\}$ and $\{C_i : i \in \mathcal{E}\}$ are a family of matrices parameterized by an index set $\mathcal{E} = \{1, 2\}$ and

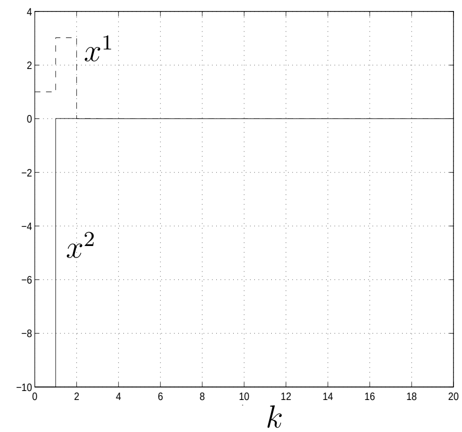

Fig. 2. Closed loop behaviour ($x_0^1 = -10$ and $x_0^2 = 1$).

$$A_1 = \begin{bmatrix} -0.84 & -0 & 0.27 \\ 0.38 & -0.33 & 0.07 \\ -0.1 & 0.55 & 0.44 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.11 \\ 0.28 \\ 0.52 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.39 & 0.07 & -0.13 \\ 0.90 & -0.39 & -0.41 \\ 0.51 & -0.32 & 0.59 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.25 \\ 0.14 \\ 0.92 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.62 & 0.77 & 0.82 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.19 & 0.62 & 0.51 \end{bmatrix}$$

The aim is to build a switched static output feedback control

$$u_k = K_\alpha y_k \tag{28}$$

which stabilizes the closed loop system for arbitrary switching signal α. Due to equality constraints, similar to the ones of Theorem 2, no solution can be provided using the single quadratic Lyapunov based approach nor extending the LMI condition in Mignone et al. (2000) to the static output feedback problem. Conditions of Theorem 2 are found to be feasible and the corresponding matrix gains are given by

$$K_1 = -0.7084, \qquad K_2 = -0.1549$$

5.3 *Example 3*

This last example illustrates the application of the switched observer conditions given in Theorem 3 to a chaos synchronization scheme.

Consider a map T of which the state space representation of the *drive* and the *response* is of the form (22) with:

- $x_k = [x_k^1\ x_k^2\ x_k^3]^T$ and $\hat{x}_k = [\hat{x}_k^1\ \hat{x}_k^2\ \hat{x}_k^3]^T$
- $A_i = \begin{bmatrix} 0 & 0.89 & 0.5 \\ h_i & 0.89 & 0 \\ -0.1 & 0 & 0.9 \end{bmatrix}$, $h_1 = -a$, $h_2 = \lambda$
- $B_1 = [0\ 0\ 0]^T$ and $B_2 = [0\ -6(a+\lambda)\ 0]^T$
- the two regions respectively associated to A_1 and A_2 are R_1 and R_2. R_1 is the set $\{x_k | x_k^1 < 6\}$ and R_2 is the set $\{x_k | x_k^1 \geq 6\}$

When they are feasible, conditions of Theorem 3 ensure synchronization independently of the kind of chaotic motion exhibited by the *drive*. The adopted "attractor" may be then totally arbitrary. For the parameters $a = 1.12$, $\lambda = 2$, the "attractor" is shown on Fig. 3.

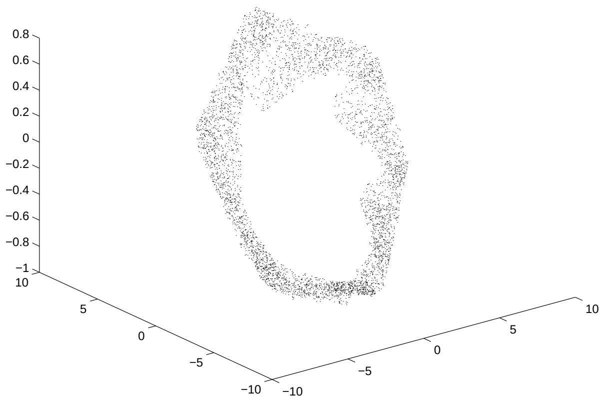

Fig. 3. Chaotic attractor generated by the *drive*

The matrices are:

$$C_1 = [-1 \quad 1 \quad -2] \quad \text{and} \quad C_2 = [-2 \quad 0.35 \quad 1]$$

Conditions of Theorem 3 are found feasible and the resulting gains are given by

$$L_1 = \begin{bmatrix} -0.3450 \\ 0.5650 \\ -0.1756 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.3033 \\ -1.3599 \\ -0.0463 \end{bmatrix}$$

The chaotic attractor generated by the *response* after convergence and the convergence of the three components of the synchronization error ε_k are depicted in Fig. 4. Notice that if we impose the constraint $S_i = G_i = S$ in the conditions of Theorem 3, which means that we look for a common quadratic Lyapunov function $V(\varepsilon_k, \xi_k) = \varepsilon_k' P \varepsilon_k$ with $P = S^{-1}$, we find that the corresponding LMI are not feasible and no such a particular solution can be provided.

6. CONCLUSION

In this paper, a link between Poly-Quadratic stability and stability of arbitrary switching systems has been established. If the class of switching signals is restricted, the application of these conditions without taking into account the properties of admissible switching signals is conservative. In such a situation, one can take benefit from the features of the switching signal and modify these conditions following reasoning as in Johansson and Rantzer (1998) and invoking the so-called $\mathcal{S}$-procedure to improve the conservatism of these conditions. One can also use similar developments as in Rubensson et al. (2000) to investigate the application of this approach to continuous-time switched systems.

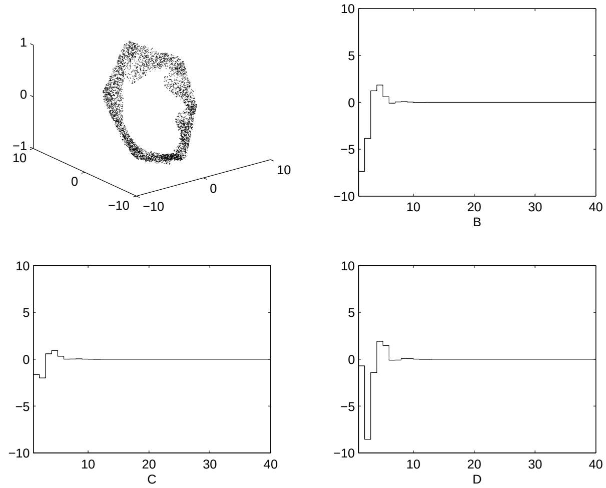

Fig. 4. Chaotic attractor generated by the *response* and convergence of ε_k

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