DOMAIN OF ATTRACTION: ESTIMATES FOR NON-POLYNOMIAL SYSTEMS VIA LMIS

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Abstract: Estimating the Domain of Attraction (DA) of equilibrium points is a problem of fundamental importance in systems engineering. Several approaches have been developed for computing the Largest Estimate of the DA (LEDA) corresponding to a Lyapunov function in the case of polynomial systems. In the case of non-polynomial systems, the computation of the LEDA is still an open problem. In this paper, an LMI technique is proposed to deal with such a problem for a class of non-polynomial systems. The key point consists of using sum of squares relaxations for taking into account the worst-case remainders corresponding to truncated Taylor expansions of the non-polynomial terms. As shown by some examples, low degree remainders may be sufficient to obtain almost tight estimates.

Keywords: Non-polynomial systems, Domain of attraction, Lyapunov function, LMI.

1. INTRODUCTION

Establishing that an equilibrium point is locally asymptotically stable is in general not sufficient to guarantee an asymptotically stable behavior of the system around such a point. In fact, except the case of linear systems, the behavior depends also on the initial condition. The local asymptotic stability ensures only that there exists a neighborhood of the equilibrium point such that the trajectory of the system converges to the point itself for any initial condition in such a neighborhood. However, further investigations on the extension of this neighborhood are required in order to allow a correct initialization of the system.

This is a problem of fundamental importance in the use of real plants where an incorrect initialization may determine a completely wrong action of the system. Obviously, this is even more crucial in expensive and/or dangerous plants, for example aircrafts, nuclear reactors, chemical plants, etc..., where unstable behaviors are not acceptable due to the disastrous consequences to which they could lead. Hence, it appears clear the importance of determining the Domain of Attraction (DA) of an equilibrium point, that is the set of initial condition from which the trajectory of the system converges to the point itself.

It is well known (see, e.g., (Genesio *et al.*, 1985; Khalil, 2001)) that the DA is a complicated set and it does not admit a tractable analytic representation in the most cases. Therefore, the inner approximation of the DA via an estimate of a simple shape has become a fundamental issue since long time. A common way of obtaining such estimates makes use of the Lyapunov stability theory. Specifically, given a Lyapunov Function (LF) for the equilibrium point, any LF level set included in the region where the LF time derivative assumes negative values is guaranteed to belong to the DA. The largest level set in such a region is called the Largest Estimate of the DA (LEDA) corresponding to the LF.

In the recent years, several approaches have been proposed for calculating the LEDA in the simpler case of polynomial systems, as for example the multidimensional gridding approach (Tibken and Dilaver, 2002) based on the use of Chebychev points, and the more powerful approaches (Parrilo, 2000; Tibken, 2000; Hachicho and Tibken, 2002; Chesi *et al.*, 2003) based on sum of squares relaxations which lead to convex optimizations constrained by LMIs. These approaches provide lower bounds of the LEDA whose conservativeness may be, in general, arbitrarily decreased by increasing the number of relaxing parameters.

Unfortunately, the most real systems do not belong to the class of polynomial systems. To name but a few, consider for example simple cases as pendulums and systems with saturations in addition to complex systems as aircrafts and reactors. Recently, the problem of recasting non-polynomial systems in polynomial ones has been addressed in (Papachristodoulou and Prajna, 2002; Papachristodoulou and Prajna, 2004) which propose suitable changes of coordinates for the problem of finding LFs for establishing the stability of equilibrium points. However, it is presently not clear the degree of conservativeness that the so obtained polynomial systems may introduce in the computation of the LEDA of the original systems due to the augmented system dimension and modified system structure. The problem of estimating the DA for non-polynomial systems is still an open problem.

In this paper, a technique for computing the LEDA corresponding to a given polynomial LF is presented for a class of non-polynomial systems. Specifically, the key idea consists of substituting the non-polynomial terms with their Taylor expansions truncated at a given degree. In order to guarantee to obtain sets that are really included in the DA, i.e. DA estimates, the remainders of such truncations in the Lagrange form are taken into account. Then, it is shown how sum of squares relaxations can be used to derive a sufficient condition for establishing whether a level set of the LF is an estimate of the DA by decomposing the problem of establishing the negativity of the time derivative over the level set into a set of subproblems where the negativity has to be established over subregions of such a set corresponding to the worst-case remainders. Such a condition meanly requires to solve an LMI feasibility problem, i.e. a convex optimization (see for example (Boyd *et al.*, 1994)) for which powerful tools have been developed. A lower bound of the LEDA is readily obtained through a one-parameter search on the level of the set which can be quickly performed, for example, via a bisection algorithm.

As shown by some examples, low degree remainders may be sufficient to obtain almost tight lower bounds. The extension of the proposed technique to systems not belonging to the considered class, currently under investigation, is briefly discussed.

2. PRELIMINARIES

2.1 Problem formulation

Notation:

- $\mathbb{N}, \mathbb{R}$: natural and real numbers sets;
- $\mathbb{M}^n$: $\mathbb{R}^{n\times n}$;
- 0_n: origin of $\mathbb{R}^n$;
- I_n: identity matrix $n \times n$;
- A': transpose of matrix A;
- $A > 0$ ($A \geq 0$): symmetric positive definite (semidefinite) matrix A;
- $\mathcal{C}^n$: set of functions whose first n derivatives are continuous over all the domain.

Consider the following class of continuous-time state space models

$$\begin{cases} \dot{x}(t) = p_0(x(t)) + \sum_{i=1}^{r} p_i(x(t)) g_i(x_{\mu_i}(t)) \\ x(0) = x_{init} \end{cases} \quad (1)$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))' \in \mathbb{R}^n$ is the state, $x_{init} \in \mathbb{R}^n$ is the initial condition, and $p_j : \mathbb{R}^n \to \mathbb{R}^n$ are polynomial vectors of degree $\epsilon_j \in \mathbb{N}$ for $j = 0, \ldots, r$. The functions $g_i : \mathbb{R} \to \mathbb{R}$ are non polynomial in $\mathcal{C}^\infty$ and $\mu_i \in \{1, \ldots, n\}$ are indexes for $i = 1, \ldots, r$. In the sequel the dependence of $x(t)$ on the time t will be omitted for ease of notation.

It is assumed that the origin is the equilibrium point of interest in the system (1). Therefore, it is assumed that

$$p_0(0_n) + \sum_{i=1}^{r} p_i(0_n) g_i(0) = 0_n. \quad (2)$$

Let $\varphi(t, x_{init}) \in \mathbb{R}^n$ denote the solution of (1) at time t. The DA of the origin is

$$\Delta = \left\{ x_{init} : \lim_{t\to+\infty} \varphi(t, x_{init}) = 0_n \right\}. \quad (3)$$

Let $v : \mathbb{R}^n \to \mathbb{R}$ be a positive definite[1] and radially unbounded[2] function in $\mathcal{C}^1$, and suppose that $v(x)$ is a LF for the origin in system (1), i.e. such that the time derivative of $v(x)$ along the trajectories,

$$\dot{v}(x) = \frac{\partial v(x)}{\partial x}\left(p_0(x) + \sum_{i=1}^{r} p_i(x) g_i(x_{\mu_i})\right), \quad (4)$$

is locally negative definite. The set

$$\mathcal{V}(c) = \{x :\ v(x) \le c\} \quad (5)$$

is an estimate of the DA if $\mathcal{V}(c) \subset \mathcal{D}$ where $\mathcal{D} = \{x :\ \dot{v}(x) < 0\} \cup \{0_n\}$.

The problem dealt with in this paper is the determination of the LEDA, i.e. the computation of the maximum c such that $\mathcal{V}(c)$ is an estimate of the DA, for a given polynomial LF $v(x)$. In particular, the LEDA is given by $\mathcal{V}(\gamma)$ where

$$\begin{array}{rl} \gamma = & \inf\limits_{x \in \mathbb{R}^n \setminus \{0_n\}} v(x) \\ & \text{s.t.} \quad \dot{v}(x) = 0. \end{array} \quad (6)$$

In order to simplify the presentation of the proposed technique it is supposed that the linearized system in the origin is asymptotically stable, i.e. that

$$A = \frac{d}{dx}\left(p_0(x) + \sum_{i=1}^{r} p_i(x) g_i(x_{\mu_i})\right)\Bigg|_{x=0_n} \quad (7)$$

is Hurwitz. Moreover, it is assumed that the LF $v(x)$ is a polynomial of degree $2\delta_v$ whose quadratic part is positive definite.

2.2 *Representation of polynomials*

Before proceeding let us introduce the Complete Square Matricial Representation (CSMR) of polynomials which provides all the possible representations of a polynomial in terms of a quadratic form (see (Chesi *et al.*, 2003) where the CSMR is similarly defined for homogeneous forms; the CSMR is also known as Gram matrix (Choi *et al.*, 1995)).

Given $m \in \mathbb{N}$ let us define the vector $x^{\{m\}} = \left[x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^m\right]' \in \mathbb{R}^{\sigma(n,m)}$ containing all monomials of degree less or equal to m in x but the constant term (the reason for which we exclude the constant term will be clear in the sequel), where

$$\sigma(n,m) = \frac{(n+m)!}{n!m!} - 1. \quad (8)$$

Let $p(x)$ be a polynomial of degree less or equal to $2m$ without constant and linear terms. The CSMR of $p(x)$ with respect to the vector $x^{\{m\}}$ is defined as

$$p(x) = x^{\{m\}'} P(\alpha) x^{\{m\}} \quad (9)$$

$$P(\alpha) = P + L(\alpha) \quad (10)$$

where $P \in \mathbb{M}^{\sigma(n,m)}$ is any symmetric matrix such that $p(x) = x^{\{m\}'} P x^{\{m\}}$, $\alpha \in \mathbb{R}^{\tau(n,m)}$ is a vector of free parameters and $L(\alpha)$ is a linear parameterization of the set $\mathcal{L} = \left\{L = L' :\ x^{\{m\}'} L x^{\{m\}} = 0 \ \forall x \in \mathbb{R}^n\right\}$. It can be verified that

$$\begin{array}{rl} \tau(n,m) = & \frac{1}{2}\sigma(n,m)(\sigma(n,m)+1) + n \\ & -\sigma(n,2m). \end{array} \quad (11)$$

3. LEDA COMPUTATION

Let us write $g_i(x_{\mu_i})$ via the Taylor expansion up to the a_i-th power for a given $a_i \in \mathbb{N}$ and expressing the remainder in the Lagrange form:

$$g_i(x_{\mu_i}) = h_i(x_{\mu_i}) + l_i(x_{\mu_i})\theta_i(y_i) \quad (12)$$

where

$$h_i(x_{\mu_i}) = \sum_{j=0}^{a_i} \frac{x_{\mu_i}^j}{j!} \frac{d^j g_i(x_{\mu_i})}{dx_{\mu_i}^j}\Bigg|_{x_{\mu_i}=0} \quad (13)$$

$$l_i(x_{\mu_i}) = \frac{x_{\mu_i}^{a_i+1}}{(a_i+1)!} \quad (14)$$

$$\theta_i(y_i) = \frac{d^{a_i+1} g_i(x_{\mu_i})}{dx_{\mu_i}^{a_i+1}}\Bigg|_{x_{\mu_i}=y_i} \quad (15)$$

$$y_i \in \left\{\begin{array}{ll} [0, x_{\mu_i}] & \text{if } x_{\mu_i} \ge 0 \\ [x_{\mu_i}, 0] & \text{otherwise.} \end{array}\right. \quad (16)$$

The time derivative $\dot{v}(x)$ becomes:

$$\dot{v}(x) = f_0(x) + \sum_{i=1}^{r} f_i(x)\theta_i(y_i) \quad (17)$$

where

$$f_0(x) = \frac{\partial v(x)}{\partial x}\left(p_0(x) + \sum_{i=1}^{r} p_i(x) h_i(x_{\mu_i})\right) \quad (18)$$

$$f_i(x) = \frac{\partial v(x)}{\partial x} p_i(x) l_i(x_{\mu_i}). \quad (19)$$

Let $\delta_q \in \mathbb{N}$ and define

[1] A function $v(x)$ is said to be positive definite if $v(0_n) = 0$ and $v(x) > 0$ for all $x \neq 0_n$. It is said to be locally positive definite if $v(0_n) = 0$ and there exists $\delta > 0$ such that $v(x) > 0$ for all $x \neq 0_n$ such that $\|x\| < \delta$.

[2] A function $v(x)$ is radially unbounded if $\lim_{\|x\| \to +\infty} v(x) = +\infty$.

$$m = \left\lceil \frac{\delta_d}{2} \right\rceil + \delta_q \quad (20)$$

$$\delta_s = m - \delta_v \quad (21)$$

where δ_d is the degree of $\dot{v}(x)$

$$\delta_d = 2\delta_v + \max\{\epsilon_0 - 1, \epsilon_1 + a_1, \ldots, \epsilon_r + a_r\}. \quad (22)$$

Let $S_\psi \in \mathbb{M}^{\sigma(n,\delta_s)}$ and $Q_{i,\psi} \in \mathbb{M}^{\sigma(n,\delta_q)+1}$ be free matrices for $\psi \in \{0,1\}^r$ and $i = 1, \ldots, r$, and define the polynomials

$$s_\psi(x) = x^{\{\delta_s\}'} S_\psi x^{\{\delta_s\}} \quad (23)$$

$$q_{i,\psi}(x) = \left[1, x^{\{\delta_q\}'}\right] Q_{i,\psi} \left[1, x^{\{\delta_q\}'}\right]'. \quad (24)$$

Let $\tilde{S}(S_\psi) \in \mathbb{M}^m$, $\tilde{V}(S_\psi) \in \mathbb{M}^m$, $\tilde{F}_i(Q_{i,\psi}) \in \mathbb{M}^m$ for $i = 1, \ldots, r$, and $F_j \in \mathbb{M}^m$ for $j = 0, \ldots, r$ be any SMR matrices satisfying

$$s_\psi(x) = x^{\{m\}'} \tilde{S}(S_\psi) x^{\{m\}} \quad (25)$$

$$v(x) s_\psi(x) = x^{\{m\}'} \tilde{V}(S_\psi) x^{\{m\}} \quad (26)$$

$$f_i(x) q_{i,\psi}(x) = x^{\{m\}'} \tilde{F}_i(Q_{i,\psi}) x^{\{m\}} \quad (27)$$

$$f_j(x) = x^{\{m\}'} F_j x^{\{m\}}. \quad (28)$$

Define for $i = 1, \ldots, r$ and $c \in \mathbb{R}$

$$\mathcal{X}_i(c) = \left[\min_{x \in \mathcal{V}(c)} x_{\mu_i}, \max_{x \in \mathcal{V}(c)} x_{\mu_i}\right] \quad (29)$$

$$\xi_i(0, c) = \max_{y_i \in \mathcal{X}_i(c)} \theta_i(y_i) \quad (30)$$

$$\xi_i(1, c) = \min_{y_i \in \mathcal{X}_i(c)} \theta_i(y_i) \quad (31)$$

The following result provides a sufficient condition for establishing whether a level set is an estimate of the DA through an LMI feasibility test.

Theorem 1. Given $c \in \mathbb{R}$, suppose that for all $\psi \in \{0,1\}^r$ there exist $S_\psi \in \mathbb{M}^{\sigma(n,\delta_s)}$, $Q_{i,\psi} \in \mathbb{M}^{\sigma(n,\delta_q)+1}$ and $\alpha_\psi \in \mathbb{R}^{\tau(n,m)}$ such that the set of LMIs

$$\begin{cases} 0 < S_\psi \\ 0 < Q_{i,\psi}, \quad i = 1, \ldots, r \\ 0 > L(\alpha_\psi) + c\tilde{S}(S_\psi) - \tilde{V}(S_\psi) + F_0 \\ \qquad + \sum_{i=1}^{r} \left(\xi_i(\psi_i, c) F_i + (-1)^{\psi_i} \tilde{F}_i(Q_{i,\psi})\right) \end{cases} \quad (32)$$

is satisfied. Then, $\mathcal{V}(c) \subseteq \Delta$.

Proof Suppose that (32) is satisfied for all $\psi \in \{0,1\}^r$ and let $\bar{x}$ be any point in $\mathcal{V}(c)$. Define

$$\bar{\psi} = [\beta_1, \beta_2, \ldots, \beta_r]'$$

$$\beta_i = \begin{cases} 0 & \text{if } f_i(\bar{x}) \geq 0 \\ 1 & \text{otherwise} \end{cases}$$

and consider the LMI set (32) relative to the index vector $\bar{\psi}$. Let us multiply both sides of the third inequality times $\bar{x}^{\{m\}'}$ at left and $\bar{x}^{\{m\}}$ at right. Since $\bar{x}^{\{m\}'} L(\alpha_{\bar{\psi}}) \bar{x}^{\{m\}} = 0$ we obtain

$$\begin{aligned} 0 &> (c - v(\bar{x})) s_{\bar{\psi}}(\bar{x}) + f_0(\bar{x}) + \sum_{i=1}^{r} \Big(\xi_i(\bar{\psi}_i, c) f_i(\bar{x}) \\ &\quad + (-1)^{\bar{\psi}_i} f_i(\bar{x}) q_{i,\bar{\psi}}(\bar{x})\Big) \\ &\geq (c - v(\bar{x})) s_{\bar{\psi}}(\bar{x}) + f_0(\bar{x}) + \sum_{i=1}^{r} \Big(\theta_i(y_i) f_i(\bar{x}) \\ &\quad + |f_i(\bar{x})| q_{i,\bar{\psi}}(\bar{x})\Big) \\ &= (c - v(\bar{x})) s_{\bar{\psi}}(\bar{x}) + \sum_{i=1}^{r} |f_i(\bar{x})| q_{i,\bar{\psi}}(\bar{x}) + \dot{v}(\bar{x}). \end{aligned}$$

Now, let us observe that the first two inequalities in (32) imply that $s_{\bar{\psi}}(\bar{x}) > 0$ and $q_{i,\bar{\psi}}(\bar{x}) > 0$. Since $\bar{x} \subseteq \mathcal{V}(c)$ it follows that $c - v(\bar{x}) \geq 0$ and, hence, $\dot{v}(\bar{x}) < 0$. This implies that $\mathcal{V}(c) \subset \mathcal{D}$, i.e. $\mathcal{V}(c)$ is an estimate of the DA. $\square$

The condition of Theorem 1 consists of 2^r LMI feasibility tests in the form (32) which have the role of considering the worst-case effects of the remainders in the Taylor expansions. Specifically, sum of squares relaxations through the matrices $Q_{i,\psi}$ and S_ψ are used to test the negativity of $\dot{v}(x)$ in the subregions of $\mathcal{V}(c)$ where the remainders act in a certain way depending on the sign of $f_i(\bar{x})$. It is worthwhile to notice that these tests can be performed separately in order to reduce the computational burden.

In order to build the set of LMIs (32) one has to calculate the sets $\mathcal{X}_i(c)$ and, from these, the quantities $\xi_i(0, c)$ and $\xi_i(1, c)$. The sets $\mathcal{X}_i(c)$ can be trivially found in closed-form solution if the LF is quadratic, otherwise by solving a system of polynomial equations. The computation of $\xi_i(0, c)$ and $\xi_i(1, c)$ amounts to finding the minimum and maximum values of a one-variable function over an interval, and it is an easy task for standard non-polynomial terms.

Finally, define the lower bound of γ in (6)

$$\hat{\gamma} = \sup\{c : \text{ the condition in Theorem 1 is satisfied}\}. \quad (33)$$

This lower bound can be found by a sweep on the scalar c, for example via a bisection algorithm in order to speed up the convergence. The conservativeness of $\hat{\gamma}$ can be decreased by increasing:

- the degree δ_q (and hence δ_m and δ_s) since this allows one to obtain less conservative sum of squares relaxations;
- the degrees a_i of the Taylor expansions for the non-polynomial terms in (1) since this reduces the remainders and their conservative effects on the DA estimates.

Remark. The proposed technique can be extended to systems that are not in the form of (1), i.e. systems containing non-polynomial terms that are functions of more than one variable, that $g_i(x)$ instead of $g_i(x_{\mu_i})$. This can be done by using the remainder of the Taylor expansion for multivariable functions. However, each non-polynomial term determines more than one unknown parameter $\theta_i(y_i)$ in the expression of $\dot{v}(x)$ in (17) and, consequently, a larger computational burden in the LMI feasibility test (32). Moreover, the computation of $\xi_i(0,c)$ and $\xi_i(1,c)$ is more involved since the remainders depend on more than one variable. Therefore, the use of the proposed technique for systems that either are not in the form of (1) or cannot be recast in such a form may be a difficult task, and it is the subject of current research.

4. EXAMPLES

In this section some examples are reported to show the applicability and the potentialities of the proposed technique. The lower bounds $\hat{\gamma}$ have been computed in all examples by using $\delta_q = 1$ in (20).

4.1 Example 1

Let us consider the simple pendulum-system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 - \sin x_1 \end{cases}$$

and the problem of estimating the DA of the origin. To this end, let us calculate the LEDA corresponding to the LF $v(x) = 4x_1^2 + 2x_1x_2 + 3x_2^2$. The time derivative is:

$$\dot{v}(x) = 6x_1x_2 - 4x_2^2 + (-2x_1 - 6x_2)\sin x_1.$$

Table 1 shows the lower bounds $\hat{\gamma}$ for some degrees a_1 of the Taylor expansion of $\sin x_1$. Figure 1 shows the estimate $\mathcal{V}(\hat{\gamma}_{a_1=6})$ and the curve $\dot{v}(x) = 0$. As we can see, the lower bound is almost tight since the ellipse almost touches this curve.

Table 1. (Example 1) Lower bounds of γ for some values of a_1.

a_1	2	3	4	5	6
$\hat{\gamma}$	18.07	12.79	16.55	21.11	22.94

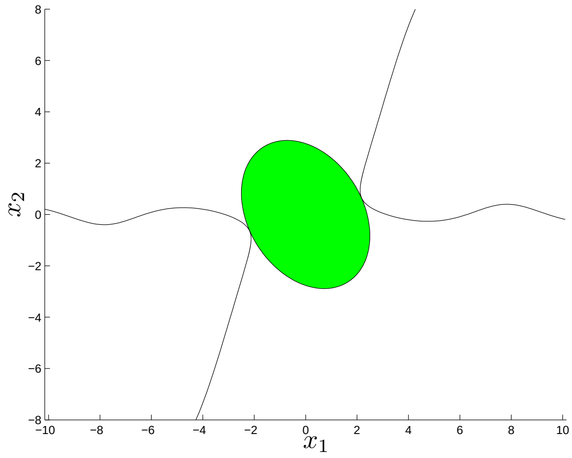

Fig. 1. Example 1. Estimate of the DA for $a_1 = 6$ (green ellipse) and curve $\dot{v}(x) = 0$ (black line). The estimate almost coincides with the LEDA, i.e. the lower bound is tight, since the ellipse is almost tangent to the curve.

4.2 Example 2

Let us consider the system

$$\begin{cases} \dot{x}_1 = -\frac{1}{4}x_1 + \ln(1 + x_2) \\ \dot{x}_2 = -\frac{3}{8}x_1 - \frac{1}{5}x_1x_2 + \left(\frac{1}{8}x_1 - x_2\right)\cos x_1 \end{cases}$$

and the problem of calculating the LEDA corresponding to the LF $v(x) = x_1^2 + x_2^2$. The time derivative is:

$$\begin{aligned} \dot{v}(x) = & -\frac{1}{2}x_1^2 - \frac{3}{4}x_1x_2 - \frac{2}{5}x_1x_2^2 \\ & + \left(\frac{1}{4}x_1x_2 - 2x_2^2\right)\cos x_1 \\ & + (2x_1)\ln(1 + x_2). \end{aligned}$$

Table 2 shows the lower bounds $\hat{\gamma}$ for some degrees a_1 and a_2. As shown by Figure 2, the lower bound for $a_1 = a_2 = 6$ is satisfactorily tight since the circle is quite close to the curve $\dot{v}(x) = 0$.

Table 2. (Example 2) Lower bounds of γ for some values of $a = a_1 = a_2$.

a	2	3	4	5	6
$\hat{\gamma}$	0.1768	0.2138	0.2367	0.2514	0.2606

4.3 Example 3

Let us consider the system

$$\begin{cases} \dot{x}_1 = 1 + x_3 + \frac{1}{8}x_3^2 - e^{x_1} \\ \dot{x}_2 = -x_2 - x_3 \\ \dot{x}_3 = -x_2 - 2x_3 - \frac{1}{2}x_1^2 \end{cases}$$

and the problem of calculating the LEDA corresponding to the LF $v(x) = x_1^2 + x_2^2 + x_3^2$. The time derivative is:

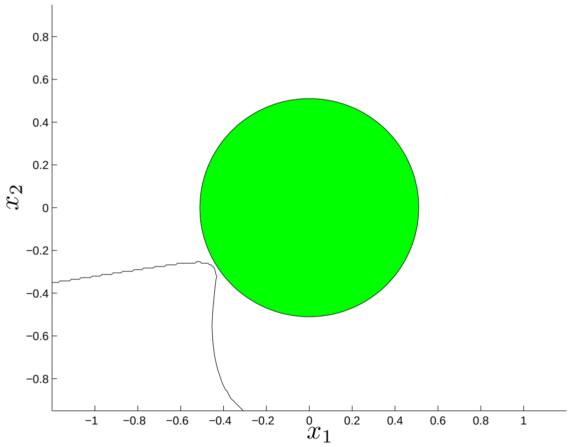

Fig. 2. Example 2. Estimate of the DA for $a_1 = a_2 = 5$ (green circle) and curve $\dot{v}(x) = 0$ (black line).

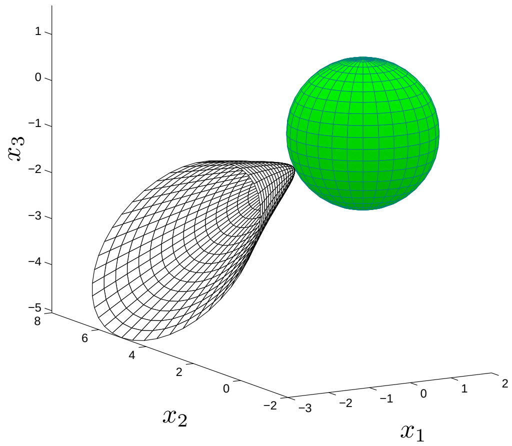

Fig. 3. Example 3. Estimate of the DA for $a_1 = 6$ (green sphere) and surface $\dot{v}(x) = 0$ (black net).

$$\dot{v}(x) = 2x_1 + 2x_1x_3 - 2x_2^2 - 4x_2x_3 - 4x_3^2 \\ -x_1^2x_3 + \frac{1}{4}x_1x_3^2 + (-2x_1)e^{x_1}.$$

Table 3 shows the lower bounds $\hat{\gamma}$ for some degrees a_1. As shown by Figure 3, the estimate for $a_1 = 6$ almost coincides with the LEDA.

Table 3. (Example 3) Lower bounds of γ for some values of a_1.

a_1	2	3	4	5	6
$\hat{\gamma}$	1.816	1.595	2.531	2.483	2.655

5. CONCLUSION

An LMI technique has been proposed to compute the Largest Estimate of the DA (LEDA) corresponding to a polynomial Lyapunov Function (LF) for a class of non-polynomial systems. This technique is based on truncated Taylor expansions of the non-polynomial terms and uses sum of squares relaxations to take into account the worst-case remainders. As shown by some examples, low degree remainders may be sufficient to obtain almost tight estimates.

Future work will deal with the problem of extending the proposed technique to all non-polynomial systems. Also the selection of the LF for obtaining better estimates of the DA is under investigation.

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