ADJUSTMENT OF HIGH-ORDER SLIDING-MODE CONTROLLERS

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Abstract: One of the main problems in the high-order sliding-mode application is the difficulty to properly adjust the controller parameters. A universal method is proposed, based on the homogeneity approach, which allows to make the finite-time convergence arbitrarily fast or slow. Another problem arises, when the dynamic system uncertainties are unbounded. In general, in that case only local uncertainty suppression is obtained. This restriction is also removed in the paper. The results are illustrated by computer simulation.

Keywords: nonlinear control, sliding mode, robustness, uncertainty, tracking systems.

1. INTRODUCTION

Sliding-mode control remains one of the most effective approaches to cope with uncertainty. The idea is to react immediately to any deviation of the system from some properly chosen constraint steering it back by a sufficiently energetic effort. Sliding mode is accurate and insensitive to disturbances (Utkin, 1992; Edwards and Spurgeon, 1998). The main drawback of the standard sliding modes is mostly related to the so-called chattering effect (Fridman, 2002).

Let the constraint be given by the equation $\sigma = s - w(t) = 0$, where s is some available output variable of an uncertain single-input-single-output (SISO) dynamic system and $w(t)$ is an unknown-in-advance smooth input to be tracked in real time. Then the standard sliding-mode control $u = - k \operatorname{sign} \sigma$ may be considered as a universal output controller applicable if the relative degree is 1, i.e. if $\dot{\sigma}$ explicitly depends on the control u and $\dot{\sigma}'_u > 0$. Higher-order sliding mode (HOSM) (Levant, 1993; 2003a) is applicable for controlling SISO uncertain systems with arbitrary relative degree r. The corresponding finite-time-convergent controllers (r-sliding controllers) (Levant, 1993; 2003a,b; Bartolini et al., 2003) require actually only the knowledge of the system relative degree. The produced control is a discontinuous function of the tracking deviation σ and of its real-time-calculated successive derivatives $\dot{\sigma}$, $\ddot{\sigma}$, ..., $\sigma^{(r-1)}$. The controllers provide also for higher accuracy with discrete sampling and, properly used, totally remove the chattering effect. In order to remove the chattering, the control derivative is to be treated as a new control.

While the second-order sliding-mode controllers are already widely used (Bartolini, *et al.*, 2003; Sira-Ramirez, 2002; Shtessel and Shkolnikov, 2003), the higher-order controllers still wait for their application. One of the main problems is the parameter adjustment. Indeed, no algebraic criterion was published for the parameter assignment, though it could be developed based on the constructive proofs (Levant 2003a, b). Such calculations would be carried out separately for each relative degree, and would produce highly conservative conditions on the parameters. Thus, the author considers such conditions practically useless. The proposed solution was to find such parameters by simulation. Tested parameter sets were published for the main practical cases $r = 2, 3, 4$. Though theoretically already one set is sufficient for any relative degree, in practice one needs to adjust these parameters, in order to hasten or to slow down the finite-time transient process. A simple algorithm is presented in this paper producing infinite number of valid parameter sets from a given one. The convergence can be made arbitrarily fast or slow.

Another known problem is the requirement of the uncertainty boundedness. In the presence of globally unbounded uncertainties the known results only provide for the local convergence to the sliding mode. This restriction is also removed in this paper for a number of controllers.

Computer simulation demonstrates the applicability of the proposed scheme on a model example.

2. THE PROBLEM STATEMENT

Consider a smooth dynamic system with a smooth output function σ, and let the system be closed by some possibly-dynamical discontinuous feedback and be understood in the Filippov sense (1988). Then, provided that successive total time derivatives σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$ are continuous functions of the closed-system state-space variables; and the set $\sigma = ... = \sigma^{(r-1)} = 0$ is a non-empty integral set, the motion on the set is called *r*-sliding (*r*th order sliding) mode (Levant, 1993; 2003a).

The standard sliding mode used in the most variable structure systems, is of the first order (σ is continuous, and $\dot{\sigma}$ is discontinuous).

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t, x), \qquad (1)$$

where $x \in \mathbf{R}^n$, a, b and σ: $\mathbf{R}^{n+1} \rightarrow \mathbf{R}$ are unknown smooth functions, $u \in \mathbf{R}$, n is also uncertain. The relative degree r of the system is assumed to be constant and known. That means that the control appears explicitly for the first time in the rth total time derivative of σ (Isidori, 1989). Full information on the system state is assumed available. In particular, t, x, σ and its r - 1 successive derivatives are measured. It is easy to check that

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \qquad (2)$$

where $h(t,x) = \sigma^{(r)}|_{u=0}$, $g(t,x) = \frac{\partial}{\partial u}\sigma^{(r)}$ are some uncertain functions, $g(t,x) \neq 0$. A locally bounded Lebesgue-measurable non-zero function $\Phi(t,x)$ is supposed to be given, such that for any d. the inequality

$$\alpha\, g(t,x)\Phi(t,x) > d + |h(t,x)| \qquad (3)$$

holds with sufficiently large α. The task is to provide in finite time for the identity $\sigma \equiv 0$.

It is also assumed that trajectories of (1) are infinitely extendible in time for any Lebesgue-measurable control $u(t, x)$ with the bounded quotient u/Φ. Actually the proposed method works for much larger class of systems, and this assumption is needed only to avoid finite-time escape. In practice the system is supposed weakly minimum phase.

Note that the traditional assumption (Levant, 2003a, Bartolini *et al*., 2003) is that

$$0 < K_{\mathrm{m}} \leq \frac{\partial}{\partial u}\sigma^{(r)} \leq K_{\mathrm{M}}, \; |\sigma^{(r)}|_{u=0}| \leq C \qquad (4)$$

for some K_{m}, K_{M}, $C > 0$. It corresponds to $\Phi = 1$. The both problem statements are further considered.

3. COPING WITH UNBOUNDED UNCERTAINTIES

Two known families of high-order sliding controllers are defined by recursive procedures. In the following $\beta_1,..., \beta_{r-1} > 0$ and $i = 1,..., r$-1.

1. The following procedure defines the "standard" *r*-sliding controller (Levant 2003a). Let p be the least common multiple of 1, 2, ..., r. Define

$$N_{i,r} = (|\sigma|^{p/r} + |\dot{\sigma}|^{p/(r-1)} + ... + |\sigma^{(i-1)}|^{p/(r-i+1)})^{(r-i)/p};$$
$$\Psi_{0,r} = \mathrm{sign}\, \sigma, \quad \Psi_{i,r} = \mathrm{sign}(\sigma^{(i)} + \beta_i N_{i,r} \Psi_{i-1,r}).$$

2. Another procedure defines the so-called quasi-continuous controller (Levant 2003b). Denote

$$\varphi_{0,r} = \sigma, \; N_{0,r} = |\sigma|, \quad \Psi_{0,r} = \varphi_{0,r} / N_{0,r} = \mathrm{sign}\, \sigma,$$
$$\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{-1/(r-i+1)} \varphi_{i-1,r},$$
$$N_{i,r} = |\sigma^{(i)}| + \beta_i N_{i-1,r}^{-1/(r-i+1)} |\varphi_{i-1,r}|,$$
$$\Psi_{i,r} = \varphi_{i,r} / N_{i,r}.$$

In the both cases the controller takes on the form

$$u = -\alpha\, \Phi(t,x)\Psi_{r-1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}), \qquad (5)$$

where $\alpha > 0$. Note that in the case of the quasi-continuous controller the function $\Psi_{r-1,r}$ can be redefined according to the continuity everywhere except the *r*-sliding set $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$. Recall that, according to the Filippov definition (1988), values of the control on any set of the zero Lebesgue measure do not influence the solutions.

Theorem 1. *Provided* $\beta_1,..., \beta_{r-1}, \alpha > 0$ *are chosen sufficiently large in the list order, controller* (5) *provides for the finite-time establishment of the identity* $\sigma \equiv 0$ *for any initial conditions. Moreover, any increase of the gain function* Φ *does not interfere with the convergence.*

In other words, the finite-time stable r-sliding mode $\sigma \equiv 0$ is established in the system (1), (5). Note that the Theorem does not claim that *all* parametric combinations providing for the finite-time convergence to the r-sliding mode, allow the arbitrary increasing of α and Φ. Such parameter combinations are called further *gain-function robust*.

A number of other HOSM controllers satisfy Theorem 1. Such controllers and parameter combinations are also called *gain-function robust*. The popular sub-optimal and twisting controllers are not gain-function robust and require special efforts to deal with unbounded uncertainties (Bartolini *et al*., 2001; Levant, 1993).

Proof. The proofs are similar for the both controllers. The main idea is that with sufficiently large α any system trajectory enters some specific region in finite time to stay in it. The region is described by some differential inequalities, which do not "remember" anything on the original process. These inequalities determine the further convergence. Consider, for example, the quasi-continuous controller. The proof is based on a number of Lemmas.

Lemma 1. Let $i = 0, ..., r-1$. *$N_{i,r}$ is positive definite, i.e. $N_{i,r} = 0$ iff $\sigma = \dot\sigma = ... = \sigma^{(i)} = 0$. The inequality $|\Psi_{i,r}| \le 1$ holds whenever $N_{i,r} > 0$. The function $\Psi_{i,r}(\sigma, \dot\sigma, ..., \sigma^{(i-1)})$ is continuous everywhere (i.e. it can be redefined by continuity) except the point $\sigma = \dot\sigma = ... = \sigma^{(i-1)} = 0$.*

Assign the weights (homogeneity degrees) $r - i$ to $\sigma^{(i)}$, $i = 0, ..., r - 1$ and the weight 1 (minus system homogeneity degree, Bacciotti and Rosier, 2001) to t, which corresponds to the r-sliding homogeneity (Levant, 2005).

Lemma 2. *The weight of $N_{i,r}$ equals $r - i$, $i = 0, ..., r - 1$. Each homogeneous locally-bounded function $\omega(\sigma, \dot\sigma, ..., \sigma^{(i)})$ of the weight $r - i$ satisfies the inequality $|\omega| \le c\, N_{i,r}$ for some $c > 0$.*

Lemma 3. *For any $\gamma > 0$ with sufficiently large $\alpha > 0$ the inequality $|\sigma^{(r-1)} + \beta_{r-1} N_{r-2,r}^{1/2}\ \Psi_{r-2,r}| \le \gamma\ N_{r-2,r}^{1/2}$ is established in finite-time and kept afterwards.*

Proof. Consider the point set $\Omega(\xi) = \{(\sigma, \dot\sigma, ..., \sigma^{(r-1)}) \mid |\Psi_{r-1,r}| \le \xi\}$ for some fixed $\xi > 0$, $\xi < \gamma/4$. Simple calculations show that $\Omega(\xi) \subset \Omega_1(\xi)$ with small ξ, where $\Omega_1(\xi)$ is defined by the inequality

$$|\sigma^{(r-1)} + \beta_{r-1}\ N_{r-2,r}^{1/2}\, \Psi_{r-2,r}| \le 4\xi\, N_{r-2,r}^{1/2}\,.$$

That inequality is equivalent to the inequality $\phi_- \le \sigma^{(r-1)} \le \phi_+$, where ϕ_-, ϕ_+ are homogeneous functions of $\sigma, \dot\sigma, ..., \sigma^{(r-2)}$ of the weight 1. Restricting ϕ_- and ϕ_+ to the homogeneous sphere $\sigma^{2p/r} + \dot\sigma^{2p/(r-1)} + ... + (\sigma^{(r-2)})^{2p} = 1$, where p is the least multiple of 1, 2, ..., $r - 1$, achieve some continuous on the sphere functions ϕ_{1-} and ϕ_{1+}. Functions ϕ_{1-} and ϕ_{1+} can be approximated on the sphere by some smooth functions ϕ_{2-} and ϕ_{2+} from beneath and from above respectively. Functions ϕ_{2-} and ϕ_{2+} are extended by homogeneity to the homogeneous functions Φ_- and Φ_+ of $\sigma, \dot\sigma, ..., \sigma^{(r-2)}$ of the weight 1, smooth everywhere except 0, so that $\Omega(\xi) \subset \Omega_2 = \{(\sigma, \dot\sigma, ..., \sigma^{(r-1)}) \mid \Phi_- \le \sigma^{(r-1)} \le \Phi_+\}$.

Thus, the inequality $|\Psi_{r-1,r}| \ge \xi$ is assured outside of Ω_2. Prove now that Ω_2 is invariant and attracts the trajectories with large α. The "upper" boundary of Ω_2 is given by the equation $\pi_+ = \sigma^{(r-1)} - \Phi_+ = 0$. Suppose that at the initial moment $\pi_+ > 0$ and, therefore, $\Psi_{r-1,r} \ge \xi$. Taking into account that $\dot\Phi_+(\sigma, \dot\sigma, ..., \sigma^{(r-1)})$ is a locally bounded homogeneous function of the zero weight, obtain $|\dot\Phi_+| \le \kappa$ for some $\kappa > 0$. Differentiating achieve that $\dot\pi_+ \le -d\,\xi + \kappa < 0$ if d is properly chosen and α is sufficiently large.

Hence, π_+ vanishes in finite time with β_{i+1} large enough. Thus, the trajectory inevitably enters the region Ω_2 in finite time. Similarly, the trajectory enters Ω_2 if the initial value of π_+ is negative and, therefore, $\Psi_{i,r} \le -\xi$. Obviously, Ω_2 is invariant.

Choosing Φ_- and Φ_+ sufficiently close to ϕ_- and ϕ_+ on the homogeneous sphere and α respectively large enough, achieve from Lemma 2 that $\Omega_2 \subset \Omega_1(\gamma_i/4)$ and the statement of Lemma 3.■

The fulfilment of the statement of Lemma 3 triggers a chain collapse as follows from the next Lemma.

Lemma 4. *Let $1 \le i \le r-2$, then for any positive β_i, γ_i, γ_{i+1} with sufficiently large $\beta_{i+1} > 0$ the inequality*

$$|\sigma^{(i+1)} + \beta_{i+1}\, N_{i,r}^{(r-i-1)/(r-i)}\, \Psi_{i,r}| \le \gamma_{i+1}\, N_{i,r}^{(r-i-1)/(r-i)}$$

provides for the finite-time establishment and keeping of the inequality

$$|\sigma^{(i)} + \beta_i\ N_{i-1,r}^{(r-i)/(r-i+1)}\, \Psi_{i-1,r}| \le \gamma_i\ N_{i-1,r}^{(r-i)/(r-i+1)}\,.$$

The proof is very similar to Lemma 3. The point set $\Omega(\xi) = \{(\sigma, \dot\sigma, ..., \sigma^{(i)}) \mid |\Psi_{i,r}| \le \xi\}$ is considered for some fixed $\xi > 0$, $\xi < \gamma_i/4$. The set $\Omega_1(\xi) \supset \Omega(\xi)$ is defined by the inequality

$$|\sigma^{(i)} + \beta_i\ N_{i-1,r}^{(r-i)/(r-i+1)}\, \Psi_{i-1,r}| \le 4\xi\, N_{i-1,r}^{(r-i)/(r-i+1)}\,.$$

The further proof uses Lemma 2 to estimate $\dot\Phi_+$ Since $N_{0,r} = |\sigma|$, $\varphi_{0,r} = \sigma$, Lemma 4 is replaced by the next simple Lemma with $i = 0$.

Lemma 5. *The inequality* $|\dot\sigma + \beta_1|\sigma|^{(r-1\}/r}\operatorname{sign}\sigma| \le$

$\gamma_1|\sigma|^{(r-1)/r}$ *provides with* $0 \le \gamma_1 < \beta_1$ *for the establishment in finite time and keeping the identity* $\sigma \equiv 0$.

This finishes the proof of the Theorem in the case of the quasi-continuous controller. In the case of the standard controller a homogeneous vicinity of the controller discontinuity set is shown to attract the trajectories in finite-time. ■

4. ADJUSTMENT OF THE PARAMETERS

Consider the problem (1), (4). Then the equality (3) implies the differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u. \qquad (6)$$

The problem is solved now building a bounded feedback control in the form

$$u = \alpha\Psi(\sigma, \dot\sigma, ..., \sigma^{(r-1)}), \qquad (7)$$

providing for the finite-time stability of the closed inclusion (6), (7).

Inclusion (6), (7) and the controller (7) are called further r-sliding homogeneous, if for any $\kappa > 0$ the combined time-coordinate transformation

$$G_\kappa: (t, \Sigma) \mapsto (\kappa t, d_\kappa \Sigma) \qquad (8)$$

where $\Sigma = (\sigma, \dot\sigma, ..., \sigma^{(r-1)})$, $d_\kappa \Sigma = (\kappa^r\sigma, \kappa^{r-1}\dot\sigma, ..., \kappa\sigma^{(r-1)})$, preserves the closed-loop inclusion (6), (7) and its solutions.

It is easy to check that (7) is r-sliding homogeneous, iff

$$\Psi(\kappa^r\sigma, \kappa^{r-1}\dot\sigma, ..., \kappa\sigma^{(r-1)}) = \Psi(\sigma, \dot\sigma, ..., \sigma^{(r-1)}).$$

Almost all known HOSM controllers are r-sliding homogeneous. Note that though the sub-optimal controller (Bartolini et al. 2003) does not exactly satisfy the described feedback form (7), it is invariant with respect to (8) with $r = 2$ and is considered here as 2-sliding homogeneous.

Denote by $T_{max}(s_1, s_2, ..., s_r)$ and $T_{min}(s_1, s_2, ..., s_r)$ the maximal and the minimal convergence times of the solutions of (6), (7) with initial conditions $\sigma = s_1$, $\dot\sigma = s_2$, ..., $\sigma^{(r-1)} = s_r$ to the origin $\sigma = \dot\sigma = ... = \sigma^{(r-1)} = 0$. It is easy to see that these functions are well defined (Filippov 1988) and continuous when r-sliding homogeneous controllers are applied. They are also homogeneous in that case with the homogeneity degree 1 (Levant 2005).

Let $\lambda > 0$. Consider the differential inclusion

$$\sigma^{(r)} \in \lambda^r[-C, C] + [K_m, K_M]u. \qquad (9)$$

and the controller

$$u = \lambda^r\alpha\Psi(\sigma, \dot\sigma/\lambda, ..., \sigma^{(r-1)}/\lambda^{r-1}).1 \qquad (10)$$

Denote by Ω_R and $\overline{\Omega}_R$ the sets $|\sigma|^{1/r} + |\dot\sigma|^{1/(r-1)} + ... + |\sigma^{(r-1)}| \le R$ and $|\sigma|^{1/r} + ... + |\sigma^{(r-1)}| \ge R$, and let $\tilde T_{\max}(\Sigma)$ and $\tilde T_{\min}(\Sigma)$ be the convergence-time functions for controller (10).

Proposition 1. *Let the differential inclusion* (6), (7) *be finite time stable and r-sliding homogeneous, then also* (9), (10) *is finite time stable and*

$$\max\{\tilde T_{\max}(\Sigma)|\Sigma \in \Omega_R\} \le \tfrac{1}{\lambda}\max\{T_{max}(\Sigma)|\Sigma \in \Omega_R\}, \quad (11)$$

$$\min\{\tilde T_{\min}(\Sigma)|\Sigma \in \overline{\Omega}_R\} \ge \tfrac{1}{\lambda}\min\{T_{min}(\Sigma)|\Sigma \in \overline{\Omega}_R\} \quad (12)$$

hold with $\lambda > 1$ *and* $\lambda < 1$ *respectively*.

Proof. Apply the time transformation $t = \lambda\tau$. Then $d/dt = \frac{1}{\lambda}\, d/d\tau$ and in the new time the closed loop inclusion takes the form (9), (10). Obviously,

$$T_{max}(\sigma, \dot\sigma, ..., \sigma^{(r-1)}) = \lambda\, \tilde T_{\max}(\sigma, \dot\sigma/\lambda, ..., \sigma^{(r-1)}/\lambda^{r-1}),$$
$$T_{min}(\sigma, \dot\sigma, ..., \sigma^{(r-1)}) = \lambda\, \tilde T_{\min}(\sigma, \dot\sigma/\lambda, ..., \sigma^{(r-1)}/\lambda^{r-1}).$$

The Proposition follows now from the fact that with $\lambda > 1$ the point $(\sigma, \dot\sigma/\lambda, ..., \sigma^{(r-1)}/\lambda^{r-1})$ belongs to Ω_R, while with $\lambda < 1$ it belongs to $\overline{\Omega}_R$. Due to the homogeneity, the minimum of T_{min} in $\overline{\Omega}_R$ exists and takes place on the set $|\sigma|^{1/r} + ... + |\sigma^{(r-1)}| = R$. ■

Note that with $\lambda > 1$ inequality (11) holds also for the inclusion (6), (10), which means that controller (10) provides for the convergence acceleration. In the case when C = 0, pure acceleration or slow down of the convergence occurs.

Obviously, if (7) is a gain-function-robust r-sliding homogeneous controller, then also (10) is gain-function robust and r-sliding homogeneous. In the special case of the quasi-continuous controller, (10) has the same form as the original controller (7).

Proposition 2. *The above-defined quasi-continuos controller preserves its form after the transformation* (10) *with* $r > 1$. *Its new parameters take on the values*

$$\tilde\beta_1 = \lambda\beta_1,\ \tilde\beta_2 = \lambda^{r/(r-1)}\beta_2,\ ...,\ \tilde\beta_{r-1} = \lambda^{r/2}\beta_{r-1},\ \tilde\alpha = \lambda^r\alpha.$$

Note that this controller is gain-function robust, which means that each λ produces a new valid combination of β_i effective for any SISO system with the given relative degree r, provided a sufficiently large gain function is taken. Following are the resulting quasi-continuous controllers with $r \le 4$, simulation-tested β_i and a general gain function Φ:

1. $u = -\alpha\,\Phi\,\mathrm{sign}\,\sigma$,
2. $u = -\alpha\,\Phi\,(|\dot\sigma| + \lambda|\sigma|^{1/2})^{-1}(\dot\sigma + \lambda|\sigma|^{1/2}\mathrm{sign}\,\sigma)$,
3. $u = -\alpha\Phi\,[\ddot\sigma + 2\lambda^{3/2}(|\dot\sigma| + \lambda|\sigma|^{2/3})^{-1/2}(\dot\sigma + \lambda|\sigma|^{2/3}\mathrm{sign}\,\sigma)] / [|\ddot\sigma| + 2\lambda^{3/2}(|\dot\sigma| + \lambda|\sigma|^{2/3})^{-1/2}|\dot\sigma + \lambda|\sigma|^{2/3}\mathrm{sign}\,\sigma|]$,

4. $$\varphi_{3,4} = \ddot\sigma + 3\lambda^2[|\ddot\sigma| + \lambda^{4/3}(|\dot\sigma| + 0.5\lambda\,|\sigma|^{3/4})^{-1/3}\,|\dot\sigma + 0.5\lambda\,|\sigma|^{3/4}\operatorname{sign}\sigma|]^{-1/2}\,[\ddot\sigma + \lambda^{4/3}(|\dot\sigma| + 0.5\lambda\,|\sigma|^{3/4})^{-1/3}\,(\dot\sigma + 0.5\lambda\,|\sigma|^{3/4}\operatorname{sign}\sigma)],$$

$$N_{3,4} = |\ddot\sigma| + 3\lambda^2[|\ddot\sigma| + \lambda^{4/3}(|\dot\sigma| + 0.5\lambda\,|\sigma|^{3/4})^{-1/3}\,|\dot\sigma + 0.5\lambda\,|\sigma|^{3/4}\operatorname{sign}\sigma|]^{-1/2}\,|\ddot\sigma + \lambda^{4/3}(|\dot\sigma| + 0.5\lambda\,|\sigma|^{3/4})^{-1/3}\,(\dot\sigma + 0.5\lambda\,|\sigma|^{3/4}\operatorname{sign}\sigma)|,$$

$$u = -\alpha\Phi\,\varphi_{3,4} / N_{3,4}.$$

As follows from Proposition 2 one needs a valid basic set of parameters to produce sets featuring different convergence rate with respect to λ. The larger λ the faster the convergence.

It is easy to show that each gain-function robust set of parameters providing for the convergence of the solutions of the differential equation $\sigma^{(r)} = u$ to $\Sigma = 0$ can be used as such a basic set. The inverse is obvious.

5. SIMULATION EXAMPLE

Consider a model example

$$\ddot x_1 = \cos 10t\,(e^{x_2} + x_1^2 \sin \dot x_1) + (2+\sin t)(x_2^2+1)\,\Phi\, u;$$
$$\dot x_2 = x_1 - x_2 + \cos t,$$

where Φ is the gain function to be specified further. Here x_1 is the output which has to track the function

$$x_{1c} = 0.08 \sin t + 0.12 \cos 0.3t.$$

Respectively, $\sigma = x_1 - x_{1c}$ is taken. The standard 3-sliding controller has the form

$$u = -\alpha\Phi\, \operatorname{sign}(\ddot\sigma + 2(|\dot\sigma|^3 + |\sigma|^2)^{1/6} \operatorname{sign}(\dot\sigma + |\sigma|^{2/3}\operatorname{sign}\sigma)),$$

also the 3-sliding quasi-continuous controller was applied listed in Section 4. The initial conditions $x_1 = 6$, $\dot x_1 = 1$, $\ddot x_1 = 15$, $x_2 = 10$ were taken at $t = 0$. The gain functions

$$\Phi = (e^{x_2} + x_1^2)/(x_2^2+1) + 1 \qquad (13)$$

and

$$\Phi = e^{x_2} + x_1^2 + 1 \qquad (14)$$

were considered. In all the cases $\alpha = 5$ is taken. The integration was carried out according to the Euler method (the only integration method possible with discontinuous dynamics) with the integration step 10^{-5}.

The both considered controllers were applied with the listed gain functions. With the smaller gain-function (13) the controllers demonstrate their standard transient features (Fig. 1 and Fig. 2, $\lambda = 1$). With the redundantly large gain-function (14) some large but quickly decreasing chattering of the control and $\ddot\sigma$ arises. One cannot distinguish between the joint graphs of σ, $\dot\sigma$, $\ddot\sigma$ for the both controllers in that case. It is interesting to mark that the graphs of σ and $\dot\sigma$ do not change drastically. Also the transient time does not change (Fig. 1). This is explained by the common dynamics in the "configuration" space σ, $\dot\sigma$ (see the proof of Theorem 1).

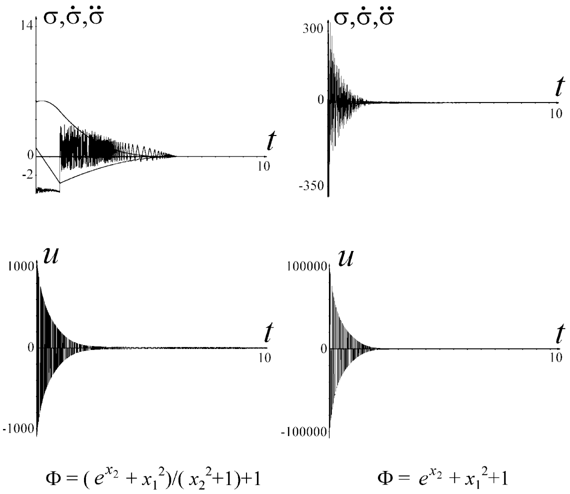

Fig. 1: Standard 3-sliding controller with different gain functions

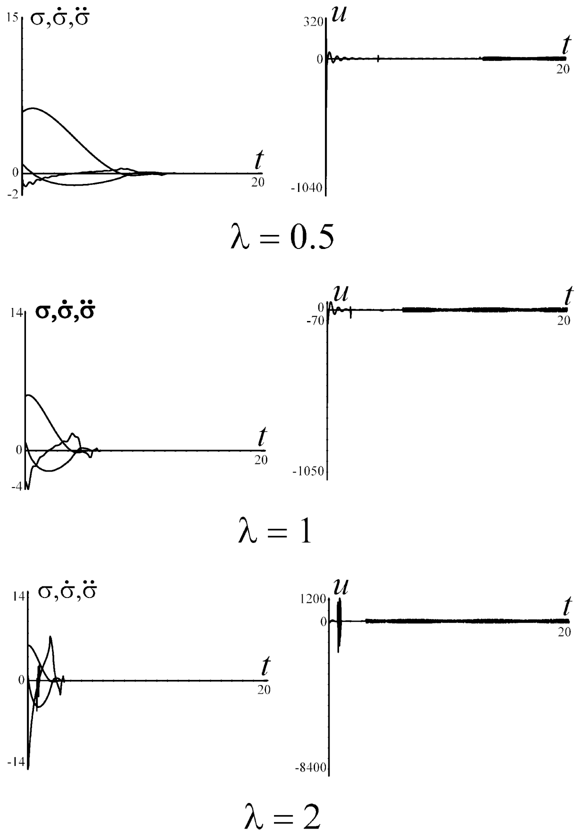

Fig. 2: Adjustment of the quasi-continuous controller with the gain function $\Phi = (e^{x_2} + x_1^2)/(x_2^2+1)+1$

It is seen in Fig. 2 that the control magnitude drops instantly from very large values. After the sliding mode is established, i.e. the trajectory approaches the control discontinuity set $\sigma = \dot{\sigma} = \ddot{\sigma} = 0$, the character sliding-mode control chattering arises with the magnitude $\alpha\Phi(t, x(t))$.

In all the cases almost the same sliding accuracy is obtained $|\sigma| \le 2\cdot10^{-12}$, $|\dot{\sigma}| \le 3\cdot10^{-8}$, $|\ddot{\sigma}| \le 1.\cdot10^{-3}$ for the standard controller and $|\sigma| \le 6\cdot10^{-13}$, $|\dot{\sigma}| \le 2\cdot10^{-8}$, $|\ddot{\sigma}| \le 8\cdot10^{-4}$ for the quasi-continuous controller. After the integration step was changed to 10^{-6} the accuracy of the standard controller changed to $|\sigma| \le 2\cdot10^{-15}$, $|\dot{\sigma}| \le 4\cdot10^{-10}$, $|\ddot{\sigma}| \le 1.\cdot10^{-4}$ which corresponds to the classical 3-sliding accuracy.

The parametric adjustment is demonstrated for the quasi-continuous controller. It is seen that with $\lambda = 0.5$ the transient is 2 times longer, while with $\lambda = 2$ it is 2 times shorter. In the latter case, with respect to (10), also α was changed to the value $2\cdot5 = 10$.

7. CONCLUSIONS

Two long lasted problems of the high-order sliding mode control are solved in this paper. It is shown that the both main types of HOSM controllers allow functional gains of very general form, providing for the suppression of unbounded uncertainties. The relative degree can be artificially increased, producing arbitrarily smooth control and removing the chattering effect.

The convergence rate is not much influenced by the large gain determining the control magnitude. It is defined mostly by the other controller parameters. In their turn those parameters can be adjusted providing for the faster or slower convergence (Propositions 1, 2). Thus, having one valid parameter set, one obtains a whole family of parameter sets with different convergence rates.

The main method of building such basic parameters' sets remains the computer simulation. It is sufficient to carry out such simulation for the simplest equation $\sigma^{(r)} = u$.

A list of quasi-continuous controllers is presented in Section 4 with relative degrees less or equal 4 and simulation tested gain-function-robust parameters. Since in the most practically important problems of output control the relative degree r does not exceed 4, this list constitutes a base for easy application of higher order sliding mode controllers.

Arbitrary-order real-time exact differentiation is known to provide for the output-feedback control of the SISO systems with *bounded* uncertainties (Levant, 2003a). Unfortunately, its application needs the boundedness of $\sigma^{(r)}$, which is not true in the considered case. The development of differentiators with a known functional bound is a challenge for the future.

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