APPROXIMATED OPTIMAL CONTROL OF SINGULARLY PERTURBED SYSTEMS VIA HAAR WAVELETS

Karimi H.R.[1] Jabedar Maralani P.[1] Moshiri B.[1] Lohmann B.[2]

[1] *Control and Intelligent Processing, Center of Excellence, Dept. of Electrical and Computer Engineering, University of Tehran, Tehran, Iran.*
E-mail: hrkarimi@ut.ac.ir
[2] *Institute of Automation, University of Bremen, Otto-Hahn-Allee, NW1, D-28359 Bremen, Germany.*

Abstract: This paper deals with the implementation of Haar wavelet to the optimal control of linear singularly perturbed systems. The approximated composite control and the slow and fast trajectories with respect to a quadratic cost function by solving only the linear algebraic equations are calculated. The results are illustrated with a simple example.

Keywords: Singularly perturbed systems, Haar wavelet, and optimal control.

1. INTRODUCTION

Orthogonal functions like Walsh (Chen and Hsiao, 1965; Rao, 1983), block pulse (Krueger and Knoop, 1990; Rao and Rao, 1979), Laguerre (Hwang and Shin, 1981), Legendre (Chang and Wang, 1984) play an important role to establish algebraic methods for the solution of problems described by differential equations, such as analysis of linear time invariant or time varying systems, model reduction, optimal control and system identification.

Time domain techniques for solving differential equations have received increased attention in the literature in the comparison to frequency domain methods. Wavelet transform (Burrus, et al., 1998) as a new technique for time domain simulations based on the time-frequency localization, or multiresolution property, has been developed into a more and more complete system and found great success in practical engineering problems. Recently, some of the attempts are made in solving surface integral equations, improving the finite difference time domain method, solving linear differential equations and nonlinear partial differential equations and modelling nonlinear semiconductor devices (Ohkita and Kobayashi, 1986; Razzaghi and Ordokhani, 2002). Recently, in (Karimi, et al., 2004a; Karimi, et al. 2004b) a computational method based on Haar wavelet in time-domain for solving optimal control and parameter estimation of the linear time invariant systems for any finite time interval was proposed.
In the sequel of the work by (Karimi, et al., 2004b), we extend the computational method based on Haar wavelet to the optimal control problem of linear singularly perturbed systems. Singularly perturbed systems often occur naturally because of the presence of small parasitic parameters multiplying the time derivatives of some of the system states. Singularly perturbed control systems have been intensively studied for the past three decades and a popular approach adopted to handle these systems is based on the so-called reduced technique; see (Kokotovic, et al., 1986). The composite design based on separate designs for slow and fast subsystems has been systematically reviewed by (Saksena, et al., 1984). In this paper, by utilizing the properties of Haar functions and the integral operation matrix and Kronecker product, we can find the approximated optimal slow and fast dynamics and approximated optimal composite control with respect to a quadratic cost function by solving only the linear algebraic equations instead of solving two Riccati differential equations of the slow and fast subsystems. We demonstrate the applicability of the proposed technique in a simple example.

2. COMPOSITE CONTROL FOR SINGULARLY PERTURBED SYSTEMS

In this paper continuous-time singularly perturbed linear system is given by the state-space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \varepsilon \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \quad (1)$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$ and n ($= n_1 + n_2$) is the order of the whole system, and $u(t) \in \Re^l$ is control vector. The matrices $A_{11} \in \Re^{n_1 \times n_1}$, $A_{12} \in \Re^{n_1 \times n_2}$, $A_{21} \in \Re^{n_2 \times n_1}$, $A_{22} \in \Re^{n_2 \times n_2}$, $B_1 \in \Re^{n_1 \times l}$, $B_2 \in \Re^{n_2 \times l}$ are constant and $\varepsilon \geq 0$ is scalar and real. The quadratic cost function to be minimized is given by

$$J = \int_0^{T_f} (x^T(t) Q x(t) + u^T(t) R u(t))\, dt \quad (2)$$

where $x = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T \in \Re^{n_1 + n_2}$ and the matrices $Q = diagonal(Q_1, Q_2)$ and R are positive semi-definite and positive definite matrices, respectively.
Using the singular perturbation method (Kokotovic, et al., 1986), we establish slow and fast subsystems, and we will derive slow and fast cost functions for the each subsystem. A composite control for the singularly perturbed system is obtained as a combination of optimal control laws of the slow and fast subsystems; i.e.

$$u := u_c = u_s + u_f \quad (3)$$

where u_s and u_f are optimal controls for the slow and fast subsystems, respectively.

2.1. Slow subsystem

First we consider the optimal control for the slow subsystem. Let $\varepsilon = 0$ and assume that A_{22} is non-singular, then we obtain the slow subsystem as

$$\dot{x}_{1s}(t) = A_0 x_{1s}(t) + B_{1s} u_s(t) \quad (4)$$

$$x_{2s}(t) = -A_{22}^{-1}(A_{21} x_{1s}(t) + B_{2s} u_s(t)) \quad (5)$$

where $x_{1s}(0) = x_1(0)$ and $x_{1s}(t)$ and $x_{2s}(t)$ are vectors in $\Re^{n_1}$ and $\Re^{n2}$ respectively, and $u_s(t) \in \Re^l$ represents the control input vector. And the coefficient matrices B_{1s}, A_0 and B_{2s} are given by

$$B_{1s} = B_1 - A_{12} A_{22}^{-1} B_2\ ,\ A_0 = A_{11} - A_{12} A_{22}^{-1} A_{21}\ ,\ B_{2s} = B_2\ . \quad (6)$$

Replacing $x(t)$ with $x_s(t) = \begin{bmatrix} x_{1s}^T(t) & x_{2s}^T(t) \end{bmatrix}^T$ yields the following quadratic cost function for the slow subsystem

$$J_s = \int_0^{T_f} (x_{1s}^T(t) \overline{Q}_0 x_{1s}(t) + 2 x_{1s}^T(t) \overline{Q}_1 u_s(t) + u_s^T(t) \overline{R}\, u_s(t))\, dt \quad (7)$$

where

$$\begin{aligned} \overline{Q}_0 &= Q_1 + A_{21}^T (A_{22}^T)^{-1} Q_2 A_{22}^{-1} A_{21} \\ \overline{Q}_1 &= A_{21}^T (A_{22}^T)^{-1} Q_2 A_{22}^{-1} B_{2s} \\ \overline{R} &= R + B_{2s}^T (A_{22}^T)^{-1} Q_2 A_{22}^{-1} B_{2s}\ . \end{aligned} \quad (8)$$

To remove the cross product term, $2 x_{1s}^T(t) \overline{Q}_1 u_s(t)$, in the cost function (7) we define

$$\overline{u}_s(t) = u_s(t) + \overline{R}^{-1} \overline{Q}_1^T x_{1s}(t)\ , \quad (9)$$

then the slow subsystem (4) and the cost function (7) are rewritten as follows:

$$\dot{x}_{1s}(t) = \overline{A}_0 x_{1s}(t) + B_{1s} \overline{u}_s(t) \quad (10)$$

$$J_s = \int_0^{T_f} (x_{1s}^T(t) Q_{1s} x_{1s}(t) + \overline{u}_s^T(t) \overline{R}\, \overline{u}_s(t))\, dt \quad (11)$$

where $\overline{A}_0 = A_0 - B_{1s} \overline{R}^{-1} \overline{Q}_1^T$ and $Q_{1s} = \overline{Q}_0 - \overline{Q}_1 \overline{R}^{-1} \overline{Q}_1^T$.

2.2. Fast subsystem

Next we consider the fast subsystem with the assumption that the slow variables are constant in the boundary layer. Redefining the fast variables as $x_{2f} = x_2 - x_{2s}$ and the fast controls $u_f = u - u_s$, then the fast subsystem is formulated as follows:

$$\dot{x}_{2f}(t) = \frac{1}{\varepsilon} A_{22} x_{2f}(t) + \frac{1}{\varepsilon} B_2 u_f(t) \quad (12)$$

with the initial condition $x_{2f}(0) = x_2(0) - x_{2s}(0)$. The performance criterion for the fast subsystem is given by

$$J_f = \int_0^{T_f} (x_{2f}^T(\tau) Q_2 x_{2f}(\tau) + u_f^T(\tau) R u_f(\tau))\, d\tau\ . \quad (13)$$

The near-optimality of the composite control law is stated in the following lemma.
Lemma 1 (Kokotovic, et al., 1986). The composite control law of the system (1) with respect to the quadratic cost function (2) is suboptimal in the sense

$$u^{opt}(t) = u_c(t) + \mathrm{O}(\varepsilon), \quad t \geq 0$$

$$x_1(t) = x_{1s}(t) + \mathrm{O}(\varepsilon), \quad t \geq 0$$

$$x_2(t) = x_{2s}(t) + x_{2f}(t) + \mathrm{O}(\varepsilon), \quad t \geq 0$$

where the composite control, $u_c(t)$, is defined as $u_c(t) = \overline{u}_s(t) + u_f(t)$ and $O(\varepsilon)$ is high degree terms of ε parameter.

3. HAAR WAVELETS

The oldest and most basic of the wavelet systems is named Haar wavelet that is a group of square waves with magnitude of ± 1 in certain intervals and zeros elsewhere (Haar, 1910), in other words,

$$\psi(t) = \begin{cases} 1 & for \quad 0 \leq t < \frac{1}{2} \\ -1 & for \quad \frac{1}{2} \leq t < 1 \\ 0 & otherwise \end{cases} \quad (14)$$

and the normalized scaling function is also defined as $\phi(t) = 1$ for $0 \leq t < 1$ and zeros elsewhere. We can easily see that the $\phi(.)$ and $\psi(.)$ are compactly supported, they give a local description, at different scales j, of the considered function. The wavelet series representation of the one-dimensional function $y(t)$ in terms of an orthonormal basis in the interval $[0, 1)$ is given by

$$y(t) = \sum_{i=0}^{\infty} a_i \psi_i(t), \quad (15)$$

where $\psi_i(t)=\psi(2^j t-k)$ for $i\ge 1$ and we write $i=2^j+k$ for $j\ge 0$ and $0\le k<2^j$ and also defined $\psi_0(t)=\phi(t)$. Since, it is not realistic to use an infinite number of wavelets to represent the function $y(t)$ and if $y(t)$ is piecewise constant by itself, or maybe approximated as piecewise constant during each subinterval, then (2) will be terminated at finite terms and we consider the following wavelet representation of the function $y(t)$, namely $\hat{y}(t)$ as follows:

$$\hat{y}(t)=\sum_{i=0}^{m-1} a_i\,\psi_i(t) \;:=\; a^T\,\Psi_m(t) \tag{16}$$

where $a:=[a_0\; a_1 \cdots a_{m-1}]^T$ and $\Psi_m(t):=[\psi_0(t)\;\psi_1(t)\cdots\psi_{m-1}(t)]^T$ for $m=2^j$ and the Haar coefficients a_i are determined to minimize the mean integral square error $\varepsilon=\int_0^1 (y(t)-a^T\,\Psi_m(t))^2\,dt$ and are given by

$$a_i=m\int_0^1 y(t)\,\psi_i(t)\;dt\,. \tag{17}$$

If $\Xi_y(m):=y(t)-\hat{y}(t)$ is the approximation error, then it means that $\hat{y}(t)$ can approximate $y(t)$ to any desired accuracy, which depended on the resolution m.
The matrix H_m can be represented as

$$H_m:=[\Psi_m(t_0),\Psi_m(t_1),\cdots,\Psi_m(t_{m-1})] \tag{18}$$

where $\frac{i}{m}\le t_i<\frac{i+1}{m}$ and using (3), we get

$$[\hat{y}(t_0)\;\hat{y}(t_1)\ldots\hat{y}(t_{m-1})]=a^T\,H_m\,. \tag{19}$$

For further information see (Hsiao and Wang, 2000).
The matrix $P_m=<\int_0^t \Psi_m(r)\,dr,\Psi_m>=\int_0^1\int_0^t \Psi_m(r)\,dr\;\Psi_m^T(t)\;dt$ represents the integral operator for wavelets on the interval at the resolution m. Hence the wavelet integral operational matrix P_m is obtained by

$$\int_0^t \Psi_m(t)\,dt=P_m\,\Psi_m(t)\,. \tag{20}$$

For Haar functions, the square matrix P_m satisfies the following recursive formula (Chen and Hsiao, 1997):

$$P_m=\frac{1}{2m}\begin{bmatrix} 2m\,P_{\frac{m}{2}} & -H_{\frac{m}{2}} \\ H_{\frac{m}{2}}^{-1} & 0\end{bmatrix} \tag{21}$$

with $P_1=\frac{1}{2}$ and $H_m^{-1}=\frac{1}{m}H_m^T\,diagonal(r)$ where H_m defined in (6) and $r:=(1,1,2,2,4,4,4,4,\cdots,\underbrace{(\frac{m}{2}),(\frac{m}{2}),\cdots,(\frac{m}{2})}_{(\frac{m}{2})\;elements})^T$ for $m>2$.

In this paper, we need to evaluate the integration of $\Psi_m(t)\Psi_m^T(t)$ and according to (Hsiao, 2004) the integration relation is as follows:

$$\int_0^1 \Psi_m(\sigma)\Psi_m^T(\sigma)\;d\sigma=\frac{H_m\,H_m^T}{m}\,. \tag{22}$$

4. SOLUTION OF SLOW AND FAST SUBSYSTEMS

Since for finding solution of slow and fast subsystems we use Haar wavelets defined on the interval $[0,1]$, we have to rescale the finite time interval; this can be done by considering the variable σ with $t=\mathrm{T}_f\;\sigma$. In the sequel, we present an algebraic method to calculate solutions of the slow and fast subsystems approximately.

4.1. Solution of slow subsystem

First we focus on solving the slow subsystem. By normalizing (10) with the time scale, we find

$$\dot{x}_{1s}(\sigma)=\mathrm{T}_f(\overline{A}_0\,x_{1s}(\sigma)+B_{1s}\,\overline{u}_s(\sigma))\,. \tag{23}$$

By using the methodology introduced by (Karimi, et al., 2004b), we find the solution of (23) in terms of Haar wavelet basis functions in this form

$$vec(X_{1s})=K_{1s}vec(\overline{U}_s)+K_{2s}\;vec(X_{0s}) \tag{24}$$

where

$$K_{1s}=\mathrm{T}_f\;(I_{m\,n_1}-\mathrm{T}_f\,(P_m^T\otimes\overline{A}_0))^{-1}(P_m^T\otimes B_{1s})\,, \tag{25}$$

$$K_2=(I_{m\,n_1}-\mathrm{T}_f\,(P_m^T\otimes\overline{A}_0))^{-1} \tag{26}$$

with $x_{1s}(\sigma)=X_{1s}\;\Psi_m(\sigma)$ and $\overline{u}_s(\sigma)=\overline{U}_s\;\Psi_m(\sigma)$, which $X_{1s}:n_1\times m$ and $\overline{U}_s:l\times m$ denote the wavelet coefficients of $x_{1s}(\sigma)$ and $\overline{u}_s(\sigma)$ after their expansions in terms of wavelet basis functions and initial condition $x_{1s}(0)=X_{0s}\;\Psi_m(\sigma)$ where the matrix $X_{0s}:n_1\times m$ is defined as $X_{0s}:=\begin{bmatrix} x_{1s}(0) & \underbrace{0\;\ldots\;0}_{(m-1)}\end{bmatrix}$.

Consequently, using (24) and the property of Kronecker product, $vec(ABC)=(C^T\otimes A)\,vec(B)$, the solution of slow subsystem (23) is

$$x_{1s}(\sigma)=(\Psi_m^T(\sigma)\otimes I_{n_1})\;vec(X_{1s})\,. \tag{27}$$

According to (25) and (26), it is trivial that we have to calculate the inverse matrix $(I_{m\,n_1}-\mathrm{T}_f\,(P_m^T\otimes\overline{A}_0))^{-1}$, with dimension $m\,n_1$.

4.2. Solution of fast subsystem

We consider the fast subsystem and using the related time scale, we find the normalized system (12) as follows:

$$\dot{x}_{2f}(\sigma)=\frac{\mathrm{T}_f}{\varepsilon}(A_{22}\,x_{2f}(\sigma)+B_2\,u_f(\sigma))\,. \tag{28}$$

Similar to Section 4.1, we find the solution of the fast subsystem as

$$vec(X_{2f})=K_{1f}vec(U_f)+K_{2f}\;vec(X_{0f}) \tag{29}$$

where

$$K_{1f}=\frac{\mathrm{T}_f}{\varepsilon}\,(I_{m\,n_2}-\frac{\mathrm{T}_f}{\varepsilon}(P_m^T\otimes A_{22}))^{-1}(P_m^T\otimes B_2)\,, \tag{30}$$

$$K_2=(I_{m\,n_2}-\frac{\mathrm{T}_f}{\varepsilon}(P_m^T\otimes A_{22}))^{-1} \tag{31}$$

such $x_{2f}(0) = X_{0f}\ \Psi_m(\sigma)$ with $X_{0f} := \begin{bmatrix} x_{2f}(0) & \underbrace{0 \ \ldots \ 0}_{(m-1)} \end{bmatrix}$ and $x_{2f}(\sigma) = X_{2f}\ \Psi_m(\sigma)$, then the solution of (28) is

$$x_{2f}(\sigma) = (\Psi_m^T(\sigma) \otimes I_{n_2})\ vec(X_{2f})\ . \qquad (32)$$

From (30) and (31), it is clear that we have to once calculate the inverse matrix, $(I_{m n_2} - \frac{\mathrm{T}_f}{\varepsilon}(P_m^T \otimes A_{22}))^{-1}$, with dimension $m n_2$.

5. APPROXIMATED OPTIMAL COMPOSITE CONTROL

In this section, the problem is to find the optimal control of the linear singularly perturbed system of (1) with respect to a quadratic cost functional (2) approximately.

5.1. Optimal control of slow subsystem

For the slow subsystem, we normalize (11) with the related time scale as follows

$$J_s = \mathrm{T}_f \int_0^1 (x_{1s}^T(\sigma) Q_{1s}\, x_{1s}(\sigma) + \overline{u}_s^T(\sigma) \overline{R}\, \overline{u}_s(\sigma))\ d\sigma\ . \qquad (33)$$

Using wavelet transformations of $x_{1s}(\sigma)$ and $\overline{u}_s(\sigma)$, we have

$$J_s = \mathrm{T}_f \int_0^1 (\Psi_m^T(\sigma) X_{1s}^T Q_{1s}\, X_{1s}\, \Psi_m(\sigma) + \Psi_m^T(\sigma) \overline{U}_s^T \overline{R}\, \overline{U}_s\, \Psi_m(\sigma))\ d\sigma \qquad (34)$$

using the property of trace operator, the cost function above can be rewritten as follows

$$J_s = \mathrm{T}_f (tr(M\, X_{1s}^T Q_{1s}\, X_{1s}) + tr(M\, \overline{U}_s^T \overline{R}\, \overline{U}_s)) \qquad (35)$$

where $M = \int_0^1 \Psi_m(\sigma) \Psi_m^T(\sigma)\ d\sigma$. Using the property of Kronecker product, we can write (33) as follows:

$$J_s = (vec^T(X_{1s}) \Pi_{1s}\, vec(X_{1s}) + vec^T(\overline{U}_s) \Pi_{2s}\ vec(\overline{U}_s) \qquad (36)$$

where $\Pi_{1s} = \mathrm{T}_f (M^T \otimes Q_{1s})$ and $\Pi_{2s} = \mathrm{T}_f (M^T \otimes \overline{R})$.

Since the cost functional of J_s becomes a function of $vec(\overline{U}_s)$, then for finding the optimal control law to minimize the cost functional J_s we have to satisfy the following necessary condition

$$\frac{\partial J_s}{\partial vec(\overline{U}_s)} = 0\ . \qquad (37)$$

We find the approximated optimal control by solving (37) as follows (Karimi, et al., 2004b):

$$vec(\overline{U}_s) = -\Pi_{2s}^{-1} K_{1s}^T \Pi_{1s}\ vec(X_{1s}) \qquad (38)$$

and from (24) and (38), we can find

$$vec(X_{1s}) = (I_{m n_1} + K_{1s}\, \Pi_{2s}^{-1} K_{1s}^T\, \Pi_{1s})^{-1} K_{2s} \begin{bmatrix} I_{n_1} & 0 & \cdots & 0 \end{bmatrix}^T x_{1s}(0) \qquad (39)$$

$$vec(\overline{U}_s) = -\Pi_{2s}^{-1} K_{1s}^T \Pi_{1s} (I_{m n_1} + K_{1s}\, \Pi_{2s}^{-1} K_{1s}^T\, \Pi_{1s})^{-1} K_{2s} \times \begin{bmatrix} I_{n_1} & 0 & \cdots & 0 \end{bmatrix}^T x_{1s}(0) \qquad (40)$$

and according to (9) the wavelet coefficients of $u_s(\sigma)$ after expansion in terms of wavelet basis functions will be

$$vec(U_s) = vec(\overline{U}_s) - (I_m \otimes \overline{R}^{-1} \overline{Q}_1^T)\, vec(X_{1s}) \qquad (41)$$

then using $\overline{u}_s(\sigma) = (\Psi_m^T(\sigma) \otimes I_l)\ vec(\overline{U}_s)$ and $x_{1s}(\sigma) = (\Psi_m^T(\sigma) \otimes I_{n_1})\ vec(X_{1s})$, we can calculate the approximated optimal control $\overline{u}_s(\sigma)$ and optimal trajectory $x_{1s}(\sigma)$, respectively. Then, by eliminating $x_{1s}(0)$, we obtain the following result for the slow subsystem:

Theorem 1. Consider the slow subsystem (23) with the cost function J_s in (33). By using the Haar wavelets, the approximated optimal feedback control of the slow subsystem is obtained as follows:

$$\begin{aligned} \overline{u}_s(\sigma) = &-(\Psi_m^T(\sigma) \otimes I_l) \Pi_{2s}^{-1} K_{1s}^T \Pi_{1s}\ (I_{m n_1} + K_{1s}\, \Pi_{2s}^{-1} K_{1s}^T\, \Pi_{1s})^{-1} \\ &\times K_{2s} \begin{bmatrix} I_{n_1} & 0 & \cdots & 0 \end{bmatrix}^T ((\Psi_m^T(\sigma) \otimes I_l)(I_{m n_1} + K_{1s}\, \Pi_{2s}^{-1} K_{1s}^T\, \Pi_{1s})^{-1} \\ &\times K_{2s} \begin{bmatrix} I_{n_1} & 0 & \cdots & 0 \end{bmatrix}^T)^{-1}\ x_{1s}(\sigma) \end{aligned} \qquad (42)$$

and also the approximated slow dynamics will be

$$x_{1s}(\sigma) = (\Psi_m^T(\sigma) \otimes I_{n_1})\ (I_{m n_1} + K_{1s}\, \Pi_{2s}^{-1} K_{1s}^T\, \Pi_{1s})^{-1} K_{2s} \times \begin{bmatrix} I_{n_1} & 0 & \cdots & 0 \end{bmatrix}^T x_{1s}(0)\ . \qquad (43)$$

5.2. Optimal control of fast subsystem

Next we consider the fast subsystem, then normalizing (13) with the related time scale as

$$J_f = \mathrm{T}_f \int_0^1 (x_{2f}^T(\sigma) Q_2\, x_{2f}(\sigma) + u_f^T(\sigma) R u_f(\sigma))\ d\sigma \qquad (44)$$

Using wavelet transformation $x_{2f}(\sigma) = X_{2f}\ \Psi_m(\sigma)$ and $u_f(\sigma) = U_f\ \Psi_m(\sigma)$, which $X_{2f} : n_2 \times m$ and $U_f : l \times m$ denote the wavelet coefficients of $x_{1f}(\sigma)$ and $u_f(\sigma)$ after their expansions in terms of wavelet basis functions, we have

$$J_f = \mathrm{T}_f \int_0^1 (\Psi_m^T(\sigma) X_{2f}^T Q_2\, X_{2f}\, \Psi_m(\sigma) + \Psi_m^T(\sigma) U_f^T R U_f\, \Psi_m(\sigma))\ d\sigma \qquad (45)$$

using the property of trace operator, the cost function above can be rewritten as follows:

$$J_f = \mathrm{T}_f (tr(M\, X_{2f}^T Q_2\, X_{2f}) + tr(M\, U_f^T R U_f))\ . \qquad (46)$$

Using the property of Kronecker product, $tr(ABC) = vec^T(A^T)\,(I_p \otimes B)\ vec(C)$, we can write (46) as

$$J_f = (vec^T(X_{2f}) \Pi_{1f}\ vec(X_{2f}) + vec^T(U_f) \Pi_{2f}\ vec(U_f) \qquad (47)$$

where $\Pi_{1f} = \mathrm{T}_f (M^T \otimes Q_2)$ and $\Pi_{2f} = \mathrm{T}_f (M^T \otimes R)$.

Since the cost functional of J_f becomes a function of $vec(U_f)$, then for finding the optimal control law to minimize the cost functional J_f we have to satisfy the following necessary condition

$$\frac{\partial J_f}{\partial vec(U_f)} = 0\ . \qquad (48)$$

According to (Karimi, et al., 2004b), we find the approximated optimal control by solving (48) as

$$vec(U_f) = -\Pi_{2f}^{-1} K_{1f}^T \Pi_{1f}\ vec(X_{2f}) \qquad (49)$$

and from (29) and (49), we can find $vec(X_{2f})$ and $vec(U_f)$ as

$$vec(X_{2f}) = (I_{m n_2} + K_{1f}\,\Pi_{2f}^{-1}\,K_{1f}^T\,\Pi_{1f})^{-1}\,K_{2f} \times \begin{bmatrix} I_{n_2} & 0 & \cdots & 0 \end{bmatrix}^T x_{2f}(0) \tag{50}$$

$$vec(U_f) = -\Pi_{2f}^{-1}\,K_{1f}^T\,\Pi_{1f}\,(I_{m n_2} + K_{1f}\,\Pi_{2f}^{-1}\,K_{1f}^T\,\Pi_{1f})^{-1} \times K_{2f} \begin{bmatrix} I_{n_2} & 0 & \cdots & 0 \end{bmatrix}^T x_{2f}(0) \tag{51}$$

then using $u_f(\sigma) = (\Psi_m^T(\sigma) \otimes I_l)\, vec(U_f)$ and $x_{2f}(\sigma) = (\Psi_m^T(\sigma) \otimes I_{n2})\, vec(X_{2f})$, the approximated optimal control $u_f(\sigma)$ and optimal trajectory $x_{2f}(\sigma)$ are calculated. Then, by eliminating $x_{2f}(0)$, we obtain the following result for the fast subsystem:

Theorem 2. Consider the fast subsystem (28) with the cost function J_f in (44). By using the Haar wavelets, the approximated optimal feedback control of the fast subsystem is obtained as follows:

$$u_f(\sigma) = -(\Psi_m^T(\sigma) \otimes I_l)\,\Pi_{2f}^{-1}\,K_{1f}^T\,\Pi_{1f}\,(I_{m n_2} + K_{1f}\,\Pi_{2f}^{-1}\,K_{1f}^T\,\Pi_{1f})^{-1} \times K_{2f} \begin{bmatrix} I_{n_2} & 0 & \cdots & 0 \end{bmatrix}^T ((\Psi_m^T(\sigma) \otimes I_l)(I_{m n_2} + K_{1f}\,\Pi_{2f}^{-1}\,K_{1f}^T\,\Pi_{1f})^{-1} \times K_{2f} \begin{bmatrix} I_{n_2} & 0 & \cdots & 0 \end{bmatrix}^T)^{-1}\, x_{2f}(\sigma) \tag{52}$$

and also the approximated fast dynamics will be

$$x_{2f}(\sigma) = (\Psi_m^T(\sigma) \otimes I_{n_2})\,(I_{m n_2} + K_{1f}\,\Pi_{2f}^{-1}\,K_{1f}^T\,\Pi_{1f})^{-1}\,K_{2f} \times \begin{bmatrix} I_{n_2} & 0 & \cdots & 0 \end{bmatrix}^T x_{2f}(0)\,. \tag{53}$$

Using Lemma1, Theorem 1 and Theorem 2, the composite control law of the system (1) with respect to the quadratic cost function (2) can be obtained as follows:

$$u^{opt}(t) = \bar{u}_s(t) + u_f(t) + \mathrm{O}(\varepsilon), \quad 0 \le t \le \mathrm{T}_f \tag{54}$$

$$x_1(t) = x_{1s}(t) + \mathrm{O}(\varepsilon), \quad 0 \le t \le \mathrm{T}_f \tag{55}$$

$$x_2(t) = x_{2s}(t) + x_{2f}(t) + \mathrm{O}(\varepsilon), \quad 0 \le t \le \mathrm{T}_f\,. \tag{56}$$

where $x_{2s}(t)$ can be calculated with (5).

6. NUMERICAL RESULTS

Let us consider the singularly perturbed system and related quadratic cost function with initial condition $x_1(0) = x_2(0) = 1$ and the perturbed parameter $\varepsilon = 0.1$ as

$$\begin{bmatrix} \dot{x}_1(t) \\ \varepsilon \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \tag{57}$$

$$J = \int_0^1 (x^2(t) + u^2(t))\, dt\,. \tag{58}$$

According to section 2.1, the slow subsystem of (57) and its cost function, $x_{1s}(0) = x_1(0)$, are as follows:

$$\dot{x}_{1s}(t) = -5\,x_{1s}(t) - \bar{u}_s(t)\,, \tag{59}$$

$$J_s = \int_0^1 (1.5\,x_{1s}^2(t) + 0.5\,\bar{u}_s^2(t))\, dt \tag{60}$$

By solving the Riccati differential equation, the analytical solution of (59-60) is obtained as (Athans and Flab, 1996)

$$x_s(t) = \frac{1.5\,(1 - e^{4\sqrt{7}(t-1)})}{5 + 2\sqrt{7} + (2\sqrt{7} - 5)e^{4\sqrt{7}(t-1)}}\, u_s(t)\,. \tag{61}$$

To calculate the approximate solution of the slow sate, $x_{1s}(t)$, and compare with the analytic solution (61), we choose the resolution level $j = 3$ and both the approximate values of state and optimal control using Haar functions and the exact values of state and optimal control are tabulated in Table 1 for comparison. Referring to Section 2.2, the fast subsystem of (57) and its cost function would be found as:

$$\dot{x}_{2f}(t) = -\frac{1}{\varepsilon} x_{2f}(t) + \frac{1}{\varepsilon} u_f(t)\,, \tag{62}$$

$$J_f = \int_0^1 (x_{2f}^2(t) + u_f^2(t))\, dt\,, \tag{63}$$

where the initial condition of $x_{2f}(t)$, $x_{2f}(0) = x_2(0) - x_{2s}(0)$, after some calculations will be $x_{2f}(0) = L_s(0) = 0.1457$. The analytical solution of (62-63) is stated as

$$x_{2f}(t) = \frac{\varepsilon\,(1 - e^{\frac{2\sqrt{2}}{\varepsilon}(t-1)})}{1 + \sqrt{2} + (\sqrt{2} - 1)\,e^{\frac{2\sqrt{2}}{\varepsilon}(t-1)}}\, u_f(t)\,. \tag{64}$$

Also, according to Lemma 1 the approximated optimal composite control and approximated states of the system (57), i.e. $x_1(t)$ and $x_2(t) = x_{2s}(t) + x_{2f}(t)$, with respect to the cost function (58) are given in Table 2 and compared with those exact values.

Table 1 Approximate values of $x_{1s}(t)$ and $\bar{u}_s(t)$ by Haar wavelets at resolution level $j = 3$ and those exact values

		Haar wavelets		*Analytic solution*	
i	t_i	$x_{1s}(t_i)$	$\bar{u}_s(t_i)$	$x_{1s}(t_i)$	$\bar{u}_s(t_i)$
0	0.0000	0.9972	-0.1435	1.0000	-0.1458
1	0.1250	0.5235	-0.0750	0.5256	-0.0795
2	0.2500	0.2715	-0.0412	0.2763	-0.0434
3	0.3750	0.1410	-0.0218	0.1452	-0.0236
4	0.5000	0.0705	-0.0105	0.0763	-0.0128
5	0.6250	0.0400	-0.0045	0.0401	-0.0068
6	0.7500	0.0205	-0.0018	0.0211	-0.0035
7	0.8750	0.0100	-0.0005	0.0111	-0.0015

Table 2. Approximate values of $x_1(t)$, $x_2(t)$ and $u^{opt}(t)$ by Haar wavelets at resolution level $j=3$ and those exact values

		Haar wavelets			*Analytic solution*		
i	t_i	$x_1(t_i)$	$x_2(t_i)$	$u^{opt}(t_i)$	$x_1(t_i)$	$x_2(t_i)$	$u^{opt}(t_i)$
0	0.0000	0.9972	0.9950	-0.1505	1.0000	1.0000	-0.1518
1	0.1250	0.5235	0.4630	-0.0750	0.5256	0.4657	-0.0.795
2	0.2500	0.2715	0.2518	-0.0412	0.2763	0.2539	-0.0433
3	0.3750	0.1410	0.1365	-0.0218	0.1452	0.1385	-0.0236
4	0.5000	0.0705	0.0700	-0.0105	0.0763	0.0755	-0.0128
5	0.6250	0.0400	0.0400	-0.0045	0.0401	0.0413	-0.0068
6	0.7500	0.0205	0.0210	-0.0018	0.0211	0.0227	-0.0035
7	0.8750	0.0100	0.0105	-0.0005	0.0111	0.0127	-0.0015

Since the Riccati differential equation is a nonlinear equation, the merits of the Haar wavelets should be appreciated from the transformation of these nonlinear differential equations into algebraic equations systematically.

7. CONCLUSION

An implementation of the Haar wavelet to the optimal control of linear singularly perturbed systems for any finite time interval was proposed. By using results of (Karimi, et al., 2004b), the approximated composite control with respect to a quadratic cost function by solving only the linear algebraic equations were calculated and the results were illustrated with a simple example.

REFERENCES

Athans M., and P.L., Flab (1996). *Optimal control*. McGraw-Hill, New York.

Burrus C.S., Gopinath R.A. and H., Guo (1998). *Introduction to wavelets and wavelet transforms*. Prentice Hall, Upper Saddle River, New Jersey.

Chang R.Y. and M.L., Wang (1984). *Legendre polynomials approximation to dynamical linear state-space equations with initial and boundary value conditions*. Int. J. Control, Vol. 40, pp. 215-232.

Chen C.F. and C.H., Hsiao (1997). *Haar wavelet method for solving lumped and distributed-parameter systems.* IEE Proc. Control Theory Appl., Vol. 144, no. 1, pp. 87-94.

Chen C.F. and C.H., Hsiao (1965). *A state-space approach to walsh series solution of linear systems.* Int. J. System Sci., Vol. 6, no. 9, pp. 833-858.

Haar A., (1910). *Zur theorie der orthogonalen funktionensysteme*. Math. Annalen, Vol. 69, pp. 331-371.

Hsiao C.H., (2004). *Solution of variational problems via haar orthonormal wavelet direct method.* Int. J. Computer Mathematics, Vol. 81, no. 7, pp. 871-887.

Hsiao C.H. and W.J., Wang (2000). *State analysis and parameter estimation of bilinear systems via haar wavelets*. IEEE Trans. Circuits and Systems I: Fundamental Theory and Applications, Vol. 47, no. 2, pp. 246-250.

Hwang C. and Y.P., Shin (1981). *Laguerre operational matrices for fractional calculus and applications*. Int. J. Control, Vol. 34, pp. 557-584.

Karimi H.R., Lohmann B., Jabedar Maralani P. and B., Moshiri (2004a). *A wavelet-based approach for optimal control of second-order linear systems*' 12th Mediterranean Conference on Control and Automation, Turkey.

Karimi H.R., Lohmann B., Jabedar Maralani P. and B., Moshiri (2004b). *A computational method for solving optimal control and parameter estimation of linear systems using haar wavelets*. Int. J. Computer Mathematics, Vol. 81, no. 9, pp. 1121-1132.

Kokotovic P.V., R.E. Khalil and J. Oreilly (1986). *Singular perturbation methods in control: analysis design.* New York: Academic.

Krueger K. and M., Knoop (1990). *Design of an optimal time-varying state variable feedback by means of an algebraisation method*. Automatisierungstechnik, Vol. 38, no. 9, pp. 343-350.

Ohkita M. and Y. Kobayashi (1986). *An application of rationalized haar functions to solution of linear differential equations*. IEEE Trans. On Circuit and Systems, Vol. 9, pp. 853-862.

Rao, G.P., (1983). *Piecewise constant orthpgonal functions and their application to systems and control.* Springer-Verlag, Berlin, Heidelberg.

Rao, V.P. and K.R., Rao (1979). *Optimal feedback control via block-pulse functions*. IEEE Trans. Automatic Control, Vol. 24, no. 2, pp. 372-374.

Razzaghi M. and Y. Ordokhani (2002). *A rationalized haar functions method for nonlinear fredholm-hammerstein integral equations.* Inter. J. Computer Math., Vol. 79 (3), pp. 333-343.

Saksena V.R., J. OReilly and P.V. Kokotovic (1984). *Singular perturbations and time-scale methods in control theory: Survey 1976-1983*. Automatica, Vol. 20, no. 2, pp. 273-293.