ON POLES AND ZEROS OF INPUT-OUTPUT AND CHAIN-SCATTERING SYSTEMS

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Abstract: A system is frequently represented by transfer functions in an input-output characterization. However, such a system (under mild assumptions) can also be represented by transfer functions in a port characterization, frequently referred to as a chain-scattering representation. Due to its cascade properties, the chain-scattering representation is used throughout many fields of engineering. This paper studies the relationship between poles and zeros of input-output and chain-scattering representations of the same plant.

Keywords: Input-output representation, chain-scattering representation, poles, zeros, cascade connection

NOTATION

$\mathbb{R}[s]$: polynomial ring
$\mathcal{R}_P(s)$: proper real rational transfer function matrices
$\mathbb{C}$: field of complex numbers
Ω: subset in $\mathbb{C}$
$pole(G(s))$: the set of all poles of $G(s) \subset \mathcal{R}_P(s)$
$zero(G(s))$: the set of all transmission zeros of $G(s) \subset \mathcal{R}_P(s)$

1. INTRODUCTION

The chain-scattering representation is used extensively in various fields of engineering to represent the scattering properties of a physical system(Kimura, 1997), especially in circuit theory where it has been widely used to deal with the cascade connection of circuits originating in analysis and synthesis problems (Fettweis and Basu, 1991; Pauli, 1990; Maffucci and Miano, 2002). In circuit theory, the chain-scattering representation is also called a scattering matrix of a two-port network(Zha and Chen, 1997). Compared with the usual input-output (I/O) representation (Fig. 1), the chain-scattering representation (Fig. 2) is in fact an alternative way of representing a plant. Cascade structure is the main property of the chain-scattering representation, which enables feedback in the I/O representation (Fig. 3) to be represented simply as a matrix multiplication in the chain-scattering representation (Fig.

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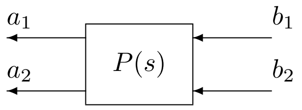

Fig. 1. Input-output representation

Fig. 2. Chain-scattering representation

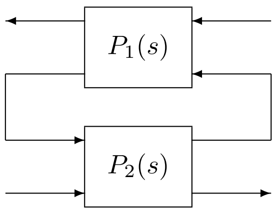

Fig. 3. Feedback connection in I/O representation

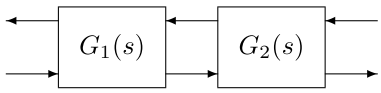

Fig. 4. Cascade connection in chain-scattering representation

4). Duality of transformation between the chain-scattering transformation and its inverse is its another useful property in the analysis of such systems(Kimura, 1995; Potirakis *et al.*, 1996). Due to these features, Kimura(Kimura, 1995) and others(Lanzon *et al.*, 2003; Le *et al.*, 1998; Pugh and Tan, 2002) used the chain-scattering representation to provide a unified framework of cascade synthesis for H^∞ control theory. Within this cascade framework, the H^∞ control problem is reduced to a factorization problem called a J-lossless factorization.

Pole-zero analysis is one of the most elementary tools of control theory to study the properties of a system. It is consequently desirable to understand the connection between the poles and zeros of an I/O representation with the poles and zeros of the corresponding chain-scattering representation. For example, in deriving necessary and sufficient conditions for the solvability of the H^∞ control problem in terms of a J-lossless factorizations, one would typically need to impose certain conditions on the poles and zeros of the chain-scattering plant(Kimura, 1997). It is natural to seek to understand what these conditions correspond to in the I/O representation.

This paper will study the relationship between poles and zeros of I/O and chain-scattering representations in detail. Firstly, the I/O and chain-scattering representations are presented. Secondly, some simple transfer function matrix results are given, which are needed for the pole-zero study in the subsequent sections. Lastly, explicit relationships between poles and zeros of I/O and chain-scattering representations are derived.

2. I/O AND CHAIN-SCATTERING REPRESENTATIONS

Consider a MIMO system with two kinds of inputs (b_1, b_2) and two kinds of outputs (a_1, a_2), as shown in Fig. 1, represented as

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = P(s) \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

The chain-scattering representation of $P(s)$, as shown in Fig. 2, is

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = G(s) \begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}$$

where

$$\begin{aligned} G(s) &:= CHAIN(P(s)) \\ &= \begin{bmatrix} P_{12} - P_{11}P_{21}^{-1}P_{22} & P_{11}P_{21}^{-1} \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{12} & 0 \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{12} & 0 \\ 0 & P_{21}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -P_{22} & I \end{bmatrix} \end{aligned} \tag{1}$$

and exists if $P_{21}(s)$ is invertible in $\mathcal{R}_P(s)$.

Then the mapping from chain-scattering representation to I/O representation is

$$\begin{aligned} P(s) &= CHAIN^{-1}(G(s)) \\ &= \begin{bmatrix} G_{12}G_{22}^{-1} & G_{11} - G_{12}G_{22}^{-1}G_{21} \\ G_{22}^{-1} & -G_{22}^{-1}G_{21} \end{bmatrix} \\ &= \begin{bmatrix} I & G_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & G_{11} \\ G_{22}^{-1} & -G_{22}^{-1}G_{21} \end{bmatrix} \\ &= \begin{bmatrix} I & G_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22}^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -G_{21} \end{bmatrix} \end{aligned} \tag{2}$$

where $G_{22}(s) = P_{21}^{-1}(s)$ is invertible in $\mathcal{R}_P(s)$.

3. SOME SIMPLE TRANSFER FUNCTION MATRIX RESULTS

In this section, some simple transfer function matrix results are given, which are needed for the pole-zero study in the next section.

Poles and zeros of any real rational transfer function matrix are obtained from its so-called McMillan decomposition(Zhou *et al.*, 1996).

Lemma 1. (McMillan form) Let $G(s) \in \mathcal{R}_P(s)$ be any proper real rational transfer function matrix, then there exist unimodular polynomial matrices $U(s), V(s) \in \mathbb{R}[s]$ such that

$$U(s)G(s)V(s) = M(s)$$
$$:= \begin{bmatrix} \frac{\alpha_1(s)}{\beta_1(s)} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \frac{\alpha_r(s)}{\beta_r(s)} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \qquad (3)$$

and $\alpha_i(s)$ divides $\alpha_{i+1}(s)$ and $\beta_{i+1}(s)$ divides $\beta_i(s)$, where $\alpha_i(s)$ and $\beta_i(s)$ are scalar polynomials. The roots of all the polynomials $\beta_i(s)$ in the McMillan form for $G(s)$ are called the poles of $G(s)$; the roots of all the polynomials $\alpha_i(s)$ in the McMillan form for $G(s)$ are called the transmission zeros of $G(s)$.

From (3), it is easy to see that

$$G(s) = U^{-1}(s)M(s)V^{-1}(s)$$

where $U^{-1}(s)$ and $V^{-1}(s)$ are also unimodular polynomial matrices.

The following lemma studies the poles and zeros of a cascade connection of MIMO systems.

Lemma 2. Given a cascade connection $G(s) = G_1(s)G_2(s)$.

(i) If $G_1(s)$ has full column normal rank[3] OR $G_2(s)$ has full row normal rank, then
$$zero(G(s)) \subset \{zero(G_1(s)), zero(G_2(s))\}.$$
(ii) $pole(G(s)) \subset \{pole(G_1(s)), pole(G_2(s))\}$.

Proof. (i) Suppose $G_1(s)$ has full column normal rank. Using Lemma 1, there exist unimodular polynomial matrices $U_1(s), V_1(s), U_2(s), V_2(s)$ such that

$$G_1(s) = U_1^{-1}(s)\begin{bmatrix} A_{r\times r} \\ 0 \end{bmatrix} V_1^{-1}(s)$$
$$G_2(s) = U_2^{-1}(s)\begin{bmatrix} B_{k\times k} & 0 \\ 0 & 0 \end{bmatrix} V_2^{-1}(s)$$

where $A_{r\times r}, B_{k\times k}$ are diagonal square matrices with full normal rank. Then

$$\begin{aligned} G(s) &= G_1(s)G_2(s) \\ &= U_1^{-1}\begin{bmatrix} A \\ 0 \end{bmatrix} V_1^{-1}U_2^{-1}\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} V_2^{-1} \\ &= U_1^{-1}\begin{bmatrix} AV_1^{-1}U_2^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} V_2^{-1}. \qquad (4) \end{aligned}$$

[3] The normal rank of $G(s)$, denoted $normalrank(G(s))$, is the maximally possible rank of $G(s)$ for at least one $s \in \mathbb{C}$.

It can be easily seen from (4) that the poles and transmission zeros of $G(s)$ are the same as the poles and transmission zeros of $AV_1^{-1}U_2^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix}$ respectively. Since B has full normal rank, $\begin{bmatrix} B \\ 0 \end{bmatrix}$ has full column normal rank. Note that $V_1^{-1}U_2^{-1}$ is a unimodular polynomial matrix and A has full normal rank. Then $AV_1^{-1}U_2^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix}$ has full column rank at $s \notin \{zero(A), zero(B)\}$. Thus, clearly $AV_1^{-1}U_2^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix}$ has full column normal rank.

Now suppose $z_0 \in \mathbb{C}$ is a transmission zero of $G(s)$ and hence also of $AV_1^{-1}U_2^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix}$. Then there exists a $0 \neq u_0 \in \mathbb{C}^k$ (Zhou *et al.*, 1996) such that $\left(AV_1^{-1}U_2^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix}\right)_{s=z_0} u_0 = 0$. If z_0 is not a transmission zero of B, then $0 \neq \begin{bmatrix} B(z_0) \\ 0 \end{bmatrix} u_0 \in \mathbb{C}^r$ and thus $0 \neq \left(V_1^{-1}U_2^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix}\right)_{s=z_0} u_0 \in \mathbb{C}^r$. Since $\left(AV_1^{-1}U_2^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix}\right)_{s=z_0} u_0 = 0$ implies
$A(z_0)\left[\left(V_1^{-1}U_2^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix}\right)_{s=z_0} u_0\right] = 0$, z_0 must be a transmission zero of A. Hence a transmission zero of $AV_1^{-1}U_2^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix}$ is a transmission zero of either A or B. This is equivalent to the statement that a transmission zero of $G(s)$ is a transmission zero of either $G_1(s)$ or $G_2(s)$. Thus,

$$zero(G(s)) \subset \{zero(G_1(s)), zero(G_2(s))\}.$$

Similarly, a dual result can be proved that if $G_2(s)$ has full row normal rank, then

$$zero(G(s)) \subset \{zero(G_1(s)), zero(G_2(s))\}.$$

(ii) It is trivial to show that

$$pole(G(s)) \subset \{pole(G_1(s)), pole(G_2(s))\}.$$

Since the I/O and chain-scattering representations of a plant are often presented in the form of block transfer function matrices, pole-zero analysis of several block transfer function matrices are studied as follows.

Lemma 3. Given a proper real rational transfer function matrix with the following block matrix partitioning

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}.$$

Then

$$pole(G(s)) \subset \{pole(G_{11}(s)), pole(G_{12}(s)), \\ pole(G_{21}(s)), pole(G_{22}(s))\},$$
$$pole(G_{ij}(s)) \subset pole(G(s)), \quad i,j = 1,2.$$

If furthermore $G_{12}(s) = 0$ and $G_{21}(s) = 0$ such that

$$G(s) = \begin{bmatrix} G_{11}(s) & 0 \\ 0 & G_{22}(s) \end{bmatrix},$$

then

$$pole(G(s)) = \{pole(G_{11}(s)), pole(G_{22}(s))\},$$
$$zero(G(s)) = \{zero(G_{11}(s)), zero(G_{22}(s))\}.$$

Proof. This proof is trivial.

Lemma 4. Given a proper real rational transfer function matrix with the following block matrix partitioning

$$G(s) = \begin{bmatrix} I & G_{12}(s) \\ 0 & I \end{bmatrix}.$$

Then

$$pole(G(s)) = pole(G_{12}(s)),$$
$$zero(G(s)) = pole(G_{12}(s)).$$

Proof. Using Lemma 3, it is clear that

$$pole(G(s)) = pole(G_{12}(s)).$$

Since

$$G^{-1}(s) = \begin{bmatrix} I & -G_{12}(s) \\ 0 & I \end{bmatrix},$$

it is easy to see that $pole(G^{-1}(s)) = pole(G_{12}(s))$. Hence

$$zero(G(s)) = pole(G^{-1}(s)) = pole(G_{12}(s)).$$

Lemma 5. Given a proper real rational transfer function matrix with the following block matrix partitioning

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ 0 & G_{22}(s) \end{bmatrix}.$$

(i) If $G_{22}(s)$ has full column normal rank, then $zero(G_{11}(s)) \subset zero(G(s))$.
(ii) If $G_{11}(s)$ has full row normal rank, then $zero(G_{22}(s)) \subset zero(G(s))$.

Proof. Follows easily after some tedious algebra. Full proof will be published elsewhere.

4. POLE-ZERO RELATIONS BETWEEN I/O AND CHAIN-SCATTERING SYSTEMS

In this section, pole-zero relations between I/O and chain-scattering representations are studied.

Theorem 6. The poles and transmission zeros of chain-scattering system $G(s) = CHAIN(P)$ have the following relations with the poles and transmission zeros of I/O system $P(s)$:

(i) $zero(G(s)) \subset \{pole(P_{11}(s)), zero(P_{12}(s)), pole(P_{21}(s)), pole(P_{22}(s))\}$,
(ii) $pole(G(s)) \subset \{pole(P_{11}(s)), pole(P_{12}(s)), zero(P_{21}(s)), pole(P_{22}(s))\}$,
(iii) $zero(P_{21}(s)) \subset pole(G(s))$,
(iv) $zero(P_{12}(s)) \subset zero(G(s))$.

Proof. (i) From (1),

$$G(s) = \begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{12} & 0 \\ 0 & P_{21}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -P_{22} & I \end{bmatrix}.$$

Since both $\begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ -P_{22} & I \end{bmatrix}$ have full normal rank, using Lemma 2,

$$zero(G(s)) \subset \{zero(\begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix}), \\ zero(\begin{bmatrix} P_{12} & 0 \\ 0 & P_{21}^{-1} \end{bmatrix}), zero(\begin{bmatrix} I & 0 \\ -P_{22} & I \end{bmatrix})\}.$$

Using Lemma 3 and 4, it is easy to see that

$$zero(\begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix}) = pole(P_{11}),$$
$$zero(\begin{bmatrix} P_{12} & 0 \\ 0 & P_{21}^{-1} \end{bmatrix}) = \{zero(P_{12}), pole(P_{21})\},$$
$$zero(\begin{bmatrix} I & 0 \\ -P_{22} & I \end{bmatrix}) = pole(P_{22}).$$

Thus,

$$zero(G(s)) \subset \{pole(P_{11}), zero(P_{12}), \\ pole(P_{21}), pole(P_{22})\}.$$

(ii) Using Lemma 3 and 4, it is clear that

$$pole(\begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix}) = pole(P_{11}),$$
$$pole(\begin{bmatrix} P_{12} & 0 \\ 0 & P_{21}^{-1} \end{bmatrix}) = \{pole(P_{12}), zero(P_{21})\},$$
$$pole(\begin{bmatrix} I & 0 \\ -P_{22} & I \end{bmatrix}) = pole(P_{22}).$$

Then using Lemma 2,

$$pole(G(s)) \subset \{pole(P_{11}), pole(P_{12}), \\ zero(P_{21}), pole(P_{22})\}.$$

(iii) Note that

$$G(s) = \begin{bmatrix} P_{12} - P_{11}P_{21}^{-1}P_{22} & P_{11}P_{21}^{-1} \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix}.$$

Then using Lemma 3, $pole(P_{21}^{-1}) \subset pole(G(s))$. That is

$$zero(P_{21}(s)) \subset pole(G(s)).$$

(iv) Using Lemma 1, there exist unimodular polynomial matrices $U(s), V(s)$ such that

$$\begin{aligned} P_{11}(s) &= U^{-1}(s)M(s)V^{-1}(s) \\ &= U^{-1} \begin{bmatrix} \frac{\alpha_1(s)}{\beta_1(s)} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \frac{\alpha_r(s)}{\beta_r(s)} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{m\times n} V^{-1} \end{aligned}$$

where $M(s)$ is the McMillan form for $P_{11}(s)$ and $\alpha_i(s)$, $\beta_i(s)$ are scalar polynomials. If suppose

$$N_\alpha := \begin{bmatrix} \alpha_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \alpha_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{m\times n},$$

$$N_\beta := \begin{bmatrix} \beta_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \beta_r(s) & 0 \\ 0 & \cdots & 0 & I \end{bmatrix}_{n\times n},$$

then

$$P_{11}(s) = U^{-1}N_\alpha N_\beta^{-1}V^{-1}.$$

And then from (1),

$$\begin{aligned} G(s) &= \begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{12} & 0 \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & U^{-1}N_\alpha N_\beta^{-1}V^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{12} & 0 \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} U^{-1} & 0 \\ 0 & VN_\beta \end{bmatrix} \begin{bmatrix} I & N_\alpha \\ 0 & I \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & N_\beta^{-1}V^{-1} \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} P_{12} & 0 \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix}. \end{aligned} \tag{5}$$

Thus,

$$\begin{aligned} &\begin{bmatrix} U & 0 \\ 0 & N_\beta^{-1}V^{-1} \end{bmatrix} G(s) \\ &= \begin{bmatrix} I & N_\alpha \\ 0 & I \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & N_\beta^{-1}V^{-1} \end{bmatrix} \begin{bmatrix} P_{12} & 0 \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix} \quad (6) \\ &= \begin{bmatrix} I & N_\alpha \\ 0 & I \end{bmatrix} \begin{bmatrix} UP_{12} & 0 \\ -N_\beta^{-1}V^{-1}P_{21}^{-1}P_{22} & N_\beta^{-1}V^{-1}P_{21}^{-1} \end{bmatrix}. \end{aligned}$$

Since $U(s)$ is a unimodular polynomial matrix and $N_\beta^{-1}V^{-1}P_{21}^{-1}$ has full normal rank, using Lemma 5, it is clear that

$$\begin{aligned} &zero(P_{12}) = zero(UP_{12}) \subset \\ &zero(\begin{bmatrix} UP_{12} & 0 \\ -N_\beta^{-1}V^{-1}P_{21}^{-1}P_{22} & N_\beta^{-1}V^{-1}P_{21}^{-1} \end{bmatrix}). \end{aligned} \tag{7}$$

Since $\begin{bmatrix} I & N_\alpha \\ 0 & I \end{bmatrix}$ is also a unimodular polynomial matrix and $\begin{bmatrix} U & 0 \\ 0 & V^{-1}N_\beta^{-1} \end{bmatrix}$ has full normal rank, using Lemma 2 from (6), it is easy to see that

$$\begin{aligned} &zero(\begin{bmatrix} UP_{12} & 0 \\ -N_\beta^{-1}V^{-1}P_{21}^{-1}P_{22} & N_\beta^{-1}V^{-1}P_{21}^{-1} \end{bmatrix}) \\ &= zero(\begin{bmatrix} U & 0 \\ 0 & N_\beta^{-1}V^{-1} \end{bmatrix} G(s)) \\ &\subset \{zero(\begin{bmatrix} U & 0 \\ 0 & N_\beta^{-1}V^{-1} \end{bmatrix}), zero(G)\}. \end{aligned} \tag{8}$$

Combining (7) and (8),

$$zero(P_{12}) \subset \{zero(\begin{bmatrix} U & 0 \\ 0 & N_\beta^{-1}V^{-1} \end{bmatrix}), zero(G)\}. \tag{9}$$

Since $U(s)$, $V(s)$ are unimodular polynomial matrices and N_β^{-1} has no transmission zeros but poles, using Lemma 3, it is clear that

$$zero(\begin{bmatrix} U & 0 \\ 0 & N_\beta^{-1}V^{-1} \end{bmatrix}) = \{\ \}.$$

Hence from(9), $zero(P_{12}(s)) \subset zero(G(s))$.

Theorem 7. (Dual result) The poles and transmission zeros of I/O system $P(s)$ have the following relations with the poles and transmission zeros of chain-scattering system $G(s) = CHAIN(P)$:

(i) $zero(P(s)) \subset \{zero(G_{11}(s)), pole(G_{12}(s)), pole(G_{21}(s)), pole(G_{22}(s))\}$,
(ii) $pole(P(s)) \subset \{pole(G_{11}(s)), pole(G_{12}(s)), pole(G_{21}(s)), zero(G_{22}(s))\}$,
(iii) $zero(G_{22}(s)) \subset pole(P(s))$,
(iv) $zero(G_{11}(s)) \subset zero(P(s))$.

Proof. The proof is similar to that of Theorem 6 from (2).

In order to visualize the relationship between poles and zeros of I/O and chain-scattering representations, next analyze situations where $P(s)$ or $G(s)$ has no poles or zeros in some region in the complex plane $\mathbb{C}$. Suppose Ω is a subset of $\mathbb{C}$, as shown in Fig. 5, which can be any region of the s-plane.

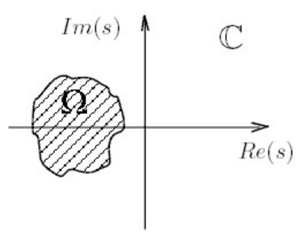

Fig. 5. Subset Ω in $\mathbb{C}$

Theorem 8. Suppose $P(s)$ has no poles in Ω. Then the following results hold:

(i) $G(s)$ has no transmission zeros in Ω if and only if $P_{12}(s)$ has no transmission zeros in Ω;
(ii) $G(s)$ has no poles in Ω if and only if $P_{21}(s)$ has no transmission zeros in Ω.

Proof. Easily follows from Theorem 6. Full proof will be published elsewhere.

Theorem 9. (Dual result) Suppose $G(s)$ has no poles in Ω. Then the following results hold:

(i) $P(s)$ has no transmission zeros in Ω if and only if $G_{11}(s)$ has no transmission zeros in Ω;
(ii) $P(s)$ has no poles in Ω if and only if $G_{22}(s)$ has no transmission zeros in Ω.

Proof. Just a dual result of Theorem 8. Full proof will be published elsewhere.

Now a slightly different result is given, which requires a milder assumption in the theorem statement. This result does not give separate necessary and sufficient conditions for $G(s)$ to have no poles OR no zeros in Ω. Instead, it considers the situation where $G(s)$ has neither poles nor zeros in Ω and gives a necessary and sufficient condition for the case.

Theorem 10. . Suppose $P_{21}(s)$ has no poles in Ω. Then $G(s)$ has no poles nor transmission zeros in Ω if and only if $P(s)$ has no poles in Ω and $P_{12}(s), P_{21}(s)$ have no transmission zeros in Ω.

Proof. Easily follows from Theorem 6 and 8. Full proof will be published elsewhere.

Theorem 11. (Dual result) Suppose $G_{22}(s)$ has no poles in Ω. Then $P(s)$ has no poles nor transmission zeros in Ω if and only if $G(s)$ has no poles in Ω and $G_{11}(s), G_{22}(s)$ have no transmission zeros in Ω.

Proof. Just a dual result of Theorem 10. Full proof will be published elsewhere.

5. CONCLUSIONS

This paper studies the relationship between poles and zeros of input-output and chain-scattering representations. Duality properties between the mapping from an I/O representation to a chain-scattering representation and its inverse are exploited to obtain the required results.

If $P_{12}(s)$ rather than $P_{21}(s)$ is invertible in $\mathcal{R}_P(s)$, a dual chain-scattering representation of $P(s)$ exists, denoted $DCHAIN(P(s))$. Dual results on poles and zeros of I/O and dual chain-scattering systems can very easily be derived in the same way.

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