J-SPECTRAL FACTORIZATION
VIA SIMILARITY TRANSFORMATIONS

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Abstract   This paper characterizes a class of regular para-Hermitian transfer matrices and then studies the J-spectral factorization of this class using similarity transformations. A transfer matrix Λ in this class admits a J-spectral factorization if and only if there exists a common nonsingular matrix to similarly transform the A-matrices of Λ and Λ^{-1}, resp., into 2×2 lower (upper, resp.) triangular block matrices with the $(1,1)$-block including all the stable modes of Λ (Λ^{-1}, resp.). For a transfer matrix in a smaller subset, this nonsingular matrix is formulated in terms of the stabilizing solutions of two algebraic Riccati equations. The J-spectral factor is formulated in terms of the original realization of the transfer matrix. *Copyright$^{©}$ 2005 IFAC.*

Keywords: J-spectral factorization, para-Hermitian transfer matrix, H_∞ control, algebraic Riccati equation (ARE)

1. INTRODUCTION

A *J-spectral factorization* is a factorization of a para-Hermitian matrix into the form of $W^\sim JW$, where W is bistable and J is a signature matrix. It plays an important role in H_∞ control of finite-dimensional systems [Francis, 1987, Green et al., 1990, Green, 1992, Green and Limebeer, 1995] as well as infinite-dimensional systems [Meinsma and Zwart, 2000, Oostveen and Curtain, 1998, Iftime and Zwart, 2002]. The necessary and sufficient condition of the J-spectral factorization has been well understood [Bart et al., 1979, Francis, 1987, Green et al., 1990, Meinsma, 1995, Ran, 2003]. The J-spectral factorizations involved in the literature are done for matrices in the form $G^\sim JG$, mostly with a stable G. For the case with an unstable G, the following three steps can be used to find the J-spectral factor of $G^\sim JG$, by applying the results in [Meinsma, 1995, Corollary 3.1]:

(i) to find the modal factorization of $\Lambda = G^\sim JG$;

(ii) to construct a stable G_- such that $\Lambda = G_-^\sim JG_-$;

(iii) to derive the J-spectral factor of $G_-^\sim JG_-$, i.e., of $G^\sim JG$.

For example, if $\Lambda = G^\sim JG$ (with G unstable) is factorized as $\Lambda = T + T^\sim$ with T stable, then $G_- = \begin{bmatrix} I + \frac{T}{2} \\ I - \frac{T}{2} \end{bmatrix}$ is stable [1] and $\Lambda = G_-^\sim JG_-$. It can then be factorized by applying Theorem 2.4 in [Meinsma, 1995]. However, at some cases, a para-Hermitian transfer matrix Λ is given in the form of a state-space realization and cannot be explicitly written in the form $G^\sim JG$, e.g., in the context of

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[1] For $Z = T + T^\sim$ with a stable T, a stable G_- such that $Z = G_-^\sim JG_-$ is as given above but not $G_- = \frac{1}{2} \begin{bmatrix} I + T \\ I - T \end{bmatrix}$. If the latter is the case, then Z should be factorized as $Z = \frac{1}{2}(T + T^\sim)$.

H_∞ control of time-delay systems [Zhong, 2003a, Meinsma et al., 2002]. In order to use the above-mentioned results, one would have to find a G such that $\Lambda = G^\sim J G$. It would be advantageous if this step could be avoided.

A very recent parallel work in [Ran, 2003] has dealt with this problem. A two-step procedure is proposed to find the J-spectral factor in [Ran, 2003]: (i) to transform Λ into an ordered Schur form; and then (ii) to solve an algebraic Riccati equation (ARE) when there is a stabilizing solution. There is no need to find a stable G such that $\Lambda = G^\sim J G$ any more. The advantage of this result is that the realization of Λ need not be minimal or in the Hamiltonian structure (because of the first step). This paper proposes a different approach to deal with the problem. It only involves very elementary mathematical tools, such as similarity transformations, so that it is easy to understand. The approach developed here has been found crucial to solve the delay-type Nehari problem [Zhong, 2003a,b].

A better literature review about this topic can be found in [Ran, 2003, Green et al., 1990] and the references therein. For a wider topic, the symmetric factorization, see [Ran and Rodman, 1991] and the references therein.

Notation

Given a matrix A, A^* denote the complex conjugate transpose and A^{-*} stands for $(A^{-1})^*$ when the inverse A^{-1} exists. A transfer matrix $G(s) = D + C(sI - A)^{-1}B$ is denoted as $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and its conjugate is defined as $G^\sim(s) = [G(-s^*)]^* = \left[\begin{array}{c|c} -A^* & -C^* \\ \hline B^* & D^* \end{array}\right]$. $J_{p,q} = \left[\begin{array}{cc} I_p & 0 \\ 0 & -I_q \end{array}\right]$ is a signature matrix and is reduced to J when p and q are obvious or irrelevant.

2. REGULAR PARA-HERMITIAN TRANSFER MATRICES

Definition 1. [Kwakernaak, 2000] A transfer matrix $\Lambda(s)$ is called a *para-Hermitian matrix* if $\Lambda^\sim(s) = \Lambda(s)$.

Definition 2. A transfer matrix $W(s)$ is a *J-spectral factor* of $\Lambda(s)$ if $W(s)$ is bistable and $\Lambda(s) = W^\sim(s)JW(s)$. Such a factorization of $\Lambda(s)$ is referred to as a *J-spectral factorization*.

Definition 3. A matrix $W(s)$ is a *J-spectral cofactor* of a matrix $\Lambda(s)$ if $W(s)$ is bistable and $\Lambda(s) = W(s)JW^\sim(s)$. Such a factorization of $\Lambda(s)$ is referred to as a *J-spectral cofactorization*.

Theorem 4. A given square, minimal, rational matrix $\Lambda(s)$, having no poles or zeros on the $j\omega$-axis including ∞, is a para-Hermitian matrix if and only if a minimal realization can be represented as

$$\Lambda = \left[\begin{array}{cc|c} A & R & -B \\ -E & -A^* & C^* \\ \hline C & B^* & D \end{array}\right] \qquad (1)$$

where $D = D^*$, $E = E^*$ and $R = R^*$.

PROOF. *Sufficiency.* It is obvious according to Definition 1.

Necessity. Since Λ is a para-Hermitian matrix, there must be $D = D^*$. By assumption D is invertible, then using similar arguments as in [Francis, 1987, pp.90-91], Λ^{-1} exists and can be minimally realized as

$$\Lambda^{-1} = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & -A_1^* & -C_1^* \\ \hline C_1 & B_1^* & D^{-1} \end{array}\right]$$

where (A_1, B_1, C_1) is a stable minimal realization. Hence,

$$\Lambda = \left[\begin{array}{cc|c} A_1 - B_1 D C_1 & -B_1 D B_1^* & -B_1 \\ C_1^* D C_1 & -(A_1 - B_1 D C_1)^* & C_1^* \\ \hline C_1 & B_1^* & D \end{array}\right]. \qquad (2)$$

This matrix is in the form of (1) where $E = E^* = -C_1^* D C_1$ and $R = R^* = -B_1 D B_1^*$. This completes the proof.

Remark 5. For the realization of Λ in (2), $R = R^*$ is equal to $-B_1 D B_1^*$. However, this is not true for the realization of Λ in (1), where $R = R^*$ is in general not equal to $-BDB^*$. Similar argument applies to $E = E^*$.

Denote $T = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & \frac{1}{2}D^{-1} \end{array}\right]$, then the result proposed in [Meinsma, 1995, Corollary 3.1] can be directly used. However, as explained before, this will result in a J-spectral factor in terms of A_1, B_1, C_1 and D but not in terms of the original realization in A, R, E, B, C and D. This is good enough for numerical computation, but not enough for further analysis, as in the case of [Zhong, 2003a].

In order to simplify later expositions, denote

$$H_p = \left[\begin{array}{cc} A & R \\ -E & -A^* \end{array}\right],$$

$$H_z \doteq \left[\begin{array}{cc} A_z & R_z \\ -E_z & -A_z^* \end{array}\right]$$
$$= \left[\begin{array}{cc} A & R \\ -E & -A^* \end{array}\right] - \left[\begin{array}{c} -B \\ C^* \end{array}\right] D^{-1} \left[\begin{array}{cc} C & B^* \end{array}\right]$$

3. J-SPECTRAL FACTORIZATION

Assume that a para-Hermitian matrix Λ as given in (1) is minimal and has no poles or zeros on the $j\omega$-axis including ∞. There always exist nonsingular matrices Δ_p and Δ_z (e.g. via Schur decomposition) such that [2]

$$\Delta_p^{-1} H_p \Delta_p = \begin{bmatrix} ? & 0 \\ ? & A_+ \end{bmatrix} \qquad (3)$$

and

$$\Delta_z^{-1} H_z \Delta_z = \begin{bmatrix} A_- & ? \\ 0 & ? \end{bmatrix}, \qquad (4)$$

where A_+ is antistable and A_- is stable (A_+ and A_- have the same dimension as A).

Lemma 6. The Λ as described above has a $J_{p,q}$-spectral factorization for some unique $J_{p,q}$ (where p is the number of the positive eigenvalues of D and q is the number of the negative eigenvalues of D) if and only if

$$\Delta = \begin{bmatrix} \Delta_z \begin{bmatrix} I \\ 0 \end{bmatrix} & \Delta_p \begin{bmatrix} 0 \\ I \end{bmatrix} \end{bmatrix} \qquad (5)$$

is nonsingular. If this condition is satisfied, then a $J-$spectral factor is formulated as

$$W = \left[\begin{array}{c|c} \begin{bmatrix} I & 0 \end{bmatrix} \Delta^{-1} H_p \Delta \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} I & 0 \end{bmatrix} \Delta^{-1} \begin{bmatrix} -B \\ C^* \end{bmatrix} \\ \hline J_{p,q} D_W^{-*} \begin{bmatrix} C & B^* \end{bmatrix} \Delta \begin{bmatrix} I \\ 0 \end{bmatrix} & D_W \end{array} \right], (6)$$

where D_W is nonsingular and $D_W^* J_{p,q} D_W = D$.

PROOF. Formulae (3) and (4) mean that

$$H_p \Delta_p \begin{bmatrix} 0 \\ I \end{bmatrix} = \Delta_p \begin{bmatrix} 0 \\ I \end{bmatrix} A_+$$

and

$$H_z \Delta_z \begin{bmatrix} I \\ 0 \end{bmatrix} = \Delta_z \begin{bmatrix} I \\ 0 \end{bmatrix} A_-.$$

Hence, $\Delta_p \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $\Delta_z \begin{bmatrix} I \\ 0 \end{bmatrix}$ span the antistable eigenspace $\mathcal{M}$ of H_p and the stable eigenspace $\mathcal{M}^\times$ of H_z, respectively. As is well known [Bart et al., 1979, Francis, 1987, Green et al., 1990, Meinsma, 1995, Ran, 2003], there exists a J-spectral factorization iff $\mathcal{M} \cap \mathcal{M}^\times = \{0\}$, which is equivalent to that the Δ given in (5) is nonsingular.

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[2] The elements denoted by "?" are irrelevant.

When this condition holds, there exists a projection P onto $\mathcal{M}^\times$ along $\mathcal{M}$. The projection matrix P is given by (see Appendix for more details)

$$P = \Delta \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Delta^{-1}. \qquad (7)$$

So a J-spectral factor is (see e.g. [Ran, 2003])

$$W = \left[\begin{array}{c|c} P H_p P & P \begin{bmatrix} -B \\ C^* \end{bmatrix} \\ \hline J_{p,q} D_W^{-*} \begin{bmatrix} C & B^* \end{bmatrix} P & D_W \end{array} \right]. \qquad (8)$$

This realization is not minimal since the A-matrix has the same dimension as H_p. Substitute (7) into (8) and apply a similarity transformation with Δ, then

$$W = \left[\begin{array}{c|c} \Delta^{-1} P H_p \Delta \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} & \Delta^{-1} P \begin{bmatrix} -B \\ C^* \end{bmatrix} \\ \hline J_{p,q} D_W^{-*} \begin{bmatrix} C & B^* \end{bmatrix} \Delta \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} & D_W \end{array} \right].$$

After removing the unobservable states by deleting the second row and the second column, W becomes

$$W = \left[\begin{array}{c|c} \begin{bmatrix} I & 0 \end{bmatrix} \Delta^{-1} P H_p \Delta \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} I & 0 \end{bmatrix} \Delta^{-1} P \begin{bmatrix} -B \\ C^* \end{bmatrix} \\ \hline J_{p,q} D_W^{-*} \begin{bmatrix} C & B^* \end{bmatrix} \Delta \begin{bmatrix} I \\ 0 \end{bmatrix} & D_W \end{array} \right].$$

Since $\begin{bmatrix} I & 0 \end{bmatrix} \Delta^{-1} P = \begin{bmatrix} I & 0 \end{bmatrix} \Delta^{-1}$, W can be further simplified as given in (6). This completes the proof.

In general, $\Delta_z \neq \Delta_p$. However, these two can be the same.

Theorem 7. Assume that a para-Hermitian matrix Λ as given in (1) is minimal and has no poles or zeros on the $j\omega$-axis including ∞. Then Λ admits a J-spectral factorization if and only if there exists a nonsingular matrix Δ such that

$$\Delta^{-1} H_p \Delta = \begin{bmatrix} A_-^p & 0 \\ ? & A_+^p \end{bmatrix} \qquad (9)$$

and

$$\Delta^{-1} H_z \Delta = \begin{bmatrix} A_-^z & ? \\ 0 & A_+^z \end{bmatrix}, \qquad (10)$$

where A_-^z and A_-^p are stable, and A_+^z and A_+^p are antistable. In this case, a J-spectral factor W is given in (6).

PROOF. *Sufficiency.* It is obvious according to Lemma 6. In this case, $\Delta_z = \Delta_p = \Delta$. *Necessity.* According to Lemma 6, if there exists a J-spectral

factorization then there does exist a nonsingular Δ, as given in (5), which satisfies (9) and (10).

Since the Δ is the same as that in Lemma 6, the J-spectral factor is the same as in (6). This completes the proof.

Remark 8. The A-matrix of W is

$$\begin{bmatrix} I & 0 \end{bmatrix} \Delta^{-1} H_p \Delta \begin{bmatrix} I \\ 0 \end{bmatrix} = A_-^p$$

and that of W^{-1} is

$$\begin{bmatrix} I & 0 \end{bmatrix} \Delta^{-1} H_z \Delta \begin{bmatrix} I \\ 0 \end{bmatrix} = A_-^z.$$

Remark 9. This theorem says that a J-spectral factorization exists if and only if there exists a common similarity transformation to transform H_p (H_z, resp.) into a 2×2 lower (upper, resp.) triangular block matrix with the $(1,1)$-block including all the stable modes of H_p (H_z, resp.). Once the similarity transformation is done, a J-spectral factor can be formulated according to (6). If there is no such a similarity transformation, then there is no J-spectral factorization.

Remark 10. The necessary and sufficient conditions in this section do not depend on the realization of Λ, which does not have to be in the form (1). If Λ is not realized in this form, only minor changes in formula (6) are needed.

4. THE CASE WITH A SMALLER SUBSET

In this section, a subset of the class of para-Hermitian matrices characterized in Theorem 4 is considered.

Theorem 11. For a para-Hermitian matrix Λ characterized in Theorem 4, assume that: (i) $(E, \; A)$ is detectable and E is sign definite; (ii) $(A_z, \; R_z)$ is stabilizable and R_z is sign definite. Then the two ARE

$$\begin{bmatrix} I & -L_o \end{bmatrix} H_p \begin{bmatrix} L_o \\ I \end{bmatrix} = 0 \qquad (11)$$

and

$$\begin{bmatrix} -L_c & I \end{bmatrix} H_z \begin{bmatrix} I \\ L_c \end{bmatrix} = 0 \qquad (12)$$

always have unique symmetric solutions L_o and L_c, respectively, such that $\begin{bmatrix} I & -L_o \end{bmatrix} H_p \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $\begin{bmatrix} I & 0 \end{bmatrix} H_z \begin{bmatrix} I \\ L_c \end{bmatrix}$ are stable. In this case, $\Lambda(s)$ has a $J_{p,q}$-spectral factorization for some unique $J_{p,q}$ (where p is the number of the positive eigenvalues

of D and q is the number of the negative eigenvalues of D) if and only if $\det(I - L_o L_c) \neq 0$. If this condition is satisfied, then one J-spectral factor is formulated as

$$W = \left[\begin{array}{c|c} A + L_o E & B + L_o C^* \\ \hline -J_{p,q} D_W^{-*}(B^* L_c + C)(I - L_o L_c)^{-1} & D_W \end{array} \right],$$

where D_W is nonsingular and $D_W^* J_{p,q} D_W = D$.

PROOF. In this case, $\Delta_z = \begin{bmatrix} I & 0 \\ L_c & I \end{bmatrix}$, $\Delta_p = \begin{bmatrix} I & L_o \\ 0 & I \end{bmatrix}$ and $\Delta = \begin{bmatrix} I & L_o \\ L_c & I \end{bmatrix}$. According to Lemma 6, there exists a J-spectral factorization iff Δ is nonsingular, i.e., $\det(I - L_o L_c) \neq 0$. Substitute Δ into (6) and apply a similarity transformation with $-(I - L_o L_c)^{-1}$, then

$$W = \left[\begin{array}{c|c} \begin{bmatrix} I & -L_o \end{bmatrix} H_p \begin{bmatrix} I \\ L_c \end{bmatrix} (I - L_o L_c)^{-1} & B + L_o C^* \\ \hline -J_{p,q} D_W^{-*}(C + B^* L_c)(I - L_o L_c)^{-1} & D_W \end{array} \right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} I & -L_o \end{bmatrix} H_p \begin{bmatrix} I \\ 0 \end{bmatrix} & B + L_o C^* \\ \hline -J_{p,q} D_W^{-*}(C + B^* L_c)(I - L_o L_c)^{-1} & D_W \end{array} \right],$$

where the ARE (11) is used. This completes the proof.

A different approach involving two similarity transformations to derive the realization of W is shown below.

Similarity Transformation 1: Stabilization

Since L_o is the stabilizing solution of (11), after applying a similarity transformation $\begin{bmatrix} I & L_o \\ 0 & I \end{bmatrix}$, Λ is equal to

$$\Lambda = \left[\begin{array}{cc|c} A + L_o E & 0 & -(B + L_o C^*) \\ -E & -A^* - E L_o & C^* \\ \hline C & B^* + C L_o & D \end{array} \right] . (13)$$

Similarity Transformation 2: Factorization

Since L_c is the stabilizing solution of (12) and $\det(I - L_c L_o) \neq 0$, the following self-adjoint matrix is well defined:

$$L_{co} = L_c(I - L_o L_c)^{-1} = (I - L_c L_o)^{-1} L_c = L_{co}^*. (14)$$

Moreover,

$$I + L_o L_{co} = (I - L_o L_c)^{-1} \quad \text{and} \quad I + L_{co} L_o = (I - L_c L_o)^{-1}$$

are nonsingular. As a result, L_c can be represented as

$$L_c = (I + L_{co} L_o)^{-1} L_{co} = L_{co}(I + L_o L_{co})^{-1},$$

and the ARE (12) is equivalent to

$$\left[\, -L_{co}\ \ I + L_{co}L_o \,\right] H_z \left[\begin{matrix} I + L_o L_{co} \\ L_{co} \end{matrix}\right] = 0, \quad (15)$$

which can be expanded as the following equality using (11):

$$L_{co}(A + L_o E) + (A^* + E L_o)L_{co} + E = -$$
$$[L_{co}B + (I + L_{co}L_o)C^*]\, D^{-1}\, [B^* L_{co} + C(I + L_o L_{co})](16)$$

$$\Lambda = \left[\begin{array}{cc|c} A + L_o E & 0 & B + L_o C^* \\ L_{co}(A + L_o E) + (A^* + E L_o)L_{co} + E & -A^* - E L_o & C^* + L_{co}(B + L_o C^*) \\ -C - (B^* + C L_o)L_{co} & B^* + C L_o & D_W^* J_{p,q} D_W \end{array}\right],$$

where an additional similarity transformation $\left[\begin{matrix} -I & 0 \\ 0 & I \end{matrix}\right]$ is applied. Due to the equality (16), the above Λ can be factorized as $\Lambda = W^{\sim} \cdot J_{p,q} \cdot W$ with

$$W \doteq \left[\begin{array}{c|c} A + L_o E & B + L_o C^* \\ \hline -J_{p,q}^{-1} D_W^{-*} [B^* L_{co} + C(I + L_o L_{co})] & D_W \end{array}\right].$$

Using (14), W can be simplified as given in Theorem 11. W is bistable because the A-matrix of W is $\left[\, I\ \ -L_o \,\right] H_p \left[\begin{matrix} I \\ 0 \end{matrix}\right] = A + L_o E$ and the A-matrix of W^{-1} is

$$A + L_o E + (B + L_o C^*)D_W^{-1} J_{p,q}^{-1} D_W^{-*}(B^* L_c + C)(I - L_o L_c)^{-1}$$
$$= \left[\, I\ \ -L_o \,\right] H_z \left[\begin{matrix} I \\ L_c \end{matrix}\right](I - L_o L_c)^{-1}$$
$$\sim (I - L_o L_c)^{-1} \left[\, I\ \ -L_o \,\right] H_z \left[\begin{matrix} I \\ L_c \end{matrix}\right]$$
$$= (I - L_o L_c)^{-1}\left(\left[\, I\ \ -L_o \,\right] H_z \left[\begin{matrix} I \\ L_c \end{matrix}\right] + L_o \left[\, -L_c\ \ I \,\right] H_z \left[\begin{matrix} I \\ L_c \end{matrix}\right]\right)$$
$$= \left[\, I\ \ 0 \,\right] H_z \left[\begin{matrix} I \\ L_c \end{matrix}\right],$$

where the "$\sim$" means "similar to" and the ARE (12) has been used. W is indeed a J-spectral factor of Λ.

Dually to Theorem 11, the following theorem holds (with proof omitted):

Theorem 12. For a para-Hermitian matrix Λ characterized in Theorem 4, assume that: (i) (A, R) is stabilizable and R is sign definite; (ii) (E_z, A_z) is detectable and E_z is sign definite. Then the two ARE

$$\left[\, -L_c\ \ I \,\right] H_p \left[\begin{matrix} I \\ L_c \end{matrix}\right] = 0$$

and

$$\left[\, I\ \ -L_o \,\right] H_z \left[\begin{matrix} L_o \\ I \end{matrix}\right] = 0$$

According to the assumptions, Λ does not have any eigenvalue on the $j\omega$ axis including ∞. Assume p is the number of the positive eigenvalues of D and q is the number of the negative eigenvalues of D, then the equation $D_W^* J_{p,q} D_W = D$ has an invertible solution D_W.

Carrying on another similarity transformation $\left[\begin{matrix} I & 0 \\ L_{co} & I \end{matrix}\right]$ with respect to (13), then

always have unique symmetric solutions L_c and L_o, respectively, such that $\left[\, I\ 0 \,\right] H_p \left[\begin{matrix} I \\ L_c \end{matrix}\right]$ and $\left[\, I\ -L_o \,\right] H_z \left[\begin{matrix} I \\ 0 \end{matrix}\right]$ are stable. In this case, $\Lambda(s)$ has a $J_{p,q}$-spectral factorization for some unique $J_{p,q}$ (where p is the number of the positive eigenvalues of D and q is the number of the negative eigenvalues of D) if and only if $\det(I - L_o L_c) \neq 0$. If this condition is satisfied, then one $J-$spectral co-factor is formulated as

$$W(s) = \left[\begin{array}{c|c} A + R L_c & -(I - L_o L_c)^{-1}(B + L_o C^*)D_W^{-*} J_{p,q} \\ \hline B^* L_c + C & D_W \end{array}\right],$$

where D_W is nonsingular and $D_W J_{p,q} D_W^* = D$.

5. CONCLUSIONS

A class of regular invertible para-Hermitian transfer matrices is characterized and then the J-spectral factorization of transfer matrices is stud-

ied. A transfer matrix Λ in this class admits a J-spectral factorization if and only if there exists a common nonsingular matrix to similarly transform the A-matrices of Λ and Λ^{-1}, resp., into 2×2 lower (upper, resp.) triangular block matrices with the $(1,1)$-block including all the stable modes of Λ (Λ^{-1}, resp.). The resulting J-spectral factor is formulated in terms of the original realization of Λ. When a transfer matrix meets additional conditions, there exists a J-spectral factorization if and only if a coupling condition related to the stabilizing solutions of two ARE holds.

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APPENDIX: PROPERTIES OF PROJECTIONS

For a given nonsingular matrix partitioned as $\begin{bmatrix} M & N \end{bmatrix}$, the projection P onto the subspace $\operatorname{Im} M$ along the subspace $\operatorname{Im} N$ is

$$P = \begin{bmatrix} M & 0 \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix}^{-1}$$

and the projection Q onto the subspace $\operatorname{Im} N$ along the subspace $\operatorname{Im} M$ is

$$Q = \begin{bmatrix} 0 & N \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix}^{-1} = \begin{bmatrix} N & 0 \end{bmatrix} \begin{bmatrix} N & M \end{bmatrix}^{-1}.$$

The following important properties hold:

(i) $P + Q = I$;

(ii) $PQ = 0$;

(iii) $P^2 = P$ and $Q^2 = Q$;

(iv) $\operatorname{Im} P = \operatorname{Im} M$;

(v) $\begin{bmatrix} M & 0 \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix}^{-1} \begin{bmatrix} M & 0 \end{bmatrix} = \begin{bmatrix} M & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & N \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix}^{-1} \begin{bmatrix} 0 & N \end{bmatrix} = \begin{bmatrix} 0 & N \end{bmatrix}$;

(vi) $\begin{bmatrix} M & 0 \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix}^{-1} \begin{bmatrix} 0 & N \end{bmatrix} = 0$.

When $M^T N = 0$,

$$\begin{bmatrix} M & N \end{bmatrix}^{-1} = \begin{bmatrix} (M^T M)^{-1} M^T \\ (N^T N)^{-1} N^T \end{bmatrix}.$$

The projections reduce to

$$P = M(M^T M)^{-1} M^T$$

and

$$Q = N(N^T N)^{-1} N^T.$$

These two formulae can be easily found in the literature.

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