CONTROLLER SYNTHESIS FOR LINEAR SYSTEMS TO IMPOSE POSITIVENESS IN CLOSED-LOOP STATES

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Abstract: This paper solves some synthesis problems for a class of linear systems, to enforce the closed-loop states to take nonnegative values whenever the initial conditions are nonnegative. In particular the following three problems are studied: the synthesis of state-feedback controllers for positive closed-loop systems, including the requirement of positiveness for the controller, the extension to uncertain plants and the presence of control signals that are positively bounded. The provided conditions are solvable in terms of Linear Programming Problems.

Keywords: Stabilization, uncertain dynamic systems, saturation control, linear programming, robust stability, positive systems.

1. INTRODUCTION

Many physical systems involve quantities that have intrinsically constant sign (For example, absolute temperatures, level of liquids in tanks and dimensions of objects are always positive). This kind of systems which have the property that the state is nonnegative whenever the initial conditions are nonnegative referred in the literature as positive systems(see Berman *et al.* (1989), Luenberger (1979) and Farina and Rinaldi (2000) for general references). The stabilization problem for this systems has been studied in Berman *et al.* (1989), Leenheer and Aeyels (2001) and Kaczorek (1999).

This paper studies a slightly different problem: if the open loop system is not restricted in sign, design a state feedback law that makes the closed-loop system positive (maybe in the pressence of uncertainty or positive controls). This problem arises when the desired value of the states in steady-state ($x = 0$) corresponds to minimum desired value of the state. For example, consider the process control problem of controlling an endothermic chemical reaction in a reactor: the temperature must always be maintained over a minimum value to get a minimum yield; at the same time this value corresponds to the most economical working point for the plant.

This paper proposes different methods for the stabilization of linear positive systems by means of state feedback, that solve different problems for these systems to enforce positiveness: nominal stability, robust stability, stability with positive controls, etc. This paper provides a new treatment for the stabilization of positive linear systems, following a common approach, where all the proposed conditions are expressed in terms of Linear Programming.

The remainder of the paper is structured as follows: Section 2 gives some preliminary results. Section 3 studies the stabilization problem. A robust stabilization problem is addressed in Section 4. The state-feedback synthesis problem with bounded positive controls is considered in section 5. Finally, section 6 gives some concluding remarks.

Notation: $\mathbf{R}_+^n$ denotes the non-negative orthant of the n-dimensional real space $\mathbf{R}^n$. M^T denotes the transpose of the real matrix M. For a real matrix (or a vector) M, $M > 0$ (resp. $M \geq 0$) means that its components are positive: $M_{ij} > 0$ (resp. $M_{ij} \geq 0$).

2. PRELIMINARIES

This section presents some definitions and preliminary results which will be used throughout the paper.

Consider the following autonomous linear system:

$$\begin{aligned} \frac{dx}{dt} &= Ax, \\ x(0) &= x_0 \in \mathbf{R}_+^n. \end{aligned} \tag{1}$$

Let us state the following definition of positive system (1):

Definition 2.1. Given any positive initial condition $x(0) = x_0 \in \mathbf{R}_+^n$, the system (1) is said to be positive if the corresponding trajectory $x(t) \in \mathbf{R}_+^n$ for all $t \geq 0$.

Next, we need to find under which condition the system (1) is positive. In fact, this condition is provided by a classical result (see Luenberger (1979)):

Lemma 2.1. The system (1) is positive if and only if the off-diagonal elements of A are positive:

$$A_{ij} \geq 0, \;\; i \neq j$$

The above result permits to determine whether a system is positive or not by simply looking at the entries of the dynamic matrix of the system. In connection with the above Lemma the following definition will be used:

Definition 2.2. A real matrix M is called a *Metzler* matrix if its off-diagonal elements are positive $M_{ij} \geq 0, \;\; i \neq j$.

The analysis of the asymptotic stability of the system (1) is presented in the following result which is well-known and simple to proof:

Theorem 2.1. Assume that the system (1) is positive, or equivalently that the dynamic matrix A is Metzler; then the following statements are equivalent:

- (i) A is a Hurwitz matrix: the real part of the eigenvalues of A is strictly negative.
- (ii) System (1) is asymptotically stable for every initial condition $x_0 \in \mathbf{R}_+^n$
- (iii) System (1) is asymptotically stable for some initial condition x_0 in the interior of $\mathbf{R}_+^n$.
- (iv) There exists $\lambda \in \mathbf{R}^n$ such that

$$A\lambda < 0, \;\; \lambda > 0. \tag{2}$$

3. STABILITY SYNTHESIS

This section studies the stabilization problem of the following governed linear system:

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu, \\ x(0) &= x_0 \in \mathbf{R}_+^n, \end{aligned} \tag{3}$$

where $A \in \mathbf{R}^{n \times n}, \;\; B \in \mathbf{R}^{n \times p}$.

In the following, we consider first the case where the control law $u \in \mathbf{R}^p$ is a state-feedback, not restricted in sign. This control law must be designed in such way that the resulting governed system is positive and asymptotically stable. In other words, with regard to the previous preliminary results, the problem reduces to look for a state-feedback law $u = Kx$, where the matrix $K \in \mathbf{R}^{p \times n}$ has to be determined to satisfy the following problem:

Find necessary and sufficient conditions on $A \in \mathbf{R}^{n \times n}$ and $B \in \mathbf{R}^{n \times p}$ such that there exists a matrix $K \in \mathbf{R}^{p \times n}$ satisfying:

- $A + BK$ is a Metzler matrix,
- $A + BK$ is a Hurwitz matrix.

Now, we are able to state the main theorem of this section:

Theorem 3.1. The following statements are equivalent:

- (i) There exists a state-feedback law $u = Kx$ such that the closed-loop system is positive and asymptotically stable.
- (ii) There exists a matrix $K \in \mathbf{R}^{p \times n}$ such that $A+BK$ is both a Metzler and a Hurwitz matrix.

- (iii) There exist $n+1$ vectors $d = [d_1 \ \ldots \ d_n]^T \in \mathbf{R}^n$ and $y_1, \ldots, y_n \in \mathbf{R}^p$ such that

$$\begin{aligned} &Ad + B\Big(\sum_{i=1}^{n} y_i\Big) < 0, \\ &d > 0, \\ &a_{ij}d_j + b_i y_j \geq 0 \text{ for } i \neq j, \end{aligned} \tag{4}$$

with $A = [a_{ij}]$ and $B^T = [b_1^T \ \ldots \ b_n^T]$.
Moreover, the gain matrix K in conditions (i) and (ii) can be calculated as follows

$$K = [d_1^{-1}y_1 \ \ldots \ d_n^{-1}y_n].$$

Proof: Regarding to the considerations discussed previously the equivalence between (i) and (ii) is straightforward. The proof will be completed by showing that (ii) and (iii) are equivalent.
Assume that condition (iii) holds and define the matrix $K = [k_1, \ldots, k_n]$ with $k_i = d_i^{-1}y_i$ for $i = 1, \ldots, n$. Then, by a simple calculation, $BKd = B\Big(\sum_{i=1}^{n} y_i\Big)$, which from condition (iii) leads to $(A + BK)d < 0$. Since $d > 0$ then by using Theorem 2.1 we can conclude that $A + BK$ is Hurwitz matrix if it is proved that it is a Metzler matrix. Now it is easy to see that $A + BK$ is a Metzler matrix since condition (iii) leads to

$$a_{ij} + b_i d_j^{-1} y_j = a_{ij} + b_i k_j = (A+BK)_{ij} \geq 0 \text{ for } i \neq j.$$

The implication $(ii) \rightarrow (iii)$ and the rest of the proof follow the same line of argument. □

The importance of the above result is relevant, because it provides not only a checkable necessary and sufficient conditions but also a simple approach to address numerically the computation of the problem. Indeed, one can see that the conditions in statement (iii) of Theorem 3.1 are linear, involving non-strict inequalities. Thus, these conditions can be solved as a standard Linear Programming problem. In addition, based on the same formulation we can take into account the positiveness of the state-feedback control law by just adding an additional constraint on the variables $y_1, \ldots, y_n$. This is shown in the following result, that is straightforward from Theorem 3.1:

Theorem 3.2. The following statements are equivalent:

- (i) There exists a positive state-feedback law $u = Kx \geq 0$ such that the closed-loop system is positive and asymptotically stable.
- (ii) There exists a matrix $K \in \mathbf{R}^{p \times n}$ such that $K \geq 0$ and $A + BK$ is both a Metzler and a Hurwitz matrix.
- (iii) The following LP problem in the variables $d = [d_1 \ \ldots \ d_n]^T \in \mathbf{R}^n$ and $y_1, \ldots, y_n \in \mathbf{R}^p$, is feasible

$$\begin{aligned} &Ad + B\Big(\sum_{i=1}^{n} y_i\Big) < 0, \\ &d > 0, \\ &y_i \geq 0 \text{ for } i = 1, \ldots, n, \\ &a_{ij}d_j + b_i y_j \geq 0 \text{ for } i \neq j. \end{aligned} \tag{5}$$

(with $A = [a_{ij}]$ and $B^T = [b_1^T \ \ldots \ b_n^T]$).
Moreover, the gain matrix K in conditions (i) and (ii) can be chosen as

$$K = [d_1^{-1}y_1 \ \ldots \ d_n^{-1}y_n]$$

where d and $y_1, \ldots, y_n$ are given by any feasible solution to the above LP problem.

At this stage some remarks are in order:

Remark 3.1. Note that if a negative state feedback control law is to be considered it suffices to impose in the previous LP formulation that $y_i \leq 0$ for $i = 1, \ldots, n$.

Remark 3.2. We stress out that our conditions do not impose any restriction on the dynamics of the governed system; that is, there is no restriction on the data matrices $A \in \mathbf{R}^{n \times n}$ and $B \in \mathbf{R}^{n \times p}$. For instance, A is not necessarily a Metzler matrix; in this case, the free system is not positive, so that the synthesis problem can be interpreted as enforcing the system to be positive.

Remark 3.3. Notice that in the specific case when A is a Metzler matrix but not Hurwitz and B is positive, then it impossible to stabilize the system by any positive state-feedback control law, because, regarding to the previous result, the existence of such control necessarily implies that $Ad < 0, \ d > 0$. Then, Theorem 2.1 implies that A must be Hurwitz, which leads to a contradiction.

4. SYNTHESIS WITH UNCERTAIN PLANT

A nice extension of the proposed approach is the possibility of handling the case when the dynamics of the system are not exactly known, as it is now presented:

Consider the following uncertain system

$$\begin{aligned} \frac{dx}{dt} &= \bar{A}x + \bar{B}u, \\ x(0) &= x_0 \in \mathbf{R}_+^n. \end{aligned} \tag{6}$$

Matrices $\bar{A} \in \mathbf{R}^{n \times n}, \ \bar{B} \in \mathbf{R}^{n \times p}$ are supposed to be not exactly determined but it is assumed that they belong to the following convex set:

$$[\bar{A} \ \ \bar{B}] \in \mathcal{P} := \Big\{ \sum_{i=1}^{l} \alpha_i [A^i \ \ B^i] \ \mid \ \sum_{i=1}^{l} \alpha_i = 1, \ \ \alpha_i \geq 0 \Big\}, \tag{7}$$

where $[A^1 \quad B^1], \dots, [A^l \quad B^l]$ are known given matrices.

Our robust synthesis problem consists in finding a fixed matrix K such that the following closed-loop system is positive and asymptotically stable for every $[\bar{A} \;\; \bar{B}] \in \mathcal{P}$:

$$\frac{dx}{dt} = (\bar{A} + \bar{B}K)x \tag{8}$$

Theorem 4.1. There exists a robust state-feedback law $u = Kx$ such that the resulting closed-loop system (8) is positive and asymptotically stable for every $[\bar{A} \;\; \bar{B}] \in \mathcal{P}$, if the following LP problem in the variables $d = [d_1 \quad \dots \quad d_n]^T \in \mathbf{R}^n$ and $y_1, \dots, y_n \in \mathbf{R}^p$, is feasible:

$$\begin{array}{l} A^k d + B^k \Big(\sum_{i=1}^{n} y_i \Big) < 0 \text{ for } k = 1, \dots, l, \\ d > 0, \\ a_{ij}^k d_j + b_i^k y_j \geq 0 \text{ for } i \neq j, k = 1, \dots, l, \end{array} \tag{9}$$

(with $A^k = [a_{ij}^k]$ and ${B^k}^T = [{b_1^k}^T \quad \dots \quad {b_n^k}^T]$, $k = 1, \dots, l$).

Moreover, the gain matrix K of the robust controller can be computed as

$$K = [d_1^{-1} y_1 \quad \dots \quad d_n^{-1} y_n],$$

where d, $y_1, \dots, y_n$ correspond to any feasible solution to the above LP problem.

Proof: By a simple convexity argument the proof is straightforward. $\Box$

5. SYNTHESIS WITH POSITIVE BOUNDED CONTROLS

This section considers the following constrained system

$$\begin{array}{rcl} \dfrac{dx}{dt} & = & Ax + Bu, \\ x \geq 0 & \text{and} & 0 \leq u \leq \bar{u}, \end{array} \tag{10}$$

that is, the trajectory of the system is positive and the input is constrained to be positive and bounded by a given value $\bar{u}$. (of course, if B has positive components then A must be Hurwitz, otherwise it is not posible to stabilize the system with only positive controls).

Although the stabilitation of constrained plants have been studied in the literature (Lin and Saberi (1995), Dahleh *et al.* (1995), Stoorvogel et al. (2002), Mesquine *et al.* (2004)), as far as the authors' knowledge, the application to positive systems is presented here for the first time.

The aim here is to address the following problem: Given $\bar{u} > 0$ find $\bar{x} > 0$ corresponding to the set of initial conditions $\chi = \{x(0) \in \mathbf{R}^{n \times n} \mid 0 \leq x(0) \leq \bar{x}\}$ for which we can determine a positive bounded state feedback control law $0 \leq u = Kx(t) \leq \bar{u}$ such that the resulting closed-loop system is positive and asymptotically stable.

The following key role lemma will be used for our purpose:

Lemma 5.1. Consider the trajectory $x(t)$ of the autonomous system $\dot{x} = Ax$; then, for a given $\bar{x} > 0$ we have $0 \leq x(t) \leq \bar{x}$ for any initial condition satisfying $0 \leq x(0) \leq \bar{x}$ if and only if A is Metzler and $A\bar{x} \leq 0$.

Proof: Sufficiency: notice that it suffices to prove the inequality $e^{tA}\bar{x} \leq \bar{x}$. Because, since A is Metzler, then $e^{tA} \geq 0, \forall t \geq 0$ so that if $0 \leq x(0) \leq \bar{x}$ we have $0 \leq x(t) = e^{tA}x(0) \leq e^{tA}\bar{x} \leq \bar{x}, \forall t \geq 0$. Now, it is shown that $e^{tA}\bar{x} \leq \bar{x}$, or equivalently

$$S(t) = (tA + \frac{t^2}{2!}A^2 + ... + \frac{t^n}{n!}A^n + ...)\bar{x} \leq 0.$$

One can see easily that $S(t) = \int_0^t e^{\tau A} d\tau A\bar{x}$. Since $e^{tA} \geq 0, \forall t \geq 0$ and $A\bar{x} \leq 0$ then we have $S(t) \leq 0$. Necessity: the system is necessarily positive and then A must be Metzler. Now, let $x(0) = \bar{x}$ then all the components of the derivative of the state at zero are negative $\dot{x}(0) = A\bar{x} \leq 0$ because the state must satisfy $x(t) \leq \bar{x}$ at any time. $\Box$ Now, we state the main result of this section.

Theorem 5.1. Consider the following LP problem in the variables $\bar{x} = [\bar{x}_1 \quad \dots \quad \bar{x}_n]^T \in \mathbf{R}^n$ and $y_1, \dots, y_n \in \mathbf{R}^p$:

$$\begin{array}{l} A\bar{x} + B\Big(\sum_{i=1}^{n} y_i \Big) < 0, \\ \bar{x} > 0, \\ y_i \geq 0 \text{ for } i = 1, \dots, n, \\ \sum_{i=1}^{n} y_i \leq \bar{u}, \\ a_{ij}\bar{x}_j + b_i y_j \geq 0 \text{ for } i \neq j, \end{array} \tag{11}$$

with $A = [a_{ij}]$ and $B^T = [b_1^T \quad \dots \quad b_n^T]$.

Define the matrix

$$K = [\bar{x}_1^{-1} y_1 \quad \dots \quad \bar{x}_n^{-1} y_n],$$

and consider the closed-loop system $\dot{x} = (A + BK)x$ under the state-feedback control $u = Kx$. Then $0 \leq u(t) \leq \bar{u}$ for any initial state satisfying $0 \leq x(0) \leq \bar{x}$. Moreover, the closed-loop system is positive and asymptotically stable.

Proof: Assume that $\bar{x} = [\bar{x}_1 \quad \dots \quad \bar{x}_n]^T$ and $y_1, \dots, y_n$ is a solution to (11) and define $K = [\bar{x}_1^{-1} y_1 \quad \dots \quad \bar{x}_n^{-1} y_n]$. Since

$$a_{ij} + b_i \bar{x}_j^{-1} y_j = a_{ij} + b_i k_j = (A + BK)_{ij} \geq 0 \text{ for } i \neq j,$$

then the matrix $A+BK$ is Metzler. The inequality $A\bar{x} + B\Big(\sum_{i=1}^{n} y_i\Big) < 0$ is equivalent to $(A + BK)\bar{x} < 0$, then by Lemma 5.1 the trajectory of the system $\dot{x} = (A + BK)x$ is such that $0 \leq x(t) \leq \bar{x}$ for any initial condition satisfying $0 \leq x(0) \leq \bar{x}$. Using this fact and recalling the inequalities $\sum_{i=1}^{n} y_i \leq \bar{u}$, $y_i \geq 0$ for $i = 1, \ldots, n$, or equivalently $K \geq 0$ and $K\bar{x} \leq \bar{u}$, it is possible to see that the state-feedback control $u = Kx$ is such that $0 \leq u(t) \leq K\bar{x} \leq \bar{u}$ for any initial state satisfying $0 \leq x(0) \leq \bar{x}$.
The closed loop system is positive because the off-diagonal components of $(A+BK)$ are positive.
Since $A\bar{x} + B\Big(\sum_{i=1}^{n} y_i\Big) < 0$, $\bar{x} > 0$ is equivalent to $(A + BK)\bar{x} < 0$, $\bar{x} > 0$ then Theorem 2.1 implies that the closed-loop system is asymptotically stable and the proof is complete. □

6. CONCLUSIONS

This paper have proposed a novel approach to solve some synthesis problems linear systems to transform them to positive systems. The stabilization problem have been considered, and necessary and sufficient conditions for its solvability have been proposed, also for the uncertain case. Moreover, the synthesis problem with positive bounded controls have been addressed. It has been shown that all the proposed conditions are solvable in terms of Linear Programming problems.

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