MODELLING AND STABILISATION OF A SPHERICAL INVERTED PENDULUM

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Abstract: We design a nonlinear control law for a four degree of freedom spherical inverted pendulum based on the forwarding technique. We first explore the forwarding structure of the spherical inverted pendulum model and then find a control law to stabilize the angle variables. Next, we develop a nested saturating controller for the whole system. The control law is evaluated through simulations.

Keywords: a spherical inverted pendulum, forwarding, nonlinear controller design

1. INTRODUCTION

The spherical inverted pendulum is a cylindrical beam attached to a horizontal plane via a universal joint (see Figure 1). The pendulum's universal joint is free to move in the horizontal plane, under the influence of a planar force. Gravity acts on the beam making the downward position the naturally stable position. The control objective considered here is to use the planar force to drive the inverted pendulum in such a way that the upright position is stable and attractive with a large domain of attraction, e.g., the upper half hemisphere. Moreover, the pendulum's universal joint has to be returned to a given point on the plane and remain there. Ignoring the spin around the symmetry axis of the cylindrical beam, the spherical pendulum has four degrees of freedom, the position of the universal joint in the plane (x, y) and the angle position, azimuth and elevation (ϕ, θ) of the beam.

Such a device is an abstraction for a rocket propelled body, and is also of interest in robotics.

From a control design point of view, the spherical inverted pendulum is a complex system because it is nonlinear and unstable. The control of a spherical inverted pendulum is considered in (Albouy and Praly, 2000; Yang *et al.*, 2000; Chung *et al.*, 2000; Bloch *et al.*, 2000; Bloch *et al.*, 2001). In (Albouy and Praly, 2000), a swing-up strategy is proposed based on passivity. Stabilizing the pendulum locally around an operating point is discussed in (Yang *et al.*, 2000; Chung *et al.*, 2000), where the pendulum is analyzed under the assumption of small deviations from the vertical upright position. We find two continuous controllers in the literature that attempt to achieve nonlocal stabilization of the pendulum (Bloch *et al.*, 2000; Bloch *et al.*, 2001). In (Bloch *et al.*, 2000; Bloch *et al.*, 2001), the authors use the controlled Lagrangian framework to derive a controller to regulate the angles for a spherical pendulum. However, to the best of our knowledge, no complete solution for the stabilization/regulation

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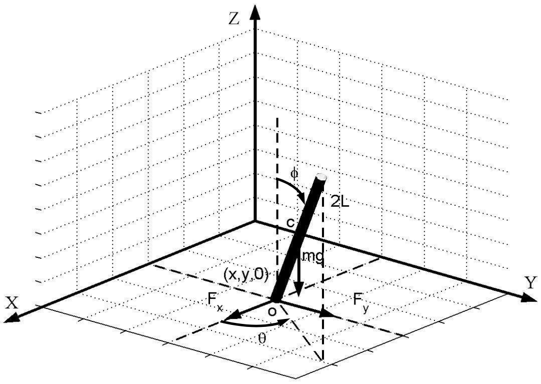

Fig. 1. The spherical inverted pendulum

of all four degrees of freedom of a spherical inverted pendulum has appeared in the literature.

In this paper, we develop a complete controller using both the forwarding technique as proposed in (Teel, 1996) such that the controlled pendulum is stabilized about the upper (unstable) equilibrium. The proposed controller brings the pendulum from any initial condition in the upper hemisphere ($\phi \in (-\pi/2, +\pi/2)$ in Figure 1) to the preferred upright position.

The paper is organized as follows. Next, we recall some results from nonlinear control theories. In section 3, we introduce the model in a from that allows us to appeal to the forwarding design methodology. Then, we complete the control design and present some simulations before offering some final observation.

2. PRELIMINARIES

A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. If $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$, the function is said to belong to class $\mathcal{K}_\infty$. C^- denotes the left hand side of the complex plane. We use the concept of "asymptotic gain" (see (Teel, 1996; Isidori, 1999)), which considers only bounds on the asymptotic behavior of the response, as $t \to \infty$. For a piecewise-continuous function $u : [0, \infty) \to R^m$, define $\|u(\cdot)\|_a = \limsup_{t\to\infty}\{\max_{1\le i\le m} |u_i(t)|\}$. The quantity thus introduced is referred to as the asymptotic "norm" of $u(\cdot)$.

The initial conditions are defined as $x_1^\circ \triangleq x_1(0) \in X_1$ and $(x_1, x_2) \triangleq (x_1^T, x_2^T)^T$ is used for convenience. A saturation function is $\sigma(s) \triangleq \begin{cases} sgn(s), & |s| > 1 \\ s, & |s| \le 1 \end{cases}$ where sgn is the sign function.

The next result is a key design tool in forwarding.

Theorem 2.1. (Teel, 1996; Isidori, 1999) Consider the system

$$\dot{z} = \mathbf{A}x + g_i(\xi_i, u), \quad \dot{\xi}_i = f_i(\xi_i, u) \tag{1}$$

in which $z \in \mathrm{R}^n$, $\xi_i \in \mathrm{R}^v$, $u \in \mathrm{R}^m$, $g_i(\xi_i, u)$ and $f_i(\xi_i, u)$ are locally Lipschitz, differentiable at $(\xi_i, u) = (0, 0)$, and $g_i(0, 0) = 0$, $f_i(0, 0) = 0$. Assume that: (i) there exists a symmetric matrix $P > 0$ such that $P\mathbf{A} + \mathbf{A}^T P \le 0$,
(ii) the linear approximation of the system at the equilibrium $(z_i, \xi_i, u) = (0, 0, 0)$ is stabilizable.
Moreover, assume that there exists a function $\begin{array}{l} \alpha_i : R^v \times R^m \to R^m \\ \quad (\xi_i, v) \mapsto \alpha_i(\xi_i, v) \end{array}$, with $\alpha_i(0, 0) = 0$, which is locally Lipschitz, differentiable at $(\xi_i, v) = (0, 0)$, with the following properties:
(iiia) the matrix $\left[\frac{\partial \alpha_i(\xi_i,v)}{\partial v}\right]_{(0,0)}$ is nonsingular,
(iiib) the matrix $\left[\frac{\partial f_i(\xi_i,\alpha_i(\xi_i,v))}{\partial \xi_i}\right]_{(0,0)}$ has all eigenvalues in C^-,
(iiic) the system $\dot{\xi}_i = f_i(\xi_i, \alpha_i(\xi_i, v))$, $y = \xi_i$ satisfies an asymptotic (input v to output y) bound , with restriction ξ_i on ξ_i°, restriction $V > 0$ on $v(\cdot)$, with linear gain function $\gamma_v(\cdot)$. Set $\xi_{i+1} = (z, \xi_i)$, $\tilde{v} = n + v$, $f_{i+1}(\xi_{i+1}, u) = \begin{pmatrix} Az + g_i(\xi_i, u) \\ f_i(\xi_i, u) \end{pmatrix}$,
$F_{i+1} = \left[\frac{\partial f_{i+1}(\xi_{i+1},\alpha_i(\xi_i,v))}{\partial \xi_{i+1}}\right]_{(0,0)}$,
$G_{i+1} = \left[\frac{\partial f_{i+1}(\xi_{i+1},\alpha_i(\xi_i,v))}{\partial v}\right]_{(0,0)}$. Then, the pair (F_{i+1}, G_{i+1}) is stabilisable. Let $\sigma(\cdot)$ be any R^m -valued saturation function. Pick a $\tilde{v} \times m$ matrix K_{i+1} such that $(F_{i+1} + G_{i+1}K_{i+1})$ has all eigenvalues in C^- and, for some $\delta' > 0$, system $\dot{x} = F_{i+1}x + G_{i+1}\sigma(K_{i+1}x + v) + w$, $y = x$ satisfies an asymptotic (input (v,w) to output y) bound, with no restriction on x° and restriction δ' on $v(\cdot)$ and $w(\cdot)$, with linear gain functions $\gamma_v(\cdot)$ and $\gamma_w(\cdot)$. Pick two $m \times m$ matrices Γ and Ω such that $\Gamma + \Omega$ is nonsingular. Consider the function
$$\begin{array}{l} \alpha_{i+1} : R^{\tilde{v}} \times R^m \to R^m \\ \quad (\xi_i, v) \mapsto \alpha_i\left(\xi_i, \lambda\sigma\left(\dfrac{K_{i+1}\xi_{i+1} + \Gamma v}{\lambda}\right) + \Omega v\right) \end{array}$$
Then, there exist numbers $\lambda > 0$ and $\tilde{v} > 0$ such that
(a) the matrix $\left[\frac{\partial \alpha_{i+1}(\xi_{i+1},v)}{\partial v}\right]_{(0,0)}$ is nonsingular,
(b) the matrix $\left[\frac{\partial f_{i+1}(\xi_{i+1},\alpha_{i+1}(\xi_{i+1},v))}{\partial \xi_{i+1}}\right]_{(0,0)}$ has all eigenvalues in C^-,
(c) the system $\dot{\xi}_{i+1} = f_{i+1}(\xi_{i+1}, \alpha_{i+1}(\xi_{i+1}, v))$, $y = \xi_{i+1}$ satisfies an asymptotic (input v - output y) bound, with restriction $\xi_{i+1} = R^n \times X_i$ on ξ_{i+1}°, restriction $\tilde{V} > 0$ on $v(\cdot)$, with linear gain function $\gamma_v(\cdot)$.

This result can be repeatedly used to globally asymptotically stabilize a system in the so called forwarding form.

3. THE DYNAMIC MODEL IN FORWARDING FORM

The setting of the pendulum is outlined in Figure 1 and Table 1. Using the Euler-Lagrange's equations for modelling of mechanical systems (Hand, 1998, pg.19)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\{\dot{q}_i\}}\right) - \frac{\partial \mathcal{L}}{\partial\{q_i\}} = \{Q_i\}, \quad i = 1, \ldots, n, \tag{2}$$

we obtain equations of motion for our system

$$D(q) \cdot \{\ddot{q}_i\} + C(q, \dot{q}) \cdot \{\dot{q}_i\} + G(q) = \{Q_i\}, \tag{3}$$

where the entries of matrix D, C, G and $\{Q_i\}$ are given in Appendix. By multiplying the inverse of inertial matrix D^{-1} to two sides of the dynamic equation (3), we rewrite equations of dynamics

$$\{\ddot{q}_i\} = D^{-1}(q) \cdot (\{Q_i\} - C(q, \dot{q}) \cdot \{\dot{q}_i\} - g(q)) \,. \tag{4}$$

Table 1. Nomenclature for the spherical inverted pendulum

Name	Symbol	Unit	Simulation Value
Generalized Coord.	(x, y, ϕ, θ)	m or rad	-
Shape Variables	$(\phi,\ \theta)$	rad	-
External Variables	(x,y)	m	-
The Length	$2 \times L$	m	0.6
The Radius	R	m	0.02
The Mass	m	kg	0.35
Gravity	g	m/s^2	9.8
Actuation Forces	F_x, F_y	N	-
Viscous Fric. Coef.	$C_{x,y}$	$N \cdot s/m$	1×10^{-4}
Viscous Fric. Coef.	C_ϕ,C_θ^2	$N \cdot s/rad$	1×10^{-4}
Torque Fric. Coef.	C_θ^1	N/rad	1×10^{-4}

We identify an upper triangular structure for the dynamics of the pendulum (4) that is suitable for controller design based on forwarding techniques. Let $\xi_{11} \triangleq (\theta, \dot{\theta})$, $\xi_{12} \triangleq (\phi, \dot{\phi})$, $\xi_1 \triangleq (\xi_{11}, \xi_{12})$ $z_1 \triangleq \dot{x}$, $z_2 \triangleq x$, $z_3 \triangleq \dot{y}$, and $z_4 \triangleq y$ be the states and $u \triangleq (F_x, F_y)$ be the input. We write the dynamics (4) in a forwarding form:

$$\dot{z}_i = A_i z_i + g_i(\xi_i, u), \quad \dot{\xi}_i = f_i(\xi_i, u), \tag{5}$$

where $i = 1, \ldots, 4$, $A_i = 0$, $g_2(\xi_1, u)= z_1$, $g_4(\xi_4, u)= z_3$ and the explicit expression of $g_1(\xi_1, u)$, $g_3(\xi_3, u)$, $f_{12}(\xi_{11}, \xi_{12}, u)$ and $f_{11}(\xi_{11}, \xi_{12}, u)$ are obtained from (4) but are omitted for space reasons.

To simplify the design of ξ_1 subsystem, the relationship between $(F_x$,$F_y)$ and $(\ddot{x}$,$\ddot{y})$ can be inverted. To this end, we let $\ddot{x} \triangleq a_x$, $\ddot{y} \triangleq a_x$ and define

$$\begin{bmatrix} u_\phi \\ u_\theta \end{bmatrix} \triangleq \frac{1}{L} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix}. \tag{6}$$

4. NONLINEAR CONTROLLER DESIGN

The forwarding controller for the planar pendulum proposed in (Angeli, 2001; Teel, 1996) can be extended to the spherical inverted pendulum. Relying on this idea, we formulate a full forwarding controller for the spherical inverted pendulum.

With this triangular structure, we can use the forwarding technique to design a stabilizing controller in the following manner:

Step 1 Derive a controller for the $\dot{\xi}_{11}$ subsystem where $\xi_{11} \triangleq (\theta, \dot{\theta}) \in R^2$ as a domain of attraction;

Step 2 Derive a controller for the $\dot{\xi}_{12}$ subsystem where $\xi_{12} \triangleq (\phi$,$\dot{\phi}) \in (-\frac{\pi}{2}, +\frac{\pi}{2}) \times R$ as a domain of attraction by using the results of Teel (Teel, 1996). Step 1 together with Step 2 stabilize subsystem $\dot{\xi}_1 = f_1(\xi_1, u)$;

Step 3-6 Use Theorem 2.1 repeatedly to design a controller for the augmented subsystem $\xi_{i+1} \triangleq (z_i, \xi_i)$ where $D = \{\xi_{i+1} \in R^{i+4} | \phi \in (-\frac{\pi}{2}, +\frac{\pi}{2}) \times R^{i+3}\}$ as a domain of attraction.

After going through Step 1-6, we obtain a nested saturation controller for the spherical inverted pendulum. In what follows, we will describe how we design the controller step by step.

4.1 Design Step 1

In our formulation of the controller, we regulate both θ and $\dot{\theta}$ to the origin in order to apply the forwarding method [2] and we also introduce frictions to assist stabilizing the pendulum about upper (unstable) equilibrium. On condition that $\dot{\theta}$ and θ converge to zero, the problem is recast as the one of stabilizing a planar inverted pendulum with the goal of making $\phi = 0$.

Taking

$$u_\theta = \frac{4}{3} \sin\phi (k_1 \theta + k_2 \dot{\theta}) - \left(\frac{8}{3} - \frac{R^2}{2L^2}\right) \dot{\phi} \dot{\theta} \cos\phi . \tag{7}$$

The controller (7) is explained as follows,

- The first term in (7) render $\sin(\phi)$ in the closed loop "unimportant" (when $R = 0$, the first term of the closed loop is independent of $\sin\phi$);
- The second term in (7) eliminates Coriolis and reduces the coupling between ϕ and θ dynamics,

[2] This means that there is a preferred orientation for the body. From a control application point of view this may or may not be important. Our explicit assumption is that the orientation is important.

which yields the closed loop dynamics

$$\dot{\xi}_{11} = A_{\xi_{11}}(\phi(t))\xi_{11} \stackrel{\triangle}{=} f_{11}, \tag{8}$$

where

$$A_{\xi_{11}}(\phi(t)) = \begin{pmatrix} 0 & 1 \\ -\psi^1(\phi(t))k_1 - \psi^2(\phi(t))C_\theta^1 & -\psi^1(\phi(t))k_2 - \psi^2(\phi(t))C_\theta^2 \end{pmatrix}$$

with $\psi^1(\phi(t)) = \frac{\frac{4}{3}L^2\sin^2\phi(t)}{\frac{4}{3}L^2\sin^2\phi(t)+\frac{1}{4}R^2(\cos^2\phi(t)+1)}$, $\psi^2(\phi(t)) = \frac{1}{m(\frac{4}{3}L^2\sin^2\phi(t)+\frac{1}{4}R^2(\cos^2\phi(t)+1))}$. It is clear that $1 > \psi^1(\phi(t)) \geq 0$ and $\frac{4}{mR^2} > \psi^2(\phi(t)) > \frac{12}{m(4L^2+3R^2)}$ hold for any $\phi(t)$. The next result concludes that the closed loop ξ_{11} subsystem is globally asymptotically stable under some mild and natural conditions.

Lemma 4.1. The closed loop subsystem (8), with bounded functions $\psi^1(\phi(t))$ and $\psi^2(\phi(t))$ such that $1 > \psi^1(\phi(t)) \geq 0$ and $\frac{4}{mR^2} > \psi^2(\phi(t)) > \frac{12}{m(4L^2+3R^2)}$ hold for any $\phi(t)$, is globally exponentially stable if k_1, k_2, C_θ^1 and C_θ^2 are positive.

Proof. Provided $k_1,\ k_2,\ C_\theta^1,\ C_\theta^2 > 0$, $-k_1 - \frac{4C_\theta^1}{mR^2} < -\psi^1(\phi(t))k_1 - \psi^2(\phi(t))C_\theta^1 \leq -|\tilde{M}| < -\frac{12C_\theta^1}{m(4L^2+3R^2)}$ and $-k_2 - \frac{4C_\theta^2}{mR^2} < -\psi^1(\phi(t))k_2 - \psi^2(\phi(t))C_\theta^2 \leq -|\tilde{N}| < -\frac{12C_\theta^2}{m(4L^2+3R^2)}$ hold for any $\phi(t)$. Because of $R << L$, the upper bounds $-|\tilde{M}|$, $-|\tilde{N}|$ are approximately $-\frac{2C_\theta^1}{mR^2}$ and $-\frac{2C_\theta^2}{mR^2}$ respectively.

It is natural that the friction coefficients C_θ^1, $C_\theta^2 > 0$ hold. Given k_1, $k_2 > 0$, it is easy to find a symmetric positive definite matrix $P_{\xi_{11}} = P_{\xi_{11}}^T > 0$, which is independent of t, and constants α_1, α_2, $\alpha_3 > 0$ such that the Lyapunov function candidate

$$V_{\xi_{11}} = \xi_{11}^T P_{\xi_{11}} \xi_{11}, \tag{9}$$

satisfies

$$\alpha_1|\xi_{11}|^2 \leq \xi_{11}^T P_{\xi_{11}} \xi_{11} \leq \alpha_2|\xi_{11}|^2, \tag{10}$$

and

$$\frac{\partial V_{\xi_{11}}(\cdot)}{\partial \xi_{11}} f_{11} = \xi_{11}^T \left[(A_{\xi_{11}}(\cdot))^T P_{\xi_{11}} + P_{\xi_{11}} (A_{\xi_{11}}(\cdot))\right]\xi_{11} \leq -\alpha_3|\xi_{11}|^2, \tag{11}$$

for any $\phi(t)$.

Therefore, the closed loop ξ_{11} subsystem is globally exponentially stable (referring to (Liu, 2004) for details). ◁

4.2 Design Step 2

We now stabilize the ξ_{12} subsystem. Setting

$$\tilde{u}_\phi = L^2 u_\phi - \left(\frac{4}{3}L^2 - \frac{1}{4}R^2\right)\sin\phi\dot{\theta}^2, \tag{12}$$

yields closed loop dynamics of $\dot{\xi}_{12} = f_{12}(\xi_{11}, \xi_{12}, u)$,

$$\dot{\xi}_{12} = \begin{pmatrix} \dot{\phi} \\ a\sin\phi - b\cos\phi\tilde{u}_\phi - \dfrac{b}{m}C_\phi\dot{\phi} \end{pmatrix}, \tag{13}$$

where $a = \frac{gL}{\frac{4}{3}L^2+\frac{1}{4}R^2}$, $b = \frac{1}{\frac{4}{3}L^2+\frac{1}{4}R^2}$.

The equation (13) coincides with the corresponding equation in the forwarding structure of the planar inverted pendulum (Angeli, 2001; Teel, 1996).

In our design, we develop a controller for the ξ_{12} subsystem by slightly modifying the controller proposed in (Teel, 1996). We choose the input signal

$$\tilde{u}_\phi = \frac{1}{b\cos\phi}\left(a\sin\phi + c_1\tan\phi + c_2\dot{\phi}\right), \tag{14}$$

where $c_1 > 0$, $c_2 > 0$ (we add c_1 and replace $\sigma(\dot{\phi})$ by $\dot{\phi}$ in the result of (Teel, 1996)). Observe that this control is smooth on the set $(\phi, \dot{\phi}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R \stackrel{\triangle}{=} \chi_2$. Then, on this set, the closed loop dynamics are governed by

$$\dot{\xi}_{12} = \begin{pmatrix} \dot{\phi} \\ -c_1\tan\phi - c_2\dot{\phi} - \dfrac{b}{m}C_\phi\dot{\phi} \end{pmatrix} \stackrel{\triangle}{=} f_{12}. \tag{15}$$

Now consider the Lyapunov function candidate

$$\begin{aligned} V_{\xi_{12}} &= c_1\int_0^\phi \tan(s)ds + \frac{1}{2}\dot{\phi}^2 \\ &= c_1 \ln|\cos^{-1}\phi| + \frac{1}{2}\dot{\phi}^2. \end{aligned} \tag{16}$$

Notice that $V_{\xi_{12}} \to \infty$ as $|\phi| \to \frac{\pi}{2}$ or as $|\dot{\phi}| \to \infty$. Taking the derivative of this function along the vector field in (15) gives

$$\begin{aligned} \frac{\partial V_{\xi_{12}}}{\partial \xi_{12}} f_{12} &= c_1(\tan\phi)\dot{\phi} + \dot{\phi}\left(-c_1\tan\phi - c_2\dot{\phi} - \frac{bC_\phi}{m}\dot{\phi}\right) = -\frac{bC_\phi}{m}\dot{\phi}^2 - c_2\dot{\phi}^2. \end{aligned} \tag{17}$$

Since $C_\phi > 0$, $c_2 > 0$, (17) is negative semi-definite. We can apply LaSalle's lemma to prove that the origin is asymptotically stable. To find $E = \{\xi_{12} \in \chi_2 | \frac{\partial V_{\xi_{12}}}{\partial \xi_{12}} f_{12} = 0\}$, note that $\frac{\partial V_{\xi_{12}}}{\partial \xi_{12}} f_{12} = 0 \iff \dot{\phi} = 0$. Hence, $E = \{\xi_{12} \in$

$\chi_2|\dot{\phi} = 0\}$. Let $\xi_{12}(t)$ be a solution that belongs to E:

$$\dot{\phi}(t) \equiv 0 \Rightarrow \ddot{\phi}(t) \equiv 0 \Rightarrow \tan(\phi(t)) \equiv 0 \Rightarrow \phi(t) \equiv 0.$$

Therefore, the only solution that can stay identically in E is the trivial solution $\xi_{12}(t) \equiv 0$. Thus, the origin is asymptotically stable with basin of attraction $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R$.

The controller (7) together with the controller (12), which includes (14), induce a unified controller (F_x^1, F_y^1) which ensures the stability of the ξ_1 subsystem.

4.3 Design Step 3-6

By applying Theorem 2.1 repeatedly, we can obtain a nested saturating controller for each augmented system ξ_{i+1}, $i = 1, \ldots, 4$.

The design task is to design some saturation function for v_i, $i = 1, \ldots, 4$ such that $\alpha_i(\xi_i, v_i)$, $i = 1, \ldots, 4$ with the external input $v_i \stackrel{\triangle}{=} (v_x^i,\ v_y^i)$, $i = 1, \ldots, 4$ ensures that the augmented system

$$f_{i+1}(\xi_{i+1}, \alpha_i(\cdot,\cdot)) = \begin{pmatrix} A_i z_i + g_i(\xi_i, \alpha_i(\cdot,\cdot)) \\ f_i(\xi_i, \alpha_i(\cdot,\cdot)) \end{pmatrix}, \tag{18}$$

satisfy an asymptotic input-output bound.

In each step of the recursive design, we must make sure that all conditions in Theorem 2.1 hold. Assumption (i) holds as $A_i = 0$, $i = 1, \ldots, 4$. Notice that the linear approximation of each augmented system at the equilibrium $(z_i, \xi_i, v_i) = (0, 0, 0)$, $i = 1, \ldots, 4$ is stabilizable. Thus, assumption (ii) holds. Assumptions (iiia-c) are automatically satisfied because they are the results of the previous step design. In summary, all conditions are satisfied.

Now, we can apply the Theorem 2.1 to design a complete control law for the pendulum. (F_{i+1}, G_{i+1}), $i = 1, \ldots, 4$ is stabilizable. We employ LQR design for all recursive design steps and obtain the optimal gain matrices K_{i+1}, $i = 1, \ldots, 4$ such that the controller $v_i = K_{i+1}\xi_{i+1}$ minimizes the cost function

$$J(\xi_{i+1}, v_i) = \int_0^\infty (\xi_{i+1}^T Q \xi_{i+1} + v_i^T R v_i) dt, \tag{19}$$

where Q and R are weight matrices. All eigenvalues of $(F_{i+1} + G_{i+1}K_{i+1})$ are in C^-.

Finally, a nested saturating controller for the whole system is obtained

$$u = \begin{pmatrix} F_x^1 \\ F_y^1 \end{pmatrix} + \sigma_{i+1}, \text{ for } i = 1, \ldots, 4, \tag{20}$$

where $\sigma_{i+1} \stackrel{\triangle}{=} \lambda_{i+1}\sigma\left(\frac{1}{\lambda_{i+1}}\left(K_{i+1}\xi_{i+1} + \Gamma_{i+1}v_{i+1}\right)\right)$, $v_{i+1} = \sigma_{i+2}$. The controller yields a domain of attraction, $D = \{(\phi^\circ, \dot{\phi}^\circ,\ \theta^\circ,\ \dot{\theta}^\circ,\ \dot{x}^\circ,\ x^\circ,\ \dot{y}^\circ,\ y^\circ) \in R^8 | -\frac{\pi}{2} \le \phi^\circ \le \frac{\pi}{2}\}$.

5. SIMULATION

The controller is evaluated through simulation. We let $\lambda_2 = 1.5$, $\lambda_3 = 1$, $\lambda_4 = 0.28$, $\lambda_5 = 0.18$, $\Gamma_{i+1} = diag(1, 1)$, $v_5(\cdot) = 0$, $c_1 = 50$ (N/rad), $c_2 = 20$ $(N \cdot s/rad)$, $k_1 = 100$ (N/rad) and $k_2 = 500$ $(N \cdot s/rad)$ for our design. Let the initial output values: $\theta^\circ = 70^\circ$, $\phi^\circ = -40^\circ$, $x^\circ = 1$, $y^\circ = 2$ and all initial velocities are zero. The simulation results are shown in Figure 2. Notice the time scales in the responses. These are due to nested saturation in the design. This indicates that perhaps a faster control may be achieved by alternative methods, and this is under investigation.

6. CONCLUSION

We identified an appropriate upper triangular structure for the dynamics of the spherical inverted pendulum allowing us to use the forwarding method to design a complete nonlinear controller. First, a controller for the shape variables is found and then we design a nested saturating controller for the whole system. The controller has a large domain of attraction. The simulation results illustrate this. Our future work is to analyze the robustness of the controller and extend the controller towards trajectory tracking.

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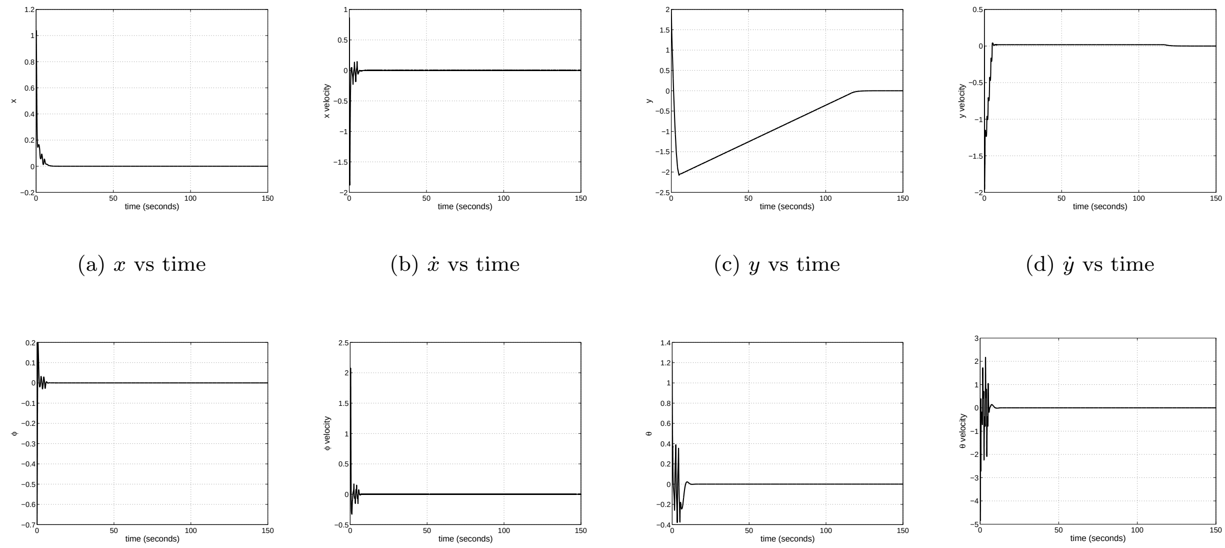

(a) x vs time (b) $\dot{x}$ vs time (c) y vs time (d) $\dot{y}$ vs time

(e) ϕ vs time (f) $\dot{\phi}$ vs time (g) θ vs time (h) $\dot{\theta}$ vs time

Fig. 2. Simulation Results

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Appendix

The inertial matrix is

$$D(q)=\begin{pmatrix} m & 0 & mL\cos\phi\cos\theta & -mL\sin\phi\sin\theta \\ 0 & m & mL\cos\phi\sin\theta & mL\sin\phi\cos\theta \\ mL\cos\phi\cos\theta & mL\cos\phi\sin\theta & \frac{4}{3}mL^2+\frac{1}{4}mR^2 & 0 \\ -mL\sin\phi\sin\theta & mL\sin\phi\cos\theta & 0 & m(\frac{4}{3}L^2\sin^2\phi+\frac{1}{4}R^2(\cos^2\phi+1)) \end{pmatrix}.$$

The Coriolis and centrifugal matrix is

$$C(q,\dot{q})=\begin{pmatrix} 0 & 0 & -mL(\dot{\phi}\sin\phi\cos\theta+\dot{\theta}\cos\phi\sin\theta) & -mL(\dot{\phi}\cos\phi\sin\theta+\dot{\theta}\sin\phi\cos\theta) \\ 0 & 0 & -mL(\dot{\phi}\sin\phi\sin\theta-\dot{\theta}\cos\phi\cos\theta) & mL(\dot{\phi}\cos\phi\cos\theta-\dot{\theta}\sin\phi\sin\theta) \\ 0 & 0 & 0 & (\frac{1}{4}mR^2-\frac{4}{3}mL^2)\dot{\theta}\sin\phi\cos\phi \\ 0 & 0 & (\frac{4}{3}mL^2-\frac{1}{4}mR^2)\dot{\theta}\sin\phi\cos\phi & (\frac{4}{3}mL^2-\frac{1}{4}mR^2)\dot{\phi}\sin\phi\cos\phi \end{pmatrix}.$$

The gravity term and the external forces are

$$G(q)=\begin{pmatrix} 0 \\ 0 \\ -mgL\sin\phi \\ 0 \end{pmatrix}, \quad \{Q_i\}=\begin{pmatrix} F_x-C_x\dot{x} \\ F_y-C_y\dot{y} \\ -C_\phi\dot{\phi} \\ -C_\theta^1\theta-C_\theta^2\dot{\theta} \end{pmatrix}.$$