ITERATIVE LEARNING CONTROL FOR A CLASS OF NONLINEAR SYSTEMS WITH PARAMETRIC UNCERTAINTIES

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Abstract: This paper addresses some issues pertinent to Iterative Learning Control (ILC) of a class of nonlinear systems with time-varying parametric uncertainties. The new control strategy combines the backstepping technique with ILC. The Energy-Function-based (EF-based) approach is employed to derive the control algorithm and analyze learning convergence. Rigorous mathematical proof shows that the proposed learning scheme is able to learn from different control targets and guarantee learning convergence for systems with high relative degree and unmatched parametric uncertainties. A numerical example is given to demonstrate the effectiveness of the proposed approach.

Keywords: Iterative Learning Control; Relative Degree; Unmatched Parametric Uncertainty; Backstepping Technique; Energy-Function-Based Approach

1. INTRODUCTION

The concept of Iterative Learning Control (ILC) was first proposed and formulated by Arimoto (Arimoto *et al.*, 1984). ILC focus on a certain category of learning problems where both the control process and the tracking task are repeatable over a finite-time interval. The ultimate control objective of ILC is to iteratively obtain perfect tracking over the entire time interval in the presence of system uncertainties. To date, a lot of ILC schemes have been developed and widely applied.Traditional ILC approaches with the target of output tracking are based on Contraction Mapping (CM) principle and generally are only applicable to global Lipschitz continuous systems with relative degree equal to zero. For systems with higher relative degree, the output derivative, equal to the system relative degree, has to be employed in the control signal (Ahn *et al.*, 1993). Moreover, to guarantee that perfect tracking can be obtained by iterations, the control target must be strictly repeatable. To further extend the implementation areas of ILC, Energy-Function-based (EF-based) iterative learnings aiming at states tracking were proposed for non-global Lipschitz continuous systems (Ham *et al.*, 1995; French and Rogers, 2000; Xu and Tan, 2002). In (Xu and Xu, 2004), EF-based ILC for learning from different control targets was developed with rigorous proof. All these works clearly demonstrate the great potential of EF-based analysis method. However, all the uncertainties considered therein must be matched with control inputs, i.e. uncertainties enter the state equations right at the point where the control actions enter. Hence, high relative degree and unmatched uncertainties are two difficult problems in ILC areas.

Several works about ILC for systems with high relative degree have been reported in (Sun and Wang, 2001; Chien and Yao, 2004). A kind of sampled-date ILC based on CM principle was proposed for global Lipschitz continuous systems with arbitrary relative degree in (Sun and Wang, 2001). However, only bounded tracking errors can be guaranteed. An adaptive output-based ILC scheme has been proposed for systems with high relative degree in (Chien and Yao, 2004). Nevertheless, they are only applicable to linear or feedback linearizable systems. Therefore, it will be interesting to explore the possibility of developing new ILC approaches for non-global Lipschitz systems with high relative degree and unmatched uncertainties.

Recently, a recursive design methodology named backstepping has been proposed for the construction of various types of control methodologies: adaptive control, robust control, neural networks control, optimal control, repetitive learning control etc. (Freeman and Kokotović, 1993; Krstić *et al.*, 1995).The main feature of the backstepping technique is that it can alleviate the restriction on system relative degree and handle unmatched uncertainties easily. In this work, we combine the backstepping technique with EF-based ILC and develop a novel ILC algorithm for a class of nonlinear systems with time-varying parametric uncertainties. The new constructed control strategy can not only learn from different control targets in the presence of unmatched time-varying parametric uncertainties but also removes the requirement of system relative degree, which greatly broadens the application domain of ILC. Rigorous mathematical proof shows that the developed ILC scheme can guarantee learning convergence of the output tracking error, i.e. perfect tracking can be obtained as the iteration number approaches to infinity.

The paper is organized as follows. Section II formulates the tracking control problem for a class of nonlinear systems with unmatched time-varying uncertainties and high relative degree. In Section III, a new ILC scheme based on the backstepping technique is proposed. Convergence analysis based on energy functions is also outlined in this section. The proposed ILC approach is implemented to a single-link robotic manipulator with a flexible joint in Section IV to illustrate its effectiveness. Finally, Section V concludes the paper.

2. PROBLEM FORMULATION

Consider the following dynamic system.

$$\dot{x}_1 = b_1(t)x_2 + \boldsymbol{\theta}_1(t)\boldsymbol{\xi}_1(x_1, t) \quad (1)$$

$$\dot{x}_2 = b_2(t)x_3 + \boldsymbol{\theta}_2(t)\boldsymbol{\xi}_2(x_1, x_2, t) \quad (2)$$

$$\dot{x}_3 = b_3(t)x_4 + \boldsymbol{\theta}_3(t)\boldsymbol{\xi}_3(x_1, x_2, x_3, t) \quad (3)$$

$$\cdots$$

$$\dot{x}_n = b_n(t)u + \boldsymbol{\theta}_n(t)\boldsymbol{\xi}_n(x_1, \cdots, x_n, t) \quad (4)$$

$$y = x_1, \quad (5)$$

where $u \in R$ is the system control input, $y \in R$ is the system output, $\forall j \in N \triangleq \{1, \cdots, n\}$, $x_j \in R$ is the measurable system state, $b_j \triangleq b_j(t)$ indicates control direction, $\boldsymbol{\theta}_j \triangleq \boldsymbol{\theta}_j(t) \in R^{1\times n_j}$ is a vector of unknown time-varying parameters, and $\boldsymbol{\xi}_j \triangleq \boldsymbol{\xi}_j(x_1, \cdots, x_j, t) \in R^{n_j}$ is a vector of known functions. Here, b_j, $\boldsymbol{\theta}_j$ and $\boldsymbol{\xi}_j$ are all smooth functions with respect to their arguments. The system dynamics (1) - (5) are repeatable over the time interval $[0, T]$, where T is a finite constant. The following assumption is further made for b_j where $j \in N$.

Assumption 1. Assume b_j $(j \in N)$ is non-singular and its sign is known and invariant over the time interval $[0, T]$.

Without loss of generality, assume that $b_j > 0$ for all $t \in [0, T]$ and $j \in N$. As part of the repeatability, the following identical initial condition (i.i.c.) is satisfied.

Assumption 2. $y_i(0) = y_{d,i}(0)$ and $x_{j,i}(0) = 0$, for any $j \in \{2, \cdots, n\}$ and $i \in Z_+ \triangleq \{1, 2, \cdots\}$, where i denotes the iteration number and $y_{d,i} \in C^n([0, T])$ represents the desired system output.

Remark 1. As our control objective is perfect tracking over the entire time interval, i.e. perfect tracking from the very beginning, $y_i(0) = y_{d,i}(0)$ is essential. $\forall j \in \{2, \cdots, n\}$, if $x_{j,i}(0) = 0$ cannot be satisfied, a simple variable replacement $x'_{j,i} = x_{j,i} - x_{j,i}(0)$ can be implemented to guarantee $x'_{j,i}(0) = 0$.

The ultimate control target is to iteratively determine a sequence of control input u_i such that the tracking error $y_{d,i} - y_i$ converges to zero as the iteration number i approaches infinity. Note that the target trajectories $y_{d,i}$ could be different from iteration to iteration.

3. ALGORITHM DEVELOPMENT AND CONVERGENCE ANALYSIS

3.1 Algorithm Development

The new ILC algorithm is constructed by means of the backstepping technique as follows:

Step 1: First, let us consider system dynamics (1). Treat $x_{2,i}$ as a virtual control input, hence the uncertainty $\boldsymbol{\theta}_1$ becomes a matched one. Define $z_{1,i} = y_{d,i} - x_{1,i}$ and from (1), we have

$$\begin{aligned}\dot{z}_{1,i} &= \dot{y}_{d,i} - b_1 x_{2,i} - \boldsymbol{\theta}_1 \boldsymbol{\xi}_{1,i} \\ &= b_1(-x_{2,i} - \boldsymbol{\theta}_1^o \boldsymbol{\xi}_{1,i}^o), \end{aligned} \tag{6}$$

where $\boldsymbol{\theta}_1^o \triangleq [b_1^{-1}, \;\; b_1^{-1}\boldsymbol{\theta}_1]$ and $\boldsymbol{\xi}_{1,i}^o \triangleq [-\dot{y}_{d,i}, \;\; \boldsymbol{\xi}_{1,i}^T]^T$. By virtue of the EF-based ILC method, the virtual control input $x_{2,i}$ can be designed as follows:

$$x_{2,i} = -\hat{\bar{\boldsymbol{\theta}}}_{1,i} \bar{\boldsymbol{\xi}}_{1,i} + k_1 z_{1,i} \triangleq \alpha_{1,i}, \tag{7}$$

$$\hat{\bar{\boldsymbol{\theta}}}_{1,i} = \hat{\bar{\boldsymbol{\theta}}}_{1,i-1} - \beta_1 z_{1,i} \bar{\boldsymbol{\xi}}_{1,i}^T, \qquad \hat{\bar{\boldsymbol{\theta}}}_{1,0} = \mathbf{0}, \tag{8}$$

where $\hat{\bar{\boldsymbol{\theta}}}_{1,i} \in R^{1\times(n_1+2)}$ is used to approximate $\bar{\boldsymbol{\theta}}_1 \triangleq [\boldsymbol{\theta}_1^o, \;\; b_1^{-2}\dot{b}_1]$, $\bar{\boldsymbol{\xi}}_{1,i} \triangleq [\boldsymbol{\xi}_{1,i}^{oT}, \;\; \frac{1}{2} z_{1,i}]^T$, $k_1 > 0$ is the feedback gain and $\beta_1 > 0$ is the learning gain. To facilitate the following development, we define the energy function as $E_{1,i}(t) = \frac{1}{2} b_1^{-1} z_{1,i}^2 + \frac{1}{2\beta_1}\int_0^t \|\delta\bar{\boldsymbol{\theta}}_{1,i}\|^2 d\tau$, where $\delta\bar{\boldsymbol{\theta}}_{1,i} \triangleq \bar{\boldsymbol{\theta}}_{1,i} - \hat{\bar{\boldsymbol{\theta}}}_{1,i}$. Hence, based on the virtual input (7), $\forall t \in [0, T]$, the difference of the energy function $\Delta E_{1,i}(t)$ is given by

$$\begin{aligned}\Delta E_{1,i}(t) &= E_{1,i}(t) - E_{1,i-1}(t) \\ &= \frac{1}{2} b_1^{-1} z_{1,i}^2 - \frac{1}{2} b_1^{-1} z_{1,i-1}^2 \\ &\quad + \frac{1}{2\beta_1}\int_0^t (\|\delta\bar{\boldsymbol{\theta}}_{1,i}\|^2 - \|\delta\bar{\boldsymbol{\theta}}_{1,i-1}\|^2) d\tau. \end{aligned} \tag{9}$$

Considering Assumption 2 and (6), the first term on the right-hand side of (9) is

$$\begin{aligned}\frac{1}{2} b_1^{-1} z_{1,i}^2 &= \int_0^t (b_1^{-1} z_{1,i}\dot{z}_{1,i} - \frac{1}{2} b_1^{-2}\dot{b}_1 z_{1,i}^2) d\tau \\ &= \int_0^t [z_{1,i}(-x_{2,i} - \boldsymbol{\theta}_1^o \boldsymbol{\xi}_{1,i}^o) - \frac{1}{2} b_1^{-2}\dot{b}_1 z_{1,i}^2] d\tau \\ &= \int_0^t z_{1,i}(-x_{2,i} - \bar{\boldsymbol{\theta}}_1 \bar{\boldsymbol{\xi}}_{1,i}) d\tau, \end{aligned} \tag{10}$$

where $\bar{\boldsymbol{\theta}}_1$ and $\bar{\boldsymbol{\xi}}_{1,i}$ are defined in (8). Substituting (7) into (10) yields

$$\frac{1}{2} b_1^{-1} z_{1,i}^2 = -\int_0^t z_{1,i}\delta\bar{\boldsymbol{\theta}}_{1,i}\bar{\boldsymbol{\xi}}_{1,i} d\tau - \int_0^t k_1 z_{1,i}^2 d\tau. \tag{11}$$

It can be shown that the second term on the right-hand side of (9) is given by

$$\begin{aligned}&\frac{1}{2\beta_1}\int_0^t (\|\delta\bar{\boldsymbol{\theta}}_{1,i}\|^2 - \|\delta\bar{\boldsymbol{\theta}}_{1,i-1}\|^2) d\tau \\ &= \frac{1}{2\beta_1}\int_0^t (\hat{\bar{\boldsymbol{\theta}}}_{1,i} - \hat{\bar{\boldsymbol{\theta}}}_{1,i-1})(\hat{\bar{\boldsymbol{\theta}}}_{1,i} + \hat{\bar{\boldsymbol{\theta}}}_{1,i-1} - 2\bar{\boldsymbol{\theta}}_1)^T d\tau \\ &\leq -\frac{1}{\beta_1}\int_0^t \delta\bar{\boldsymbol{\theta}}_{1,i}(\hat{\bar{\boldsymbol{\theta}}}_{1,i} - \hat{\bar{\boldsymbol{\theta}}}_{1,i-1})^T d\tau. \end{aligned} \tag{12}$$

Considering the updating law (8), we have

$$\begin{aligned}&\frac{1}{2\beta_1}\int_0^t (\|\delta\bar{\boldsymbol{\theta}}_{1,i}\|^2 - \|\delta\bar{\boldsymbol{\theta}}_{1,i-1}\|^2) d\tau \\ &\leq \int_0^t z_{1,i}\delta\bar{\boldsymbol{\theta}}_{1,i}\bar{\boldsymbol{\xi}}_{1,i} d\tau. \end{aligned} \tag{13}$$

Substituting (11) and (13) into (9) yields

$$\Delta E_{1,i}(t) \leq -\frac{1}{2} b_1^{-1} z_{1,i-1}^2. \tag{14}$$

Step 2: Consider system dynamics (2) and treat $x_{3,i}$ as a virtual input. Define $z_{2,i} \triangleq \alpha_{1,i} - x_{2,i}$. According to (2), (6) and (7), the following result can be established:

$$\begin{aligned}\dot{z}_{2,i} &= \dot{\alpha}_{1,i} - \dot{x}_{2,i} \\ &= -\dot{\hat{\bar{\boldsymbol{\theta}}}}_{1,i}\bar{\boldsymbol{\xi}}_{1,i} - \hat{\bar{\boldsymbol{\theta}}}_{1,i}(\frac{\partial \bar{\boldsymbol{\xi}}_{1,i}}{\partial x_{1,i}}\dot{x}_{1,i} + \frac{\partial \bar{\boldsymbol{\xi}}_{1,i}}{\partial t}) + k_1\dot{z}_{1,i} \\ &\quad - b_2 x_{3,i} - \boldsymbol{\theta}_2\boldsymbol{\xi}_{2,i} \\ &= -\dot{\hat{\bar{\boldsymbol{\theta}}}}_{1,i}\bar{\boldsymbol{\xi}}_{1,i} - \hat{\bar{\boldsymbol{\theta}}}_{1,i}[\frac{\partial \bar{\boldsymbol{\xi}}_{1,i}}{\partial x_{1,i}}(b_1 x_2 + \boldsymbol{\theta}_1\boldsymbol{\xi}_{1,i}) + \frac{\partial \bar{\boldsymbol{\xi}}_{1,i}}{\partial t}] \\ &\quad - k_1 b_1 x_{2,i} - k_1 b_1 \boldsymbol{\theta}_1^o\boldsymbol{\xi}_{1,i}^o - b_2 x_{3,i} - \boldsymbol{\theta}_2\boldsymbol{\xi}_{2,i} \\ &= b_2(-x_{3,i} - \boldsymbol{\theta}_2^o\boldsymbol{\xi}_{2,i}^o), \end{aligned} \tag{15}$$

where $\boldsymbol{\theta}_2^o \triangleq b_2^{-1}[1, \;\; b_1, \;\; \boldsymbol{\theta}_1, \;\; \boldsymbol{\theta}_2, \;\; b_1\boldsymbol{\theta}_1^o]$ and $\boldsymbol{\xi}_{2,i}^o \triangleq [(\dot{\hat{\bar{\boldsymbol{\theta}}}}_{1,i}\bar{\boldsymbol{\xi}}_{1,i} + \hat{\bar{\boldsymbol{\theta}}}_{1,i}\frac{\partial\bar{\boldsymbol{\xi}}_{1,i}}{\partial t}), \;\; (\hat{\bar{\boldsymbol{\theta}}}_{1,i}\frac{\partial\bar{\boldsymbol{\xi}}_{1,i}}{\partial x_{1,i}} + k_1)x_{2,i}, \;\; \hat{\bar{\boldsymbol{\theta}}}_{1,i}\frac{\partial\bar{\boldsymbol{\xi}}_{1,i}}{\partial x_{1,i}}\boldsymbol{\xi}_{1,i}^T, \;\; \boldsymbol{\xi}_{2,i}^T, \;\; k_1\boldsymbol{\xi}_{1,i}^{oT}]^T$. The virtual control input $x_{3,i}$ is constructed as follows:

$$x_{3,i} = -\hat{\bar{\boldsymbol{\theta}}}_{2,i}\bar{\boldsymbol{\xi}}_{2,i} + k_2 z_{2,i} + z_{1,i} \triangleq \alpha_{2,i} \tag{16}$$

$$\hat{\bar{\boldsymbol{\theta}}}_{2,i} = \hat{\bar{\boldsymbol{\theta}}}_{2,i-1} - \beta_2 z_{2,i}\bar{\boldsymbol{\xi}}_{2,i}^T, \qquad \hat{\bar{\boldsymbol{\theta}}}_{2,0} = \mathbf{0}. \tag{17}$$

Similarly, $\hat{\bar{\boldsymbol{\theta}}}_{2,i}$ is used to approximate $\bar{\boldsymbol{\theta}}_2 \triangleq [\boldsymbol{\theta}_2^o, \;\; b_2^{-2}\dot{b}_2]$, $\bar{\boldsymbol{\xi}}_{2,i} \triangleq [\boldsymbol{\xi}_{2,i}^{oT}, \;\; \frac{1}{2}z_{2,i}]^T$, $k_2 > 0$ is the feedback gain and $\beta_2 > 0$ is the learning gain.

Define $E_{2,i}(t) = E_{21,i}(t) + E_{22,i}(t)$, where $E_{21,i}(t) = E_{1,i}(t)$, $E_{22,i}(t) = \frac{1}{2}b_2^{-1}z_{2,i}^2 + \frac{1}{2\beta_2}\int_0^t \|\delta\bar{\boldsymbol{\theta}}_{2,i}\|^2 d\tau$ and $\delta\bar{\boldsymbol{\theta}}_{2,i} \triangleq \bar{\boldsymbol{\theta}}_2 - \hat{\bar{\boldsymbol{\theta}}}_{2,i}$. From the definition of $E_{1,i}(t)$, it can be shown that

$$\begin{aligned}\Delta E_{21,i}(t) &= E_{21,i}(t) - E_{21,i-1}(t) \\ &= \Delta E_{1,i}(t)|_{x_{2,i} = \alpha_{1,i} - z_{2,i}}. \end{aligned} \tag{18}$$

Substituting the relationship $x_{2,i} = \alpha_{1,i} - z_{2,i}$ into (10), we have

$$\frac{1}{2}b_1^{-1}z_{1,i}^2 = \int_0^t z_{1,i}(-\alpha_{1,i} + z_{2,i} - \bar{\boldsymbol{\theta}}_1\bar{\boldsymbol{\xi}}_{1,i})d\tau$$
$$= -\int_0^t z_{1,i}\delta\boldsymbol{\theta}_{1,i}\bar{\boldsymbol{\xi}}_{1,i}d\tau - \int_0^t k_1 z_{1,i}^2 d\tau$$
$$+ \int_0^t z_{1,i}z_{2,i}d\tau.$$

Using the result of (13), we have

$$\Delta E_{21,i}(t) = \Delta E_{1,i}(t)|_{z_{2,i}=0} + \int_0^t z_{1,i}z_{2,i}d\tau$$
$$\leq -\frac{1}{2}b_1^{-1}z_{1,i-1}^2 + \int_0^t z_{1,i}z_{2,i}d\tau. \quad (19)$$

On the other hand, we have

$$\Delta E_{22,i}(t) = \frac{1}{2\beta_2}\int_0^t (\|\delta\bar{\boldsymbol{\theta}}_{2,i}\|^2 - \|\delta\bar{\boldsymbol{\theta}}_{2,i-1}\|^2)d\tau$$
$$+ \frac{1}{2}b_2^{-1}z_{2,i}^2 - \frac{1}{2}b_2^{-1}z_{2,i-1}^2. \quad (20)$$

Analogous to (10) and (11), considering (15) and (16), the first term on the right-hand side of (20) can be rewritten as

$$\frac{1}{2}b_2^{-1}z_{2,i}^2$$
$$= \int_0^t z_{2,i}(-x_{3,i} - \bar{\boldsymbol{\theta}}_2\bar{\boldsymbol{\xi}}_{2,i})d\tau$$
$$= -\int_0^t z_{2,i}\delta\bar{\boldsymbol{\theta}}_{2,i}\bar{\boldsymbol{\xi}}_{2,i}d\tau - \int_0^t z_{1,i}z_{2,i}d\tau$$
$$- \int_0^t k_2 z_{2,i}^2 d\tau. \quad (21)$$

From the updating law (17) and the result given in (13), it can be shown that

$$\frac{1}{2\beta_2}\int_0^t (\|\delta\bar{\boldsymbol{\theta}}_{2,i}\|^2 - \|\delta\bar{\boldsymbol{\theta}}_{2,i-1}\|^2)d\tau$$
$$\leq \int_0^t z_{2,i}\delta\bar{\boldsymbol{\theta}}_{2,i}\bar{\boldsymbol{\xi}}_{2,i}d\tau. \quad (22)$$

Substituting (21) and (22) into (20) yields

$$\Delta E_{22,i}(t) \leq -\int_0^t z_{1,i}z_{2,i}d\tau - \frac{1}{2}b_2^{-1}z_{2,i-1}^2. \quad (23)$$

Therefore, (19) and (23) imply that

$$\Delta E_{2,i}(t) \leq -\frac{1}{2}b_1^{-1}z_{1,i-1}^2 - \frac{1}{2}b_2^{-1}z_{2,i-1}^2. \quad (24)$$

Step 3: Consider system dynamics (3) and treat $x_{4,i}$ as a virtual input. Define $z_{3,i} \triangleq \alpha_{2,i} - x_{3,i}$. According to (3) and (16), we have

$$\dot{z}_{3,i} = -\dot{\hat{\bar{\boldsymbol{\theta}}}}_{2,i}\bar{\boldsymbol{\xi}}_{2,i} - \hat{\bar{\boldsymbol{\theta}}}_{2,i}\Big(\frac{\partial\bar{\boldsymbol{\xi}}_{2,i}}{\partial x_{1,i}}\dot{x}_{1,i} + \frac{\partial\bar{\boldsymbol{\xi}}_{2,i}}{\partial x_{2,i}}\dot{x}_{2,i}$$
$$+ \frac{\partial\bar{\boldsymbol{\xi}}_{2,i}}{\partial t}\Big) + k_2\dot{z}_{2,i} + \dot{z}_{1,i} - b_3x_{4,i} - \boldsymbol{\theta}_3\boldsymbol{\xi}_{3,i}. \quad (25)$$

Considering system dynamics (1) - (2) and the dynamics of $z_{1,i}$ and $z_{2,i}$ ((6) and (15)), (25) can be rewritten as

$$\dot{z}_{3,i} = -\dot{\hat{\bar{\boldsymbol{\theta}}}}_{2,i}\bar{\boldsymbol{\xi}}_{2,i} - \hat{\bar{\boldsymbol{\theta}}}_{2,i}\Big[\frac{\partial\bar{\boldsymbol{\xi}}_{2,i}}{\partial x_{1,i}}(b_1x_{2,i} + \boldsymbol{\theta}_1\boldsymbol{\xi}_{1,i})$$
$$+ \frac{\partial\bar{\boldsymbol{\xi}}_{2,i}}{\partial x_{2,i}}(b_2x_{3,i} + \boldsymbol{\theta}_2\boldsymbol{\xi}_{2,i}) + \frac{\partial\bar{\boldsymbol{\xi}}_{2,i}}{\partial t}\Big] - k_2b_2x_{3,i}$$
$$-k_2b_2\boldsymbol{\theta}_2^o\boldsymbol{\xi}_{2,i}^o - b_1x_{2,i} - b_1\boldsymbol{\theta}_1^o\boldsymbol{\xi}_{1,i}^o - b_3x_{4,i} - \boldsymbol{\theta}_3\boldsymbol{\xi}_{3,i}$$
$$= b_3(-x_{4,i} - \bar{\boldsymbol{\theta}}_3^o\bar{\boldsymbol{\xi}}_{3,i}^o), \quad (26)$$

where $\bar{\boldsymbol{\theta}}_3^o \triangleq b_3^{-1}[1, b_1, b_2, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, b_1\boldsymbol{\theta}_1^o, b_2\boldsymbol{\theta}_2^o]$ and $\bar{\boldsymbol{\xi}}_{3,i}^o \triangleq [(\dot{\hat{\bar{\boldsymbol{\theta}}}}_{2,i}\bar{\boldsymbol{\xi}}_{2,i} + \hat{\bar{\boldsymbol{\theta}}}_{2,i}\frac{\partial\bar{\boldsymbol{\xi}}_{2,i}}{\partial t}), \ (\hat{\bar{\boldsymbol{\theta}}}_{2,i}\frac{\partial\bar{\boldsymbol{\xi}}_{2,i}}{\partial x_{1,i}} + 1)x_{2,i}, (\hat{\bar{\boldsymbol{\theta}}}_{2,i}\frac{\partial\bar{\boldsymbol{\xi}}_{2,i}}{\partial x_{2,i}} + k_2)x_{3,i}, \hat{\bar{\boldsymbol{\theta}}}_{2,i}\frac{\partial\bar{\boldsymbol{\xi}}_{2,i}}{\partial x_{1,i}}\boldsymbol{\xi}_{1,i}^T, \hat{\bar{\boldsymbol{\theta}}}_{2,i}\frac{\partial\bar{\boldsymbol{\xi}}_{2,i}}{\partial x_{2,i}}\boldsymbol{\xi}_{2,i}^T, \boldsymbol{\xi}_{3,i}^T, \boldsymbol{\xi}_{1,i}^{oT}, k_2\boldsymbol{\xi}_{2,i}^{oT}]^T$. The virtual control law is designed as follows:

$$x_{4,i} = -\hat{\bar{\boldsymbol{\theta}}}_{3,i}\bar{\boldsymbol{\xi}}_{3,i} + k_3z_{3,i} + z_{2,i} \triangleq \alpha_{3,i} \quad (27)$$
$$\hat{\bar{\boldsymbol{\theta}}}_{3,i} = \hat{\bar{\boldsymbol{\theta}}}_{3,i-1} - \beta_3z_{3,i}\bar{\boldsymbol{\xi}}_{3,i}^T, \quad \hat{\bar{\boldsymbol{\theta}}}_{3,0} = \mathbf{0}, \quad (28)$$

where $\hat{\bar{\boldsymbol{\theta}}}_{3,i}$ is used to approximate $\bar{\boldsymbol{\theta}}_3 \triangleq [\boldsymbol{\theta}_3^o, \ b_3^{-2}\dot{b}_3]$, $\bar{\boldsymbol{\xi}}_{3,i} \triangleq [\boldsymbol{\xi}_{3,i}^{oT}, \ \frac{1}{2}z_{3,i}]^T$, $k_3 > 0$ is the feedback gain and $\beta_3 > 0$ is the learning gain. Define $E_{3,i}(t) = E_{31,i}(t) + E_{32,i}(t)$, where $E_{31,i}(t) = E_{2,i}(t)$, $E_{32,i}(t) = \frac{1}{2}b_3^{-1}z_{3,i}^2 + \frac{1}{2\beta_3}\int_0^t \|\delta\bar{\boldsymbol{\theta}}_{3,i}\|^2 d\tau$ and $\delta\bar{\boldsymbol{\theta}}_{3,i} \triangleq \bar{\boldsymbol{\theta}}_3 - \hat{\bar{\boldsymbol{\theta}}}_{3,i}$. Analogous to the analysis in *Step 2*, the following result can be derived.

$$\Delta E_{3,i}(t) \leq -\frac{1}{2}\sum_{j=1}^{3} b_j^{-1}z_{j,i-1}^2 \quad (29)$$

Step j: Define $z_{j,i} \triangleq \alpha_{j-1,i} - x_{j,i}$. The dynamics of $z_{j,i}$ can be derived similarly to *Step 1*, *2* and *3*.

$$\dot{z}_{j,i} = b_j(-x_{j+1,i} - \boldsymbol{\theta}_j^o\boldsymbol{\xi}_{j,i}^o), \quad (30)$$

where $\boldsymbol{\theta}_j^o \triangleq b_j^{-1}[1, \ b_1, \ \cdots, \ b_{j-3}, \ b_{j-2}, \ b_{j-1}, \ \boldsymbol{\theta}_1, \ \cdots, \ \boldsymbol{\theta}_{j-1}, \ \boldsymbol{\theta}_j, \ b_{j-2}\boldsymbol{\theta}_{j-2}^o, \ b_{j-1}\boldsymbol{\theta}_{j-1}^o]$ and

$\boldsymbol{\xi}^o_{j,i} \triangleq [\dot{\hat{\bar{\boldsymbol{\theta}}}}_{j-1}\bar{\boldsymbol{\xi}}_{j-1,i} + \hat{\bar{\boldsymbol{\theta}}}_{j-1,i}\frac{\partial \bar{\boldsymbol{\xi}}_{j-1,i}}{\partial t}, \hat{\bar{\boldsymbol{\theta}}}_{j-1,i}\frac{\partial \bar{\boldsymbol{\xi}}_{j-1,i}}{\partial x_{1,i}}x_{2,i},$
$\cdots,\ \hat{\bar{\boldsymbol{\theta}}}_{j-1,i}\frac{\partial \bar{\boldsymbol{\xi}}_{j-1,i}}{\partial x_{j-3,i}}x_{j-2,i},\ (\hat{\bar{\boldsymbol{\theta}}}_{j-1,i}\frac{\partial \bar{\boldsymbol{\xi}}_{j-1,i}}{\partial x_{j-2,i}}+1)x_{j-1,i},$
$(\hat{\bar{\boldsymbol{\theta}}}_{j-1,i}\frac{\partial \bar{\boldsymbol{\xi}}_{j-1,i}}{\partial x_{j-1,i}} + k_{j-1})x_{j,i},\ \ \hat{\bar{\boldsymbol{\theta}}}_{j-1,i}\frac{\partial \bar{\boldsymbol{\xi}}_{j-1,i}}{\partial x_{1,i}}\boldsymbol{\xi}^T_{1,i},$
$\cdots,\ \hat{\bar{\boldsymbol{\theta}}}_{j-1,i}\frac{\partial \bar{\boldsymbol{\xi}}_{j-1,i}}{\partial x_{j-1,i}}\boldsymbol{\xi}^T_{j-1,i},\ \boldsymbol{\xi}^T_{j,i},\ \boldsymbol{\xi}^{oT}_{j-2,i},\ k_{j-1}\boldsymbol{\xi}^{oT}_{j-1}]^T.$
The virtual control laws are as follows:

$$x_{j+1,i} = -\hat{\bar{\boldsymbol{\theta}}}_{j,i}\bar{\boldsymbol{\xi}}_{j,i} + k_j z_{j,i} + z_{j-1,i} \triangleq \alpha_{j,i} \quad (31)$$

$$\hat{\bar{\boldsymbol{\theta}}}_{j,i} = \hat{\bar{\boldsymbol{\theta}}}_{j,i-1} - \beta_j z_{j,i}\bar{\boldsymbol{\xi}}^T_{j,i}, \quad \hat{\bar{\boldsymbol{\theta}}}_{j,0} = \mathbf{0}, \qquad (32)$$

where $\hat{\bar{\boldsymbol{\theta}}}_{j,i}$ is used to approximate $\bar{\boldsymbol{\theta}}_j \triangleq [\boldsymbol{\theta}^o_j,\ b_j^{-2}\dot{b}_j]$, $\bar{\boldsymbol{\xi}}_{j,i} \triangleq [\boldsymbol{\xi}^{oT}_{j,i},\ \frac{1}{2}z_{j,i}]^T$, $k_j > 0$ is the feedback gain and $\beta_j > 0$ is the learning gain. Define $E_{j,i}(t) = E_{j-1,i}(t) + \frac{1}{2}b_j^{-1}z^2_{j,i} + \frac{1}{2\beta_j}\int_0^t \|\delta\bar{\boldsymbol{\theta}}_{j,i}\|^2 d\tau$, where $\delta\bar{\boldsymbol{\theta}}_{j,i} \triangleq \bar{\boldsymbol{\theta}}_j - \hat{\bar{\boldsymbol{\theta}}}_{j,i}$. It follows from (18) - (24) that $\Delta E_{j,i}(t) \le -\frac{1}{2}\sum_{m=1}^{j} b_m^{-1}z^2_{m,i-1}$.

Step n: From the results in *Step j*, we have

$$\dot{z}_{n,i} = b_n(-u_i - \boldsymbol{\theta}^o_n\boldsymbol{\xi}^o_{n,i}), \qquad (33)$$

where $\boldsymbol{\theta}^o_n$ and $\boldsymbol{\xi}^o_{n,i}$ are the terms defined as in (30) with $j = n$. The real system input u_i is constructed as follows:

$$u_i = -\hat{\bar{\boldsymbol{\theta}}}_{n,i}\bar{\boldsymbol{\xi}}_{n,i} + k_n z_{n,i} + z_{n-1,i} \qquad (34)$$

$$\hat{\bar{\boldsymbol{\theta}}}_{n,i} = \hat{\bar{\boldsymbol{\theta}}}_{n,i-1} - \beta_n z_{n,i}\bar{\boldsymbol{\xi}}^T_{n,i}, \qquad \hat{\bar{\boldsymbol{\theta}}}_{n,0}(0) = \mathbf{0}, (35)$$

where $\hat{\bar{\boldsymbol{\theta}}}_{n,i}$ is used to approximate $\bar{\boldsymbol{\theta}}_{n,i} \triangleq [\boldsymbol{\theta}^o_n, b_n^{-2}\dot{b}_n]$, $\bar{\boldsymbol{\xi}} \triangleq [\boldsymbol{\xi}^{oT}_{n,i},\ \frac{1}{2}z_{n,i}]^T$, k_n is the feedback gain and β_n is the learning gain. Define $E_i(t) = E_{n-1,i}(t) + \frac{1}{2}b_n^{-1}z^2_{n,i} + \frac{1}{2\beta_n}\int_0^t \|\delta\bar{\boldsymbol{\theta}}_{n,i}\|^2 d\tau$, where $\delta\bar{\boldsymbol{\theta}}_{n,i} \triangleq \bar{\boldsymbol{\theta}}_n - \hat{\bar{\boldsymbol{\theta}}}_{n,i}$. Following the approach similar to the derivation in *Step 2*, it can be shown that

$$\Delta E_i(t) \le -\frac{1}{2}\sum_{j=1}^{n} b_j^{-1}z^2_{j,i-1}. \qquad (36)$$

3.2 Convergence Analysis

Using (36) repeatedly, we have

$$E_i(t) \le E_1(t) - \frac{1}{2}\sum_{p=1}^{i-1}\sum_{j=1}^{n} b_j^{-1}z^2_{j,p} \qquad (37)$$

If $E_1(t)$ is bounded, according to (37) and considering the positiveness of $E_i(t)$ and $b_j^{-1} \ne 0$ $(j \in N)$, it can be derived that $\lim_{i\to\infty} z_{j,i}(t) = 0$ pointwisely for any $j \in N$. Now, let us derive the finiteness of $E_1(t)$. From the definition of $E_i(t)$, we have

$$E_1(t) = \frac{1}{2}\sum_{j=1}^{n} b_j^{-1}z^2_{j,1} + \sum_{j=1}^{n}\frac{1}{2\beta_j}\int_0^t \|\delta\bar{\boldsymbol{\theta}}_{j,1}\|^2 d\tau.$$

Hence, the derivative of $E_1(t)$ is given by

$$\dot{E}_1(t) = \sum_{j=1}^{n}(b_j^{-1}z_{j,1}\dot{z}_{j,1} - \frac{1}{2}b_j^{-2}\dot{b}_j z^2_{j,1}) + \sum_{j=1}^{n}\frac{1}{2\beta_j}\|\delta\bar{\boldsymbol{\theta}}_{j,1}\|^2. \qquad (38)$$

According to *Step j*, for any $j \in \{1, \cdots, n-1\}$, we have $\dot{z}_{j,1} = -b_j x_{j+1,1} - b_j\boldsymbol{\theta}^o_j\boldsymbol{\xi}^o_{j,1}$, $x_{j+1,1} = \alpha_{j,1} - z_{j+1,1}$ and $\alpha_{j,1} = -\hat{\bar{\boldsymbol{\theta}}}_{j,1}\bar{\boldsymbol{\xi}}_{j,1} + k_j z_{j,1} + z_{j-1,1}$. Hence, the following result can be established:

$$\begin{aligned}
& b_j^{-1}z_{j,1}\dot{z}_{j,1} - \frac{1}{2}b_j^{-2}\dot{b}_j z^2_{j,1} \\
&= z_{j,1}(-x_{j+1,1} - \boldsymbol{\theta}^o_j\boldsymbol{\xi}^o_{j,1} - \frac{1}{2}b_j^{-2}\dot{b}_j z_{j,i}) \\
&= z_{j,1}(-x_{j+1,1} - \bar{\boldsymbol{\theta}}_j\bar{\boldsymbol{\xi}}_{j,1}) \\
&= z_{j,1}(-\alpha_{j,1} + z_{j+1,1} - \bar{\boldsymbol{\theta}}_j\bar{\boldsymbol{\xi}}_{j,1}) \\
&= z_{j,1}(\hat{\bar{\boldsymbol{\theta}}}_{j,1}\bar{\boldsymbol{\xi}}_{j,1} - k_j z_{j,1} - z_{j-1,1} + z_{j+1,1} - \bar{\boldsymbol{\theta}}_j\bar{\boldsymbol{\xi}}_{j,1}) \\
&= -z_{j,1}\delta\bar{\boldsymbol{\theta}}_{j,1}\bar{\boldsymbol{\xi}}_{j,1} + z_{j+1,1}z_{j,1} - z_{j,1}z_{j-1,1} - k_j z^2_{j,1}.
\end{aligned}$$

For $j = n$, we have

$$\begin{aligned}
& b_n^{-1}z_{n,1}\dot{z}_{n,1} - \frac{1}{2}b_n^{-2}\dot{b}_n z^2_{n,1} \\
&= z_{n,1}(-u_1 - \boldsymbol{\theta}^o_n\boldsymbol{\xi}^o_{n,1}) - \frac{1}{2}b_n^{-2}\dot{b}_n z^2_{n,1} \\
&= z_{n,1}(-u_1 - \bar{\boldsymbol{\theta}}_n\bar{\boldsymbol{\xi}}_{n,1}) \\
&= -z_{n,1}\delta\bar{\boldsymbol{\theta}}_{n,1}\bar{\boldsymbol{\xi}}_{n,1} - z_{n-1,1}z_{n,1} - k_n z^2_{n,1}.
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{n}(b_j^{-1}z_{j,1}\dot{z}_{j,1} - \frac{1}{2}b_j^{-2}\dot{b}_j z^2_{n,1}) = -\sum_{j=1}^{n} z_{j,1}\delta\bar{\boldsymbol{\theta}}_{j,1}\bar{\boldsymbol{\xi}}_{j,1} - \sum_{j=1}^{n} k_j z^2_{j,1}. \qquad (39)$$

For the last term on the right-hand side of (38), the following result is valid for all $j \in N$.

$$\frac{1}{2\beta_j}\|\delta\bar{\boldsymbol{\theta}}_{j,1}\|^2 = \frac{1}{2\beta_j}(\|\delta\bar{\boldsymbol{\theta}}_{j,1}\|^2 - \|\bar{\boldsymbol{\theta}}_j - \hat{\bar{\boldsymbol{\theta}}}_{j,0}\|^2) + \frac{1}{2\beta_j}\|\bar{\boldsymbol{\theta}}_j - \hat{\bar{\boldsymbol{\theta}}}_{j,0}\|^2. \qquad (40)$$

As $\bar{\boldsymbol{\theta}}_j$ is bounded over the time interval $[0, T]$ and $\hat{\bar{\boldsymbol{\theta}}}_{j,0} = 0$, a finite constant B_j can be found such that $B_j = \max_{t\in[0,T]}\{\frac{1}{2\beta_j}\|\bar{\boldsymbol{\theta}}_j - \hat{\bar{\boldsymbol{\theta}}}_{j,0}\|^2\}$. Moreover, analogous to the derivations in (12) and (13), we have $\frac{1}{2\beta_j}(\|\delta\bar{\boldsymbol{\theta}}_{j,1}\|^2 - \|\bar{\boldsymbol{\theta}}_j - \hat{\bar{\boldsymbol{\theta}}}_{j,0}\|^2) \le z_{j,1}\delta\bar{\boldsymbol{\theta}}_{j,1}\bar{\boldsymbol{\xi}}_{j,1}$. Hence, (40) can be rewritten as

$$\frac{1}{2\beta_j}\|\delta\bar{\boldsymbol{\theta}}_{j,1}\|^2 \le z_{j,1}\delta\bar{\boldsymbol{\theta}}_{j,1}\bar{\boldsymbol{\xi}}_{j,1} + B_j. \tag{41}$$

Substituting (39) and (41) into (38) yields $\dot{E}_1(t) \le \sum_{j=1}^{n} B_j$. Considering Assumption 2 and the properties of $\bar{\boldsymbol{\theta}}_j$, we can establish that $E_1(0)$ is bounded. Therefore, the finiteness of $\dot{E}_1(t)$ implies that $E_1(t)$ is finite over the time interval $[0,T]$, which ensures that $\forall j \in N$, $\lim_{i\to\infty} z_{j,i} = 0$. Furthermore, the boundedness of $E_i(t)$ leads to the boundedness of $z_{j,i}$ and $\int_0^t \|\delta\bar{\boldsymbol{\theta}}_{j,i}\|^2 d\tau$ for all $j \in N$. According to the definition of $z_{j,i}$, it can be derived that y_i and $\int_0^t x_{j,i}^2 d\tau$ $(j = \{2,\cdots,n\})$ are finite. Hence, from the control law, the term $\int_0^t u_i^2 d\tau$ is bounded, i.e. u_i is L^2 bounded over $[0,T]$.

4. ILLUSTRATIVE EXAMPLE

To illustrate the effectiveness of the proposed control strategy, we consider a single-link robotic manipulator with a flexible joint.

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{Mgl}{J_1}\sin x_1 - \frac{k}{J_1}(x_1 - x_3) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{k}{J_m}(x_1 - x_3) + \frac{1}{J_m}u \\
y &= x_1,
\end{aligned} \tag{42}$$

where x_1 is the link angle, x_3 is the motor shaft angle, J_1, J_m are the load and motor inertias, k is the joint stiffness, and u is the input torque. Assume that the system is repeatable over $[0, 2\pi]$. Let $y_{d,i} = A_i \sin t$, which is iteration-dependent. We choose A_i randomly from the interval $[0.5, 1]$. To satisfy the i.i.c., let $x_{1,i}(0) = y_{d,i}(0) = 0$ and $x_{2,i}(0) = x_{3,i}(0) = x_{4,i}(0) = 0$. Choose $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 5$ and $k_1 = k_2 = k_3 = k_4 = 1$. Applying the proposed ILC law developed in Section 3, we obtain the simulation result depicted in Fig. 1. In Fig. 1, the horizontal axis is the iteration number i and the vertical axis is the sup-norm of $y_{d,i} - y_i$, i.e. $\max_{t\in[0,T]} |y_{d,i} - y_i|$. Obviously, the proposed ILC approach works quite well for systems with unmatched time-varying parametric uncertainties and non-uniform tracking targets.

5. CONCLUSIONS

A novel ILC strategy based on the backstepping technique has been developed in this paper. The new scheme extends current ILC to systems with unmatched time-varying uncertainties and high relative degree, which greatly widens the application domain of ILC.

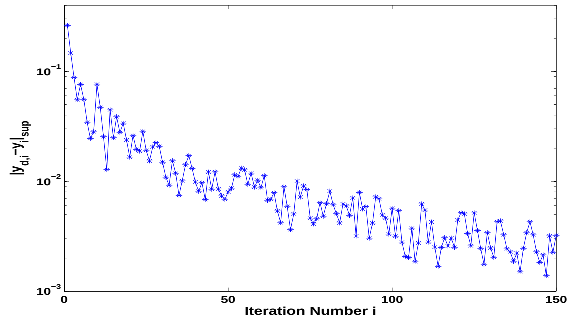

Fig. 1. Convergence of the output tracking error.

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