EXTENSION OF S-PROCEDURE IN THE ANALYSIS OF MULTIVARIABLE CONTROL SYSTEMS

Lev B. Rapoport [*,1]

[*] *Institute for Control Sciences RAS, Profsoyuznaya, 65, Topcon Positioning Systems CIS, Stasovoy 4, Moscow, Russia*

Abstract: Let $q(x)$ and $p_i(x), i = 1, \ldots, m$ be quadratic forms of real variables $x \in R^n$. In many problems of investigation of stability and estimation of the attraction domain and attainability sets, one faces the following question: under what conditions, do inequalities $p_i(x) \geq 0, i = 1, \ldots, m, x \neq 0$ imply the inequality $q(x) > 0$? The commonly used S-procedure method (Yakubovich, 1977) consists in checking of whether there exist values $\tau_i \geq 0$ such that the quadratic form

$$q(x) - \sum_{i=1}^{m} \tau_i p_i(x) \tag{1}$$

is positive definite. It is well known that, if $m \geq 2$, the S-procedure gives us only sufficient conditions for positive definiteness of the quadratic form $q(x)$ under the constraints $p_i(x) \geq 0, i = 1, \ldots, m$. These conditions are necessary only for $m = 1$. This property is called "lossness" of the S-procedure for multiple constraints. The use of only sufficient conditions leads to additional conservatism of stability criteria and attraction domains estimation. Necessary and sufficient conditions are obtained in (Rapoport, 1989) and (Rapoport, 1996) for a special case of quadratic constraints represented as products of two linear forms. This paper further extends those results. The special case of $m = 2$, where additional conditions were imposed on the quadratic forms $p_1(x)$ and $p_2(x)$ to make conditions (1) necessary and sufficient, has been addressed in (Polyak, 1998). In this paper, the losslessness of the S-procedure for $m = 2$ is proved under less restrictive additional conditions. A case of one general-form quadratic constraint and $m - 1$ constraints presented as products of two linear forms is also considered.

Keywords: S-procedure, quadratic constraints, Lur'e system

1. CONSTRAINTS OF SPECIAL FORM

The problem of positive definiteness of quadratic forms in "sector-like" regions arises in the analysis of stability of the Lur'e systems, estimation of attainability sets for the Lur'e systems, and in other applications. The "sector-like" constraints are inequalities imposed on the products of two linear forms:

$$p_i(x) = (f_i' x)(g_i' x) \geq 0, \tag{2}$$

where $f_i, g_i \in R^n$, $i \in 1, \ldots, m$. Let F and G be $R^{n \times m}$ matrices composed of the columns f_i and g_i respectively. Results presented in the

[1] The work is supported by the Russian Foundation for Basic Research (project no. 04-01-00391a) and RAS Presidium complex program (19-1.5)

following subsection allow us to analyze positive definiteness of $q(x)$ under constraints (2) using inductive dimensions reduction.

1.1 Preliminary results

Let us denote $\Omega = \{x \in R^n \setminus \{0\} : p_i(x) \geq 0, i = 1, \ldots, m\}$ and $\Omega_i = \{x \in \Omega : p_i(x) = 0\}$. Note that some of the sets Ω_i may be empty. Let $'$ denotes transposition. The quadratic form $q(x) = \frac{1}{2}x'Qx$ is assumed to be not positive definite in the entire space R^n; i.e., $q(y) \leq 0$ for some $y \in R^n$, $y \neq 0$. Otherwise, there is no need to examine problem to investigate positive definiteness of $q(x)$ in the set Ω.

Lemma 1. Let the quadratic form $q(x)$ be not positive definite in the entire space R^n; i.e., there exists a vector $y \in R^n$, $y \neq 0$, such that $q(y) \leq 0$. Then, $q(x) > 0$ in Ω if and only if the following conditions hold:
(a) $q(x) > 0$ in sets Ω_i, $i = 1, \ldots, m$,
(b) $y \notin \Omega$.

Remark 1. Lemma 1 says that, if $q(x) > 0$ on the "facets" Ω_i of the set Ω, then, together with a certain vector y for which $q(y) \leq 0$, the set Ω does not contain any other vectors z with $q(z) \leq 0$. In other words, $y \notin \Omega$ implies $\{z : q(z) \leq 0\} \cap \Omega = \emptyset$.

Let $\pi(q)$ and $\pi(p_i)$ be the numbers of positive eigenvalues of the quadratic forms $q(x)$ and $p_i(x)$, respectively.

Lemma 2. Let $\pi(q) + \pi(p_j) < n$ for some j. Then, $q(x) > 0$ in Ω if and only if condition (a) of Lemma 1 holds.

Proofs of Lemmas 1 and 2 are similar to those of two lemmas presented in (Rapoport, 1989).

1.2 Lossless extension of the S-procedure

Let $s(w, u)$ be a quadratic form of $2m$ variables $w, u \in R^m$ of the following form

$$s(w, u) = \frac{1}{2}w'Vw + w'Tu + \frac{1}{2}u'Uu, \qquad (3)$$

where V, T, U are $m \times m$ matrices, with V and U being symmetric. The following theorem gives "lossless" extension of the S-procedure to quadratic forms $p_i(x)$ of form (2). Let $\Pi = \{(w, u) : w = F'x, u = G'x, x \in R^n\}$. The linear space Π coincides with R^{2m} if and only if $rank[F \mid G] = 2m \leq n$.

Theorem 1. $q(x) > 0$ in the region Ω defined by quadratic forms (2) if and only if there exists a quadratic form $s(w, u)$ of form (3) such that the following conditions hold:
(a) $q(x) - s(F'x, G'x) > 0$ for $x \neq 0$
(b) $s(w, u) > 0$ for $(w, u) \in \Pi$ satisfying conditions $w_i u_i \geq 0$, $i = 1, \ldots, m$, and $(w, u) \neq 0$.

Proof. Sufficiency is obvious. Let us prove necessity. Positive definiteness of $q(x)$ in the region Ω implies its positive definiteness on the subspace $L = \{x : F'x = 0, G'x = 0\}$. The case of $L = \{0\}$ is not excluded and means positive definiteness of $s(w, u)$ on $L = \{0\}$. Let $(w, u) \in \Pi$. Then, the linear algebraic system $F'x = w, G'x = u$ is feasible, and the function $q(x)$ has a unique minimum under the constraints $F'x = w, G'x = u$. The vector $x^*(w, u)$ on which the minimum is attained depends linearly on w and u. Then, $s_0(w, u) = q(x^*(w, u))$ is a quadratic form, and $s_0(w, u) > 0$ for nonzero (w, u) satisfying conditions $(w, u) \in \Pi$ and $w_i u_i \geq 0$. Hence, for sufficiently small $\varepsilon > 0$, the quadratic form $s(w, u) = s_0(w, u) - \varepsilon(\| w \|^2 + \| u \|^2)$ also satisfies condition (b) of the theorem. Further,

$$\begin{aligned} & q(x) - s(F'x, G'x) \geq \\ & q(x^*(F'x, G'x)) - s(F'x, G'x) = \\ & s_0(F'x, G'x) - s(F'x, G'x) = \\ & \varepsilon(\| F'x \|^2 + \| G'x \|^2), \end{aligned}$$

from which inequality (a) follows for $x \notin L$. If $x \neq 0$ and $x \in L$, then inequality (a) follows from positive definiteness of $q(x)$ on L. Theorem is proved. ■

Remark 2. If the condition $rank[F \mid G] = 2m \leq n$ holds, then $\Pi = R^{2m}$, and condition $(w, u) \in \Pi$ may be removed.

Let us consider the question of verification of condition (b) of Theorem 1. For the sake of brevity, we suppose that $rank[F \mid G] = 2m \leq n$. In this case,

$$\begin{aligned} \Omega = & \{(w, u) \in R^{2m} \setminus \{0\} : \\ & p_i(w, u) = w_i u_i \geq 0, i = 1, \ldots, m\} \end{aligned} \qquad (4)$$

and

$$\pi(p_i) = 1, i = 1, \ldots, m$$

Application of Lemmas 1 and 2 requires the verification of positive definiteness of $s(w, u)$ in the regions Ω_i. This problem has the same form as the original one and differs from it only in that it does not contain the constraint i and either w_i or u_i is equal to zero. In other words,

$$\begin{aligned} \Omega_i = & \{(w, u) \in R^{2m} : w_j u_j \geq 0, j \neq i, w_i = 0\} \\ & \cup \{(w, u) \in R^{2m} : w_j u_j \geq 0, j \neq i, u_i = 0\}. \end{aligned}$$

In order to apply Lemmas 1 and 2, let us introduce the notation

$$M = \{1, \ldots, m\}$$

and, for three nonintersecting subsets $N_0, N_1, N_2 \subseteq M$ satisfying the condition $N_0 \cup N_1 \cup N_2 = M$,

$$\Omega(N_0, N_1, N_2) = \left\{ (w,u) \in R^{2m} \setminus \{0\} : \begin{array}{lll} w_i u_i \geq 0 & \text{for} & i \in N_0, \\ w_i = 0 & \text{for} & i \in N_1, \\ u_i = 0 & \text{for} & i \in N_2. \end{array} \right\}$$

In particular,

$$\Omega = \Omega(M, \varnothing, \varnothing),$$

$$\Omega_i = \Omega(M \setminus \{i\}, \{i\}, \varnothing) \cup \Omega(M \setminus \{i\}, \varnothing, \{i\}).$$

Lemmas 1 and 2 take the following form. Suppose that the quadratic form $s(w,u)$ is not positive definite in the entire space R^{2m} and (w_0, u_0) is a nonzero vector for which $s(w_0, u_0) \leq 0$.

Lemma 3. $s(w,u) > 0$ in the region Ω of form (4) if and only if the following conditions hold:
(a) $s(w,u) > 0$ in the regions $\Omega(M \setminus \{i\}, \{i\}, \varnothing)$ and $\Omega(M \setminus \{i\}, \varnothing, \{i\})$, $i \in M$,
(b) $(w_0, u_0) \notin \Omega$.

Lemma 4. Let $\pi(s) < n - 1$. Then, $s(w,u) > 0$ in the region Ω of form (4) if and only if condition (a) of Lemma 3 holds.

The matrix of the quadratic form $s(w,u)$ is given by

$$S = \begin{bmatrix} V & T \\ T' & U \end{bmatrix}. \tag{5}$$

For subsets of indices $N_1, N_2 \subseteq M$,$N_1 \cap N_2 = \varnothing$ let $S(N_1, N_2)$ be the matrix obtained from S of form (5) by deleting the rows and columns with indices $i \in N_1$ and $m + i$ for $i \in N_2$. Analysis of positive definiteness of the quadratic form $s(w,u)$ in the set Ω_i is equivalent to two problems: analysis of positive definiteness in the set $\Omega(M \setminus \{i\}, \{i\}, \varnothing)$ and analysis of positive definiteness in the set $\Omega(M \setminus \{i\}, \varnothing, \{i\})$. Both problems have one constraint less than the initial problem has ($w_j u_j \geq 0, j \neq i$) and one variable less. The matrix of the quadratic form of the first problem is obtained from S by striking out the row and the column with the index i. The matrix of the quadratic form of the second problem is obtained from S by deleting the row and the column with the index $m + i$.

When reducing the dimension of the quadratic form by one the number of positive eigenvalues either reduces by one or remains unchanged. Consider the inequality $\pi(s) < n - 1$ in the condition of Lemma 4. When eliminating one variable, the number $n - 1$ on the right-hand side of the inequality decreases by one. The number $\pi(s)$ either decreases or remains unchanged. Thus, condition of Lemma 4, which was satisfied for the initial quadratic form, may be violated after reducing the dimensions by one. Hence, together with verifying the condition (a) of Lemma 3, there may arise necessity to verify also condition (b) for some vector (w_0, u_0) for which $s(w_0, u_0) \leq 0$.

On the other hand, suppose that this condition was satisfied when testing positive definiteness of the quadratic form $s(w,u)$ in the set $\Omega(M \setminus \{i\}, \{i\}, \varnothing)$ or $\Omega(M \setminus \{i\}, \varnothing, \{i\})$ by means of Lemma 3. Then, this condition will also be satisfied when verifying positive definiteness $s(w,u)$ in the original set Ω.

Let us call the matrix $S(N_1, N_2)$ minimal if it has only one nonpositive eigenvalue and all matrices $S(N_1 \cup \{i\}, N_2)$ and $S(N_1, N_2 \cup \{i\})$, obtained from $S(N_1, N_2)$ deleting the row and the column with any of indices i or $m + i$, are positive definite.

For every minimal matrix, one can find a nonzero vector y of the corresponding dimension such that

$$y' S(N_1, N_2) y \leq 0;$$

i.e., there exists $(\bar{w}, \bar{u})$ such that

$$\begin{array}{l} s(\bar{w}, \bar{u}) y \leq 0, \\ \bar{w}_j = 0 \text{ for } j \in N_1, \\ \bar{u}_k = 0 \text{ for } k \in N_2. \end{array} \tag{6}$$

The above discussion, together with inductive application of Lemmas 3 and 4, lead to the following result.

Theorem 2. Let the quadratic form $s(w,u)$ be not positive definite in the entire space R^{2m}. Then, $s(w,u)$ is positive definite in the region Ω of form (4) if and only if the following conditions hold:
(a) every matrix $S(N, M \setminus N)$ is positive definite, $N \subseteq M$,
(b) for every minimal matrix $S(N_1, N_2)$, the corresponding vector $(\bar{w}, \bar{u})$ satisfying condition (6) does not belong to the set $\Omega(M \setminus (N_1 \cup N_2), N_1, N_2)$.

Remark 3. Condition (a) means positive definiteness of the quadratic form $s(w,u)$ in the regions $\Omega(\varnothing, N, M \setminus N)$, which are subspaces $w_i = 0$ for $i \in N$, $u_i = 0$ for $i \in M \setminus N$. In other words, this condition means positive definiteness of all 2^m $m \times m$ matrices obtained from S given by (5) by deleting the rows and columns with indices $i \in N$ and $m + i$ for $i \in M \setminus N$ for all subsets $N \subseteq M$. This is necessary for positive definiteness of $s(w,u)$ in the region Ω.

Remark 4. The case of positive definiteness of the quadratic form $s(w,u)$ is trivial. That is why the matrix S is supposed to be not positive definite. Together with condition (a), this means the existence of minimal matrices.

Theorems 1 and 2 can be applied to the analysis of the Lur'e systems to obtain the results reported in (Rapoport, 1989), (Rapoport, 1996). Another possible application is analysis of attainable sets for linear control systems with several component-wise bounded control inputs.

2. TWO GENERAL FORM CONSTRAINTS

Let $m = 2$. Consider conditions on the quadratic forms $q(x), p_1(x), p_2(x)$ that guarantee losslesness of the S-procedure. This problem has been studied in (Polyak, 1998). We prove a closely related result under less restrictive assumptions using a different method. Namely, condition $n \geq 3$ is removed. Let us denote

$$\hat{\Omega}_i = \{x \in R^n : p_i(x) \geq 0\} \text{ for } i = 1, 2.$$

Then, $\Omega = \hat{\Omega}_1 \cap \hat{\Omega}_2$. For any subset $X \subseteq R^n$, let $T(X)$ denote the mapping $R^n \mapsto R^2$ defined by the relations

$$T(X) = \{(\xi, \eta) \in R^2 : \xi = q(x), \eta = p_1(x), \forall x \in X\}.$$

The well-known Dines theorem (Yakubovich, 1977) says that $T(R^n)$ is a convex cone. Losslessness of the S-procedure for $m = 1$ follows from just this fact. Let us study the set $T(\hat{\Omega}_2)$, which is obviously a closed cone.

Lemma 5. Let $q(x) > 0$ in the set $\Omega = \{x \in R^n : p_1(x) \geq 0, p_2(x) \geq 0\}$. Then, the cone $T(\hat{\Omega}_2) \setminus \{0\}$ is linearly connected.

Proof. Let $y_1 = (\xi_1, \eta_1) \in T(\hat{\Omega}_2) \setminus \{0\}$ and $y_2 = (\xi_2, \eta_2) \in T(\hat{\Omega}_2) \setminus \{0\}$. First, let us assume that $y_1 \neq \alpha y_2$ for all real α and construct the path $y(\tau) \in T(\hat{\Omega}_2) \setminus \{0\}$, where $\tau \in [0,1], y(0) = y_1, y(1) = y_2$. The case of $y_1 = \alpha y_2$ reduces to the first case by choosing y_3 such that $y_1 \neq \alpha y_3$ and $y_3 \neq \alpha y_2$. Then, we construct two paths $\bar{y}(\tau)$ and $\bar{\bar{y}}(\tau)$ connecting y_1 with y_3 and y_3 with y_2, respectively.

Let $x_1 \in \hat{\Omega}_2 \setminus \{0\}$ and $x_2 \in \hat{\Omega}_2 \setminus \{0\}$ satisfy the conditions $y_i = T(x_i)$, $i = 1, 2$. The vectors x_i satisfy the condition $x_1 \neq \beta x_2$ because, otherwise, $y_1 = \beta^2 y_2$ which contradicts our assumption that $y_1 \neq \alpha y_2$. Thus, the line $x(\tau) = (1 - \tau)x_1 + \tau x_2$ does not pass through the origin:

$$x(\tau) \neq 0 \text{ for } \tau \in [0, 1].$$

Consider the function $p_2(x(\tau))$, which is quadratic with respect to τ and satisfies the conditions $p_2(x(0)) \geq 0$ and $p_2(x(1)) \geq 0$. The two following cases are possible:
(1) $p_2(x(\tau)) \geq 0$ for all $\tau \in [0, 1]$;
(2) there are two roots $\bar{\tau}$ and $\bar{\bar{\tau}}$ such that $p_2(x(\bar{\tau})) = 0$, $p_2(x(\bar{\bar{\tau}})) = 0$, and $p_2(x(\tau)) < 0$ for $\tau \in (\bar{\tau}, \bar{\bar{\tau}})$.

In the first case, $x(\tau) \in \hat{\Omega}_2 \setminus \{0\}$ for all $\tau \in [0, 1]$. Moreover, $T(x(\tau)) \neq 0$, because, otherwise, $q(x(\tau)) = 0, p_1(x(\tau)) = 0, p_2(x(\tau)) \geq 0$ for some $\tau \in [0, 1]$, which contradicts the assumptions of the lemma. Thus, $y(\tau) = T(x(\tau)) \in T(\hat{\Omega}_2) \setminus \{0\}$ for all $\tau \in [0, 1]$, which completes the proof of the lemma for the first case. Consider the second case. Let $\bar{x} = x(\bar{\tau})$. Then, $p_2(\bar{x}) = \frac{1}{2}\bar{x}'P_2\bar{x} = 0$ and $\frac{d}{d\tau}p_2(x(\tau))|_{\tau=\bar{\tau}} = \bar{x}'P_2(x_2 - x_1) < 0$. Further,

$$(1 - \bar{\tau})x'P_2(x_2 - x_1) = \bar{x}'P_2(x_2 - \bar{x}) < 0$$

since $\bar{\tau} < 1$. Taking into account $\bar{x}'P_2\bar{x} = 0$, we have

$$\bar{x}'P_2x_2 < 0. \tag{7}$$

Let us denote

$$\tilde{x}(\tau) = -\bar{x} + \frac{\tau - \bar{\tau}}{1 - \bar{\tau}}(x_2 + \bar{x}) \text{ for } \tau \in [\bar{\tau}, 1].$$

Then, the path $\tilde{x}(\tau)$ satisfies the conditions $\tilde{x}(\bar{\tau}) = -\bar{x}$ and $\tilde{x}(1) = x_2$. Further,

$$\frac{d}{d\tau}p_2(\tilde{x}(\tau))|_{\tau=\bar{\tau}} = \frac{1}{1 - \bar{\tau}}(-\bar{x})'P_2(x_2 + \bar{x}),$$

and it follows from (7) and $\bar{x}'P_2\bar{x} = 0$ that

$$\frac{d}{d\tau}p_2(\tilde{x}(\tau))|_{\tau=\bar{\tau}} > 0.$$

Thus, the function $p_2(\tilde{x}(\tau))$ has no roots on the segment $(\bar{\tau}, 1)$. In other words, $\tilde{x}(\tau) \in \hat{\Omega}_2$ for $\tau \in (\bar{\tau}, 1)$. Let

$$\hat{x}(\tau) = \begin{cases} x(\tau) & \tau \in [0, \bar{\tau}], \\ \tilde{x}(\tau) & \tau \in (\bar{\tau}, 1]. \end{cases}$$

Then, $\hat{x}(\tau) \in \hat{\Omega}_2$. Further, it is continuous at the point $\tau = \bar{\tau}$ because $T(\hat{x}(\bar{\tau})) = T(\bar{x}) = T(-\bar{x}) = T(\hat{x}(\bar{\tau} + 0))$. Thus, $T(\hat{x}(\tau)) \in T(\hat{\Omega}_2)$. Finally, it can be proved that $T(\hat{x}(\tau)) \in T(\hat{\Omega}_2) \setminus \{0\}$ in the same way as it was proved in the first case. The lemma is proved. ■

Theorem 3. Let $q(x) > 0$ in Ω. If there exist real values λ and μ such that the quadratic form $\lambda q(x) + \mu p_1(x)$ is positive definite or the quadratic form $\lambda q(x) + \mu p_2(x)$ is positive definite, then there exist $\tau_1 \geq 0$ and $\tau_2 \geq 0$ such that the quadratic form

$$q(x) - \tau_1 p_1(x) - \tau_2 p_2(x) \tag{8}$$

is positive definite.

Proof. Let us assume that the quadratic form $\lambda q(x) + \mu p_1(x)$ is positive definite. The case $\lambda q(x) + \mu p_2(x) > 0$ will follow from symmetry of the problem setup. The condition $\lambda q(x) + \mu p_1(x) > 0$ for $x \neq 0$ implies that $T(R^n) \neq R^2$. In other words, the set $T(R^n)$, which is a convex cone by virtue of the Dines theorem, does not cover the entire plane R^2. Being a convex cone, it belongs to some half plane of R^2. However, $T(\hat{\Omega}_2) \subseteq T(R^n)$ and, therefore, $int(T(\hat{\Omega}_2)) \subseteq T(R^n)$. In the two-dimensional case, the linearly connected set $int(T(\hat{\Omega}_2))$ (by Lemma 5), which belongs to the half plane of R^2, is a convex set. Hence, the set $T(\hat{\Omega}_2)$ is a convex cone. Thus, the S-procedure is lossless for this case, and there exists $\tau_1 \geq 0$ such that

$$q(x) - \tau_1 p_1(x) > 0 \text{ for } x \in \hat{\Omega}_2.$$

Applying the S-procedure that is lossless for the case of single constraint $p_2(x) \geq 0$, we finally obtain positive definiteness of (8). ■

3. THE CASE OF ONE GENERAL FORM CONSTRAINT AND SEVERAL CONSTRAINTS OF SPECIAL FORM

Let us consider the case of $m+1$ constraints, where $p_1(x)$ is a general quadratic form and $p_i(x), i = 2, \dots, m+1$ are presented in form (2). The condition $2m < n$ is also assumed to hold. Following the scheme of the previous section, let

$$\hat{\Omega} = \{x \in R^n : p_i(x) \geq 0,\ i = 2, \dots, m+1\}. \quad (9)$$

Lemma 6. Let $q(x) > 0$ in the set $\Omega = \{x \in R^n : p_i(x) \geq 0,\ i = 1, \dots, m+1\ \}$. Then, the cone $T(\hat{\Omega}) \setminus \{0\}$ is linearly connected.

Proof is similar to the proof of Lemma 5, but the path connecting points x_1 and x_2 is constructed as combination of two segments: $[x_1, x_0]$ and $[x_0, x_2]$, where $x_0 \neq 0$ satisfies the linear equations

$$(f_i' x_0) = 0 \text{ and } (g_i' x_0) = 0,\ i = 2, \dots, m+1.$$

■

Theorem 4. Let $q(x) > 0$ in Ω. If there exist real values λ and μ such that the quadratic form $\lambda q(x) + \mu p_1(x)$ is positive definite, then there exists $\tau \geq 0$ such that the quadratic form

$$q(x) - \tau p_1(x) \quad (10)$$

is positive definite in the region $\hat{\Omega}$ of form (9).

Proof is similar to the proof of Theorem 4.

4. APPLICATION TO THE ANALYSIS OF ABSOLUTE STABILITY OF LUR'E SYSTEMS

Consider the control system

$$\begin{aligned} \dot{x} &= Ax + Bu + Dr, \\ y &= C'x,\ z = E'x, \\ u_i(t) &= \varphi_i(y_i(t)),\ i = 1, \dots, m, \end{aligned} \quad (11)$$

where $x \in R^n$ is state, $u \in R^m$ - control, $r \in R^d$ is disturbance, $y \in R^m$ and $z \in R^l$ are outputs. Matrices A, B, C, D, and E have appropriate dimensions, and matrix A is Hurwitz. Let b_i be the ith column of the matrix B (c_i, d_i, and e_i are defined similarly). Nonlinear functions $\varphi_i(y)$ satisfy the "sector-like" conditions

$$0 \leq y\varphi_i(y) \leq \mu_i y^2 \quad y \in R. \quad (12)$$

Here, $0 < \mu_i < \infty$. Absolute stability of the closed system (11),(12) means stability of the origin $x = 0$ for all feedback functions $\varphi_i(y)$ satisfying (12).

Disturbances $r(\cdot)$ are supposed absent in the absolute stability problem setup. Therefore, we start with the system (11) without variables r and z. The well-known Lyapunov function

$$V(x) = \frac{1}{2}x'Px + \sum_{i=1}^{m} \theta_i \int_0^{c_i'x} \varphi_i(y)dy, \quad (13)$$

is used, where P is a positive definite matrix and θ_i are scalars. The classical Popov criterion was obtained for the case of $m = 1$: let the pair $\{A, b\}$ be controllable, the pair $\{A, c\}$ be observable, and

$$Re(i\omega\theta W_u(i\omega) + W_u(i\omega)) + \frac{1}{\mu} > 0, \quad \omega \in (-\infty, \infty), \quad (14)$$

where $W_u(i\omega) = c'(A - i\omega I)^{-1}b$, $i = \sqrt{-1}$, and I is the identity matrix. Then, the system (11), (12) is absolutely stable. It is well known that the frequency inequality (14) holds if and only if there exists Lyapunov function (13) with negative definite derivative with respect to system (11), (12):

$$\frac{dV}{dt} = (Ax + bu)'(Px + \theta cu) < 0.$$

These conditions are equivalent for $m = 1$. For $m > 1$,

$$\frac{dV}{dt} = (Ax + Bu)'(Px + C\Theta u), \quad (15)$$

where $\Theta = diag(\theta_1, \dots, \theta_m)$. Condition $\frac{dV}{dt} < 0$ leads to the problem described in the abstract, where

$$\begin{aligned} q(x,u) &= -\,(Ax + Bu)'(Px + C\Theta u) \\ p_i(x,u) &= (c_i'x - \mu_i^{-1}u_i)u_i, \quad i = 1, \dots, m. \end{aligned} \quad (16)$$

Application of the S-procedure gives only sufficient conditions of existence of matrices P and Θ that guarantee $q(x,u) > 0$ in the region $p_i(x,u) \geq 0$. The use of only sufficient conditions leads to conservative criteria of absolute stability. Necessary and sufficient conditions of existence of the Lyapunov function (13) are obtained in (Rapoport, 1989) and (Rapoport, 1996). Here, these conditions are obtained in the form of feasibility of LMI. Suppose that $rank(C) = m$. Then, Theorem 1 takes the following form.

Theorem 5. Let $rank(C) = m \leq n$. Then, $q(x,u) > 0$ in the region Ω defined by the conditions $(c_i'x - \mu_i^{-1}u_i)u_i \geq 0$, $i \in M$, if and only if there exists a quadratic form $s(w,u)$ (3) such that the following conditions hold:

(a) $q(x,u) - s(C'x - \mu^{-1}u, u) > 0$ for $(x,u) \neq 0$,
(b) $s(w,u) > 0$ for $w_i u_i \geq 0$, $i \in M$, $(w,u) \neq 0$.

Let $\mu = diag(\mu_1, \ldots, \mu_m)$. To check condition (b) of Theorem 5, Theorem 2 is used. Condition (a) of Theorem 2 requires positive definiteness of all matrices $S(N_1, N_2)$ for $N_1 \cap N_2 = \varnothing$, $N_1 \cup N_2 = M$. To check condition (b) of Theorem 2, one needs to describe all minimal matrices. For every $i \in M$ and subsets $N_1 \subseteq M$ $N_2 \subseteq M$ satisfying the conditions

$$\begin{aligned} &\{i\} \cup N_1 \cup N_2 = M, \\ &N_1 \cap N_2 = \varnothing, \\ &i \notin N_1, i \notin N_2 \end{aligned} \tag{17}$$

let us consider the matrix $S(N_1, N_2)$ and show that, if condition (a) of Theorem 5 holds, then this matrix is minimal. Indeed, $S(N_1, N_2)$ is the matrix of restriction of the quadratic form $s(w,u)$ onto the subspace $w_j = 0, j \in N_1$, $u_k = 0, k \in N_2$. Let (x,u) satisfy the conditions

$$\begin{aligned} &c_j' x - \mu_j^{-1} u_j = 0 \text{ for } j \in N_1, \\ &u_k = 0 \text{ for } k \in N_2. \end{aligned} \tag{18}$$

For every $N \subseteq M$, denote

$$\begin{aligned} A(N) &= (A + \sum_{j \in N} \mu_j b_j c_j'), \\ P(N) &= (P + \sum_{j \in N} \mu_j \theta_j c_j c_j'). \end{aligned}$$

Then, it follows from condition (a) of Theorem 5 that

$$\begin{aligned} &s(C'x - \mu^{-1}u, u) \\ &< -(A(N_1)x + b_i u_i)'(P(N_1)x + c_i \theta_i u_i). \end{aligned} \tag{19}$$

All matrices $A(N)$ are Hurwitzian. Adding the condition

$$\bar{u}_i = 1, \quad \bar{x} = -A(N_1)^{-1} b_i, \tag{20}$$

to conditions (18), we find from (18)-(20) that

$$\begin{aligned} \bar{u}_j &= 0 \quad j \in N_2, \\ \bar{u}_i &= 1, \\ \bar{u}_k &= \mu_k^{-1} c_k' \bar{x} \quad k \in N_1, \\ \bar{w} &= C'\bar{x} - \mu^{-1}\bar{u}, \\ s(\bar{w}, \bar{u}) &< 0, \\ \bar{w}_k &= 0 \quad k \in N_1, \end{aligned}$$

from which it follows that the matrix $S(N_1, N_2)$ is not positive definite. On the other hand, matrices $S(N_1 \cup \{i\}, N_2)$ and $S(N_1, N_2 \cup \{i\})$ are positive definite by condition (a) of Theorem 2. Hence, the matrix $S(N_1, N_2)$ is minimal and, to guarantee the fulfillment of condition (b) of Theorem 2, it is sufficient to require that $(\bar{w}, \bar{u}) \notin \Omega(\{i\}, N_1, N_2)$. In other words, $\bar{w}_i \bar{u}_i < 0$ or

$$-c_i' A(N_1)^{-1} b_i - \mu_i^{-1} < 0. \tag{21}$$

Let $C(N)$ and $B(N)$ be matrices consisting of columns c_j and b_j of the matrices C and B, respectively, $j \in N \subseteq M$. The diagonal matrix $\mu(N)$ is constructed similarly. By the Shur lemma,

$$\begin{aligned} &det \begin{bmatrix} -A & B(N_1 \cup \{i\}) \\ C(N_1 \cup \{i\})' & \mu(N_1 \cup \{i\})^{-1} \end{bmatrix} \\ &= det \begin{bmatrix} -A & B(N_1) \\ C(N_1)' & \mu(N_1)^{-1} \end{bmatrix} \times \\ &\left(-[c_i' \mid 0] \begin{bmatrix} -A & B(N_1) \\ C(N_1)' & \mu(N_1)^{-1} \end{bmatrix}^{-1} \begin{bmatrix} b_i \\ 0 \end{bmatrix} + \mu_i^{-1} \right) \\ &= det \begin{bmatrix} -A & B(N_1) \\ C(N_1)' & \mu(N_1)^{-1} \end{bmatrix} (c_i' A(N_1)^{-1} b_i + \mu_i^{-1}). \end{aligned} \tag{22}$$

On the other hand,

$$\begin{aligned} &det \begin{bmatrix} -A & B(N_1) \\ C(N_1)' & \mu(N_1)^{-1} \end{bmatrix} \\ &= det(\mu(N_1)^{-1} + C(N_1)' A^{-1} B(N_1)) det(-A). \end{aligned}$$

Combining the last identity with (21) and (22), we find that all the expressions

$$det(\mu(N)^{-1} + C(N)' A^{-1} B(N))$$

must have the same sign for all $N \subseteq M$. Thus, the following theorem is proved.

Theorem 6. Let $rank(C) = m \leq n$. Then, $q(x,u) > 0$ in the region Ω defined by the conditions $(c_i' x - \mu_i^{-1} u_i) u_i \geq 0$, $i \in M$, if and only if there exists a quadratic form $s(w,u)$ (3) such that the following conditions hold:
(a) $q(x,u) - s(C'x - \mu^{-1}u, u) > 0$ for $(x,u) \neq 0$,
(b) every matrix $S(N, M \setminus N)$ is positive definite for $N \subseteq M$,
c) all principal minors of the matrix $\mu^{-1} + C'A^{-1}B$ have the same sign.

The last theorem reduces the problem of existence of the Lyapunov function (13) to the problem of feasibility of the LMI.

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