# EXISTENCE AND OPTIMALITY OF NASH EQUILIBRIA IN INVENTORY GAMES ⋆

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Abstract: This paper studies the stability and optimality of a distributed consensus protocol for $n$-player repeated non cooperative games under incomplete information. At each stage, the players choose binary strategies and incur in a payoff monotonically decreasing with the number of active players. The game is specialized to an inventory application, where fixed costs are shared among all retailers, interested in whether reordering or not from a common warehouse. The authors focus on Pareto optimality as a measure of coordination of reordering strategies, proving that there exists a unique Pareto optimal Nash equilibrium that verifies certain stability conditions. 

Keywords: Inventory control, Stability, Optimality, Nash equilibrium

## 1. INTRODUCTION

The present paper aims at showing that the *consensus protocol* introduced in (Bauso *et al.*, 2003) and in (Bauso *et al.*, 2004) allows the convergence of strategies to the desired equilibrium by exploiting stability properties of Pareto optimal Nash equilibria (Shamma and Arslan, 2003). In our consensus protocol each player exchanges a limited amount of information with a subset of other players. We cast this protocol within the *minimal information paradigm* (Fax and Murray, 2002) to reduce each player's data exposure to the competitors.

Our results apply to repeated non cooperative games in which, at each stage, the payoff of the players is a monotonic function of the strategies of the others. Possible examples are the so-called *externality games* (Friedman, 1996) and *cost-sharing games* (Watts, 2002).

We consider a multi-retailer inventory application. The players, namely different competing retailers, aim at coordinating joint orders thus to share fixed transportation costs (see, e.g., (Silver *et al.*, 1998)). Our idea of selecting the best (Pareto optimal) among several Nash equilibria presents some similarities with (Cachon, 2001). However, we do not implement either central coordination or side payments as we consider a competitive environment.

## 2. THE INVENTORY GAME

Hereafter, we indicate with the same symbol $i$ both the generic player and the associated index.

We consider a set of $n$ players $\Gamma = \{1, \ldots, n\}$ where each player may exchange information only with a subset of neighbor players. More formally,

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we assume that the set $\Gamma$ induces a single component graph $G = (\Gamma, E)$ whose edgeset $E$ includes all the non oriented couples $(i, j)$ of players that exchange information with each other. In this context, we define the neighborhood of a player $i$ the set $N_i = \{j : (i, j) \in E\} \cup \{i\}$.

Players cannot hold any private inventory but share a common warehouse. At each stage $k$, each player $i$ faces a customer demand and decides whether to fulfill it or to pay a penalty $p_i$; the unfilled demand is lost. We call *active* player the one who decides to meet the demand. The active players receive the items required by their customer from the common warehouse and equally divide a fixed transportation cost $K$.

More formally, we define the function $s_i(k) \in S_i = \{0, 1\}$ as the strategy of player $i$, for each player $i \in \Gamma$. We indicate $s(k) = \{s_1(k), \ldots, s_n(k)\}$ as the vector of the players' strategies and $s_{-i} = \{s_1(k), \ldots, s_{i-1}(k), s_{i+1}(k), \ldots, s_n(k)\}$ as the vector of strategies of players $j \neq i$. At stage $k$, $s_i(k)$ is equal to 1 if player $i$ meets the demand and equal to 0 otherwise. Then $s_i(k)$ has a payoff

$$J_i(k)(s_i(k), s_{-i}(k)) = \frac{K}{1 + \|s_{-i}(k)\|_1} s_i(k) + (1 - s_i(k))p_i, \quad (1)$$

where $\|s_{-i}(k)\|_1$ is trivially equal to the number of active players other than $i$.

At stage $k$, player $i$ processes two types of public information: *pre-decision information*, $x_i(k)$, received from the neighbor players in $N_i$, and *post-decision information*, $z_i(k)$, transmitted to the neighbor players. Player $i$ selects its strategy $s_i(k) = \mu_i(x_i(k))$ on the on the basis of only its predecision information

The information evolves according to a *distributed protocol* $\Pi = \{\phi_i, h_i, \ i \in \Gamma\}$ defined by the following equations:

$$x_i(k+1) = \phi_i(z_j(k) \text{ for all } j \in N_i) \quad (2a)$$

$$z_i(k) = h_i(s_i(k), s_i(k-1), x_i(k)) \quad (2b)$$

In (Bauso *et al.*, 2003) and (Bauso *et al.*, 2004), we show that there can be defined functions $\phi_i(\cdot)$ and $h_i(\cdot)$ such that $x_i(k) = \|s_{-i}(k-1)\|_1$.

Note that if each player $i$ knew the other players' strategies $s_{-i}(k)$ it would optimize its payoff (1) choosing as *best response* (see, e.g., (Shamma and Arslan, 2003)) the following threshold strategy

$$s_i(k) = (\|s_{-i}(k)\|_1 \geq l_i), \quad (3)$$

where the threshold $l_i$ is equal to $\frac{K}{p_i} - 1$, $\|s_{-i}(k)\|_1$ is the number of all other active players, and $(\|s_{-i}(k)\|_1 \geq l_i)$ is a boolean function that returns 1 if its argument holds true, 0 otherwise. Working in an *incomplete information* context, our generic player $i$ can only estimate the number $\|s_{-i}(k)\|_1$ of all other active players. However, in the rest of the paper, we prove that the threshold strategy

$$s_i(k) = (\|s_{-i}(k-1)\|_1 \geq l_i) \quad (4)$$

allows the convergence of strategies (4) to the Pareto optimal Nash equilibrium $s^*$. To this aim, in the next section, we prove that the Pareto optimal Nash equilibrium exists and, in Section 4, we prove that such equilibrium is stable.

## 3. EXISTENCE OF A PARETO OPTIMAL NASH EQUILIBRIUM

Initially, let us make, without loss of generality, the following assumptions:

**Assumption 1** The set $\Gamma$ of players is ordered so that $l_1 \leq l_2 \leq \ldots \leq l_n$.

**Assumption 2** There may exist other players $i = n+1, n+2, \ldots$ not included in $\Gamma$, all of them with thresholds $l_i = \infty$

**Assumption 3** The players in the empty subset of $\Gamma$ have thresholds $l_i = -\infty$.

The last assumption is obviously artificial, but simplifies the proofs of most results in the rest of the paper. It allows us to prove theorems without the necessity of introducing different arguments in case the set of active players is empty.

### *3.1 Existence of Nash Equilibria*

In a Nash equilibrium $s^\star = \{s_1^\star, \ldots, s_n^\star\}$, each player $i$ selects a strategy $s_i^\star$ such that

$$J_i(s_i^\star, s_{-i}^\star) \leq J_i(s_i, s_{-i}^\star) \quad \text{for all } s_i \in S_i,\ i \in \Gamma. (5)$$

From (3), we obtain the equilibrium conditions

$$s_i^\star = (\|s_{-i}^\star\|_1 \geq l_i), \quad \text{for all}\, i \in \Gamma. \quad (6)$$

Then, given Assumption 1, we can state, the following necessary conditions on the existence of a Nash Equilibrium.

*Lemma 1.* If $s^\star$ is a Nash equilibrium then:

i) if player $i$ is active, namely $s_i^\star = 1$, then all the preceding players $1, \ldots, i-1$ are also active, i.e., $s_1^\star = \ldots = s_{i-1}^\star = 1$;
ii) if player $i$ is not active, namely $s_i^\star = 0$, then neither all successive players $i+1, \ldots, n$ are active, i.e., $s_{i+1}^\star = \ldots = s_n^\star = 0$.

**Proof -** We show that the assumption $s_i^\star = 1$ and $s_{i-1}^\star = 0$ are in contradiction to prove item i). The equilibrium condition (6) and $s_i^\star = 1$ imply $\|s_{-i}\|_1 = \sum_{j\in\Gamma,\, j\neq i} s_j^\star \geq l_i$. Since $s_{i-1}^\star = 0$, the latter inequality is equivalent to

$$\sum_{j\in\Gamma,\, j\neq i,\, j\neq i-1} s_j^\star \geq l_i. \tag{7}$$

However, condition (6) and $s_{i-1}^\star = 0$ also imply $\|s_{-(i-1)}\|_1 = \sum_{j\in\Gamma,\, j\neq i-1} s_j^\star < l_{i-1}$. But this last inequality is in contradiction with (7) since $\sum_{j\in\Gamma,\, j\neq i,\, j\neq i-1} s_j^\star \leq \sum_{j\in\Gamma,\, j\neq i-1} s_j^\star$.

A complementary argument proves item ii).

□

Let us now introduce two definitions.

*Definition 1.* A set $C \subseteq \Gamma$ of cardinality $|C| = r$ is *complete* if it contains all the first $r$ players, with $r \geq 0$, i.e., $C = \{1, \ldots, r\}$.

*Definition 2.* A set $C \subseteq \Gamma$ is *compatible* if $l_i \leq |C| - 1$ for all $i \in C$.

Note that each player of a compatible set $C$ finds convenient to meet the demand if all other players in $C$ do the same. Note also that $C = \emptyset$ is both a compatible and a complete set.

*Theorem 1.* The vector of strategies $s^\star$, defined as

$$s_i^\star = \begin{cases} 1 \text{ if } & i \in C \\ 0 \text{ otherwise} \end{cases} \tag{8}$$

is a Nash equilibrium if and only if the set $C = \{1, \ldots, r\} \subseteq \Gamma$ is both compatible and complete and the following condition holds

$$l_{r+1} > r. \tag{9}$$

**Proof -** *Sufficiency.* Assume that $s^\star$, defined as in (8), is a Nash equilibrium. Observe that if $C = \emptyset$ then it is complete and compatible by definition. Otherwise, $C$ is complete by Lemma 1 and compatible by definition of a Nash equilibrium. Finally, note that if $C = \Gamma$, condition (9) holds since $l_{n+1} = \infty$. Otherwise, condition (9) holds since the player $r+1 \notin C$ chooses a strategy $s_{r+1}^\star = 0$ that, together with (6), implies $l_{r+1} > \|s_{-(r+1)}^\star\|_1 = r$.

*Necessity.* Assume that $C$ is complete, compatible and condition (9) holds. Observe that $J_i(1, s_{-i}^\star) \leq J_i(0, s_{-i}^\star)$ and therefore $s_i^\star = 1$ holds, for all players $i \in C$, since $C$ is compatible. Then note that, since $C$ is complete, all $i \notin C$ are such that $i > r$. From condition (9) we also have $l_i > r$ for all $i > r$. Hence, $J_i(0, s_{-i}^\star) \leq J_i(1, s_{-i}^\star)$ holds and consequently $s_i^\star = 0$ for all players $i \notin C$.

□

From Theorem 1 we derive the following corollary.

*Corollary 1. (Existence of Nash equilibria)* Let $\overline{C}$ be the maximal compatible set. Then, there always exists a Nash equilibrium

$$s_i^\star = \begin{cases} 1 \text{ if } & i \in \overline{C} \\ 0 \text{ otherwise} \end{cases} \tag{10}$$

**Proof -** First observe that the set $\overline{C}$ always exists since it may possibly be the empty set.

With Assumptions **1, 2, 3** in mind, we show that if $\overline{C}$ is maximal then it is also complete. Assume by contradiction that $\overline{C}$ is not complete. Let player $i$ be in $\overline{C}$ and player $i-1$ be not in $\overline{C}$. As $i \in \overline{C}$, then $l_i < |\overline{C}|$. Since $l_{i-1} \leq l_i$, then $l_{i-1} < |\overline{C}|$ which in turn implies that also $\overline{C} \cup \{i-1\}$ is a compatible set in contradiction with the maximality hypothesis on $\overline{C}$.

Now, assume that $\overline{C}$ is equal to $\{1, \ldots, r\}$. Since $\overline{C}$ is maximal, $\overline{C} \cup \{r+1\}$ is not compatible, so $l_{r+1} > r$, i.e., condition (9). Then, even for $\overline{C} \subset \Gamma$, the hypotheses of Theorem 1 hold true, and the vector of strategies $s^\star$, defined in (10) is a Nash equilibrium. Note that due to Assumptions **2, 3** the above reasoning applies also to the cases $\overline{C} = \emptyset$ and $\overline{C} = \Gamma$.

□

Two additional considerations can be done. First, other Nash equilibria different from $s^\star$ defined in (10) may exist. They are associated to complete and compatible sets $C$ that are not maximal.

Second, if $\overline{C}$ is the maximal compatible set, it trivially holds

$$r = |\overline{C}| = \max_{\lambda} \{\lambda \in \{1, \ldots, n\} : l_\lambda < \lambda\}. \tag{11}$$

*3.2 Pareto Optimality of the Nash equilibrium associated to $\overline{C}$*

A vector of strategies $\hat{s}$ is Pareto optimal if there is no other vector of strategies $s$ such that

$$J_i(s_i, s_{-i}) \leq J_i(\hat{s}_i, \hat{s}_{-i}) \quad \text{for all } i \in \Gamma \tag{12}$$

and strict inequality holds for at least one player.

*Theorem 2.* Let $s^\star$ be the Nash equilibrium associated to the maximal compatible set $\overline{C}$. If $p_i \neq \frac{K}{|\overline{C}|}$ for all $i \in \overline{C}$, then

- *[Pareto optimality]* the vector of strategies $s^\star$ is Pareto optimal;
- *[uniqueness]* the vector of strategies $s^\star$ is the unique Pareto optimal Nash equilibrium.

**Proof** *Pareto optimality.* We show that Nash equilibrium $s^\star$ is Pareto optimal since any other vector of strategies $s$ induces a worse payoff to at least one player. In the Nash equilibrium $s^\star$, each $i \in \overline{C}$ gets a payoff $J_i(1, s^\star_{-i}) = \frac{K}{|\overline{C}|} < p_i$, each $i \notin \overline{C}$ gets a payoff $J_i(0, s^\star_{-i}) = p_i < \frac{K}{|\overline{C}|}$. Now, consider the vector of strategies $s$. Define $D = \{i \in \overline{C} : s_i = 0\}$ as the set of players with $l_i < |\overline{C}|$ that are not active in $s$ and $E = \{i \notin \overline{C} : s_i = 1\}$ as the set of players with $l_i \geq |\overline{C}|$ that are active in $s$. Trivially, $D \cup E \neq \emptyset$ as $s \neq s^\star$. We deal with $E \neq \emptyset$ and $E = \emptyset$ separately.

If $E \neq \emptyset$ and $D = \emptyset$, each $i \in E$ gets a payoff $J_i(1, s_{-i}) = \frac{K}{|\overline{C} \cup E|}$ strictly greater than $J_i(0, s^\star_{-i}) = p_i$ as $\overline{C}$ is the maximal compatible sets. The latter condition trivially holds also when $D \neq \emptyset$ since, in this case, each player $i \in E$ incurs in a higher payoff $J_i(1, s_{-i}) = \frac{K}{|(\overline{C} \cup E) \setminus D|}$.

If $E = \emptyset$, then $D \neq \emptyset$, and each $i \in \overline{C} \setminus D$, if exists, gets a payoff $J_i(1, s_{-i}) = \frac{K}{|\overline{C} \setminus D|} > J_i(0, s^\star_{-i}) = \frac{K}{|\overline{C}|}$. At the same time, each $i \in D$ gets a payoff $J_i(0, s_{-i}) = p_i > J_i(1, s^\star_{-i}) = \frac{K}{|\overline{C}|}$. Finally, each $i \in \Gamma \setminus \overline{C}$ gets a payoff $J_i(0, s_{-i}) = p_i = J_i(0, s^\star_{-i})$.

*Uniqueness.* Consider a generic Nash equilibrium $s$ associated to a complete and compatible set $C$ different form $\overline{C}$. Since $\overline{C}$ is maximal then $C \subset \overline{C}$. Then, each $i \in C$, if exists, gets a payoff $J_i(s_i, s_{-i}) = \frac{K}{|C|} > J_i(s^\star_i, s^\star_{-i}) = \frac{K}{|\overline{C}|}$; analogously, each $i \in \overline{C} \setminus C$ gets a payoff $J_i(s_i, s_{-i}) = p_i > J_i(s^\star_i, s^\star_{-i}) = \frac{K}{|\overline{C}|}$; finally, each player $i \in \Gamma \setminus \overline{C}$, gets a payoff $J_i(s_i, s_{-i}) = p_i = J_i(s^\star_i, s^\star_{-i})$. Then, any generic Nash equilibrium has payoffs not better than the ones associated to $s^\star$.

□

Observe that if $p_i = \frac{K}{|\overline{C}|}$ there exists two Pareto optimal Nash equilibria with equal payoff. They are associated respectively to the maximal compatible set $\overline{C}$ and to the empty set. In the rest of the paper, only the equilibrium $s^\star$ associated to the maximal compatible set $\overline{C}$ will be called the *Pareto optimal Nash equilibrium*.

## 4. STABILITY OF NASH EQUILIBRIA

In this section, we prove the stability of the Pareto optimal Nash equilibrium under the hypothesis that at each stage $k$, each player $i$ implements strategy (4).

Given an equilibrium $s^\star$ and the associated complete compatible set $C = \{1, \ldots, r\}$, the vector $\Delta s(0) = s(0) - s^\star \geq 0$ ($\Delta s(0) \leq 0$) is defined a *positive (negative) perturbation* at stage 0. In other words, a positive (negative) perturbation is a change of strategies of a subset of players $P = \{i \in \Gamma \setminus C : \Delta s_i(0) = 1\}$ ($P = \{i \in C : \Delta s_i(0) = -1\}$), called *perturbed set*. The cardinality of the perturbed set $|P| = \|\Delta s(0)\|_1$ is the number of players that join the set $C$ (leave the set $C$). In addition, a positive (negative) perturbation $\overline{\Delta s}(0)$ is *maximal* when $\|\overline{\Delta s}(0)\|_1 = |\Gamma \setminus C|$, ($\|\overline{\Delta s}(0)\|_1 = |C|$). In this last case, all the players in $\Gamma \setminus C$, ($C$) change strategy.

We call a Nash equilibrium $s^\star$ *stable with respect to positive perturbations* if there exists a scalar $\delta > 0$ and $\bar{k} > 0$ such that if $\|\Delta s(0)\|_1 \leq \delta$, then $s^k = s^\star$ for all $k \geq \bar{k}$. Analogously, we call a Nash equilibrium $s^\star$ *maximally stable with respect to positive perturbations* if it is stable with respect to the maximal positive perturbation $\overline{\Delta s}(0)$.

In the following, let us introduce some theorems concerning the stability of Nash equilibria.

*Theorem 3.* Consider a Nash equilibrium $s^\star$ associated to a set $C = \{1, \ldots, r\}$. The vector of strategies $s^\star$ is stable with respect to positive perturbations $\Delta s(0) : ||\Delta s(0)||_1 = j - r - 1$ if all players $i \notin C$, with $r < i \leq j$, have thresholds $l_i \geq i$.

**Proof -** Keep in mind that the players observe the other strategies with a one-step delay throughout this proof. Observe that, since $s^\star$ is a Nash equilibrium, Theorem 1 and the definition of positive perturbation implies that the following two conditions hold: i) the threshold $l_{r+1} \geq r + 1$ and, ii) at stage $k = 0$, $s_i(0) = 1$, for all $i \in C \cup P$, whereas $s_i(0) = 0$, for all $i \in \Gamma \setminus C \cup P$. Note also that a positive perturbation $\Delta s(0)$ induces players $i \in \Gamma \setminus C$, with thresholds $l_i \leq |C \cup P|$ to change strategy from $s_i(0) = 0$ to $s_i(1) = 1$. Differently, all players $i \in C$, will not change strategy, since they have thresholds $l_i \leq |C| < |C \cup P|$.

Consider now a particular perturbation $\Delta s(0)$ with $||\Delta s(0)||_1 = j - r - 1$. Then, at stage $k = 1$, all players $i \geq j$ set $s_i(1) = 0$, since they observe $j - 1$ active players, and their threshold is $l_i \geq j$. Hence, at stage $k = 2$, player $j - 1$ surely sets $s_{j-1}(2) = 0$, since it observes at most $j - 2$ active players and its threshold is $l_{j-1} \geq j - 1$. Following the same line of reasoning at the generic stage $k$ with $1 < k < j - r$, player $j - k + 1$ sets $s_{j-k+1}(k) = 0$, since it observes at most $j - k$ active players and $l_{j-k+1} \geq j - k + 1$. Hence, at most at stage $k = j - r$ the strategies converge to the desired Nash equilibrium $s^\star$.

□

Theorem 3 proves the stability of a Nash equilibrium $s^\star$ when all players $j > r$ have a threshold $l_j \geq j$. Now, let us deal with the situation in which there exist players $j > r$ with thresholds $l_j < j$. To be more specific, let us consider the particular player $j$ such that $j = arg \min\{i \in \Gamma \setminus C : l_i < i\}$.

Clearly, when player $j$ exists, it must be $j \geq r+2$, since it must necessarily be $l_{r+1} \geq r+1$ from condition (9) of Theorem 1. In addition, $l_j = j-1$ since for all $i$ such that $r < i < j$ there holds $l_i \geq i$ by minimality of $j$.

*Theorem 4.* Consider the Nash equilibrium $s^\star$ associated to a set $C = \{1, \ldots, r\}$ and assume that there exists a player $j = arg\min\{i \in \Gamma \setminus C : l_i < i\}$. The vector of strategies $s^\star$ is not stable with respect to positive perturbations $\Delta s(0) : ||\Delta s(0)||_1 = j - r$.

**Proof -** It is enough to show that the Nash equilibrium $s^\star$ is not stable with respect a perturbation $\Delta s(0) : ||\Delta s(0)||_1 = j - r$ induced by the perturbed set $P = \{r+1, \ldots, j\}$. To this aim, consider that, at stage $k = 1$, players $r+1, \ldots, j$ set $s_{r+1}(1) = \ldots = s_j(1) = 1$ since their thresholds are lower than or equal to $l_j = j - 1$. The players $r+1, \ldots, j$ do not change their strategies in the following stages, then the desired equilibrium point $s^\star$ will never be reached.

□

Given the Nash equilibrium $s^\star$ and assuming the existence of a player $j = arg\min\{i \in \Gamma \setminus C : l_i < i\}$, Theorems 3 and 4 establish that $s^\star$ is stable with respect to positive perturbation $\Delta s(0)$ if $||\Delta s(0)||_1 < j - r - 1$ and is not stable if $||\Delta s(0)||_1 \geq j - r$. The following theorem addresses the case $||\Delta s(0)||_1 = j - r - 1$.

*Theorem 5.* Consider the Nash equilibrium $s^\star$ associated to a set $C = \{1, \ldots, r\}$. Assume that there exists a player $j = arg\min\{i \in \Gamma \setminus C : l_i < i\}$ and let $\hat{\imath} = arg\min\{i \in \Gamma \setminus C : l_i = j - 1\}$. The vector of strategies $s^\star$ is not stable with respect to positive perturbations $\Delta s(0) : ||\Delta s(0)||_1 = j - r - 1$ iff at least one of the following conditions holds:
i) there exist players $j+1, \ldots, 2j - \hat{\imath} - 1$ with threshold equal to $j - 1$,
ii) there exist players $j+1, \ldots, 2j - \hat{\imath}$.

**Proof -** Let us initially observe that, due to the minimality of $j$, $\hat{\imath}$ is less than or equal to $j - 1$. If $\hat{\imath} = j - 1$ then condition i) holds since it defines an empty set.

*Sufficiency.* We first prove condition i). For doing so, let the perturbed set $P$ be equal to $\{r+1, \ldots, j-1\}$ then $s_{r+1}(0) = \ldots = s_{j-1}(0) = 1$, which implies, at stage $k = 1$, $s_{r+1}(1) = \ldots = s_{\hat{\imath}-1}(1) = 1$, $s_j(1) = \ldots = s_{2j-\hat{\imath}-1}(1) = 1$. Actually, each player $i$ such that $r+1 \leq i \leq \hat{\imath} - 1$ observes that at the previous stage, $k = 0$, other $j - 2 \geq l_i$ players are active and each player $i$ such that $j \leq i \leq 2j - \hat{\imath} - 1$ observes that other $j - 1 = l_i$ players are active. Similarly, at stage $k = 2$, it surely holds that $s_{r+1}(2) = \ldots = s_{j-1}(2) = 1$. Hence, from such a stage on, the players $r+1, \ldots, j-1$ surely decide to meet the demand on the even stages and strategies oscillate.

It is left to prove condition ii). Let the perturbed set $P$ be equal to $\{r+1, \ldots, \hat{\imath}-1, j+1, \ldots, 2j-\hat{\imath}\}$ then $s_{r+1}(0) = \ldots = s_{\hat{\imath}-1}(0) = 1$ and $s_{j+1}(0) = \ldots = s_{2j-\hat{\imath}}(0) = 1$, which implies, at stage $k = 1$, $s_{r+1}(1) = \ldots = s_j(1) = 1$. Actually, each player $i$ such that $r+1 \leq i \leq \hat{\imath} - 1$ observes that at the previous stage, $k = 0$, other $j - 2 \geq l_i$ players are active and each player $i$ such that $\hat{\imath} \leq i \leq j$ observes that other $j - 1 = l_i$ players are active. For an analogous reason, from stage $k = 2$ on, the players $r+1, \ldots, j$ surely decide to meet the demand at every stage and strategies converge to a new Nash equilibrium different from $s^\star$.

*Necessity.* Assume that condition i) and condition ii) do not hold. Then the set $\Gamma$ includes at most $2j - \hat{\imath} - 1$ players and the threshold of the last player must satisfy the following condition $l_{2j-\hat{\imath}-1} > j - 1$. Then, given a perturbation $\Delta s(0)$ with $||\Delta s(0)||_1 = j - r - 1$, at stage $k = 1$ it holds $s_i(1) = 1$ for $i$ such that either $i < \hat{\imath}$ or $l_i = j - 1$ but $i \notin P$, $s_i(1) = 0$ otherwise. Assume without loss of generality that all $i$ such that $l_i = j - 1$ but $i \notin P$ are smaller than the minimum $\hat{\jmath}$ such that $l_{\hat{\jmath}} = j - 1$ and $\hat{\jmath} \in P$, then the maximum number of active players at stage $k = 1$ may be obtained for $P = \{r+1, \ldots, \hat{\imath}-1, j, \ldots, 2j - \hat{\imath} - 1\}$. Indeed, by doing this, we preserve all players $i$ with threshold $l_i = j - 1$ from being perturbed at $k = 0$.

Having chosen such a $P$, the number of active players at stage $k = 1$ is equal to $j - 1$. Indeed, all players $i = \hat{\imath}, \ldots, j - 1$ have thresholds $l_i = j - 1$ and therefore $s_{\hat{\imath}}(1) = \ldots = s_{j-1}(1) = 1$. At the same time, all players $i = j, \ldots, 2j - \hat{\imath} - 1$ whose thresholds are $l_i \geq j - 1$ observe other $j - 2$ active players and therefore $s_j(1) = \ldots = s_{2j-\hat{\imath}-1}(1) = 0$. Now, at stage $k = 2$, since by assumption is $l_{2j-\hat{\imath}-1} > j - 1$, we have $s_{2j-\hat{\imath}-1}(2) = 0$ such as $s_{\hat{\imath}}(2) = \ldots = s_{j-1}(2) = 0$. The situation at $k = 2$ is equivalent to the one obtainable at $k = 0$ in presence of a perturbation with $||\Delta s(0)||_1 = j - r - 2$. Since for perturbations with $||\Delta s(0)|| < j - r - 1$, see Theorem 3, the Nash equilibrium $s^\star$ is stable, we can affirm that even in this case the strategies will converge to $s^\star$.

□

Now, we specialize the previous theorems to the Pareto optimal Nash equilibrium.

*Corollary 2.* The unique Pareto optimal Nash equilibrium is maximally stable with respect to positive perturbations.

**Proof -** From definition of maximal stability, we must show that $s^\star$ is stable with respect to the maximal positive perturbation $\overline{\Delta s}(0)$, with $\|\overline{\Delta s}(0)\|_1 = |\Gamma - \overline{C}|$. From maximality of $\overline{C}$ it must hold $l_i \geq i$ for all $i$, such that $r < i < n$. Then, see Theorem 3, $s^\star$ is stable with respect to $\overline{\Delta s}(0)$ and therefore it is also maximally stable.

□

Let us conclude remarking that the Pareto optimal Nash equilibrium may not be globally stable with respect to negative perturbations. It is straightforward to prove this fact when, e.g., several Nash equilibria exist.

## 5. SIMULATION RESULTS

We consider a set $\Gamma$ of 8 players. Fig. 1 reports the induced graph $G$, whereas Tab. 1 lists the players' thresholds $l_i$.

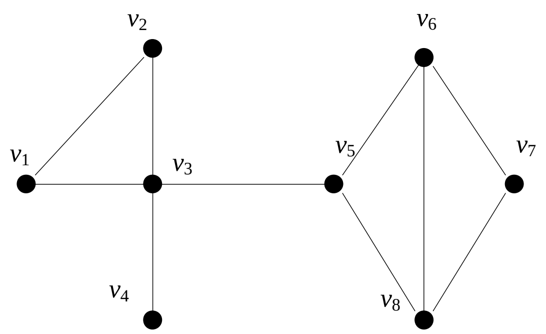


Fig. 1. An example of graph $G$ for a set $\Gamma$ of 8 players

Table 1. Players' thresholds

| players | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $l_i$ | 5 | $\infty$ | $\infty$ | $\infty$ | 2 | 1 | 4 | 1 |

Tab. 2 lists the players' states $x_i(k)$ and the strategies strategies $s_i(k)$, for $k \geq 0$. Note that, at $k = 0$, we assume that each player estimates that all the other players make its own decision. Such choice guarantees that the initial vector strategy $s_i(0)$ is not negatively perturbed with respect the Pareto optimal Nash equilibrium $s^\star$ Then, players $1 - 5 - 6 - 7 - 8$ are active, while players $2 - 3 - 4$ are not. At stage $k = 1$ all the players estimate the number of active players as equal to 5. Then, player 1 changes strategy from $s_1(0) = 1$ to $s_1(1) = 0$ since its estimate is lower than his corresponding threshold $l_1 = 5$. At $k = 2$, the players' new estimate is 4 and player 7 changes strategy, too. Finally, at stage $k = 3$, the players strategies converge to the Pareto optimal Nash equilibrium with $\|s^\star\|_1 = 3$.

Table 2. States and strategies over time

| players | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $x_i(0)$ | 8 | 0 | 0 | 0 | 8 | 8 | 8 | 8 |
| $s_i(0)$ | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $x_i(1)$ | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| $s_i(1)$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $x_i(2)$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| $s_i(2)$ | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 |
| $x_i(3)$ | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| $s_i(3)$ | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 |
| $x_i(k \geq 4)$ | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| $s_i(k \geq 4)$ | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 |

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