SYSTEM PARAMETER ESTIMATION USING TOTAL P-NORM MINIMIZATION

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Abstract: Real time system parameter estimation from the set of input-output data is usually solved by the quadratic norm minimization of system equations errors - known as least squares (LS). But measurement errors are also in the data matrix and so it is necessary to use a modification known as total least squares (TLS) or mixed LS and TLS. Instead of quadratic norm minimization other p-norms are used, for $1 \leq p \leq 2$. In the article new method is described named **Total p-norm** and **Mixed total p-norm** which is the analog to TLS and mixed LS and TLS method in the quadratic case.
The goal of the paper is to develop the method and to compare a set of parameter estimations of ARX model where each estimation is obtained by minimizing total p-norm $(1 \leq p \leq 2)$. Total p-norm and mixed total p-norm approach is used when errors are also in data matrix. If the measurement of the system output is damaged by some outliers described method gives better results than standard TLS or mixed LS and TLS approach.

Keywords: Identification, Estimation, ARX model, p-norm, Least Squares, Total Least Squares

1. INTRODUCTION

Identification of dynamic properties of a real plant is usually solved by making its model - choosing the model structure and the estimation of model unknown parameters using data measured on the real plant (Ljung, 1987).

If ARX model of the system is used, the measured data together with unknown system parameters forms the system of linear equation and parameter vector is obtained by the minimization of equations error vector by the least squares (LS) method (Björck, 1996). But the errors are also in the data matrix and such problem leads to the total least squares (TLS) method (Hueffel *et al.*, 1991). Solution of such problem is usually done by Singular Value Decomposition (SVD) of data matrix. But usually the errors are only in measured output of the system and not in the input, which is realized by controller and so it is known without errors. In such case only part of the data matrix - some of its columns - is measured with errors and such problem can be solved by mixed LS and TLS. Mixed LS and TLS can also be solved by SVD decomposition of the data matrix. The problem in such case is that updating of the results when new data are obtained is difficult.

The contribution of the article is in the new method of TLS and mixed LS and TLS problem solution using iterative solution of weighted least squares based on the rotating system of coordinates (Pachner, 2002).

The main idea of the method is based on the fact that in some special cases the LS and TLS solutions are identical.

The advantage of the new method is in simple updating of the TLS and mixed LS and TLS when new data are given. Such approach opens new possibilities to solve TLS and mixed LS and TLS problems.

Equation error minimization can also be realized using general p-norm instead of quadratic norm. Algorithm of the solution is known as Iteratively Reweighted Least Squares (IRLS) (Björck, 1996). The utilization of this algorithm together with the method using rotating system of coordinates results in a new method of p-norm minimization when errors are also in the data matrix. Such method is named **Total p-norm** and **Mixed total p-norm** which is the analog to TLS and mixed LS and TLS method in the quadratic case. It is known, that utilization of p norm $(1 \leq p < 2)$ instead of quadratic norm suppress the wrong measurements (outliers) in the data.

The paper is organized as follows:
The second section shows identification of ARX model using p-norm. The algorithm for minimization of p-norm $(1 < p < 2)$ is recapitulated in the section 3. This algorithm is known as Iteratively Reweighed Least Squares (IRLS). Original method of TLS problem solution based on rotating system of coordinates is described in section 4 and its variant when only part of data matrix is corrupted by the noise is given in subsection 4.1.

Because TLS method using rotating system of coordinates is based on iterative LS solutions so the method can be modified in such a way that in each iteration instead LS solution Iteratively Reweighed Least Squares are used. Total p-norm method is obtained in this way as analogy to known TLS method.

In section 5 the examples of system parameter estimation in different variant of total p-norm are shown.

2. IDENTIFICATION OF ARX MODEL

Usually chosen structure is the ARX model of the system described by equation

$$\begin{aligned} y(t) = {} & a_1 y(t-1) + \ldots + a_{n-1} y(t-n-1) + \\ & + b_0 u(t) + \ldots + b_{n-1} u(t-n-1) + e(t) \end{aligned} \qquad (1)$$

where $y(t)$ is the output of the system in time t and $u(t)$ is the system input, $e(t)$ is equation error and a_i, b_j are system parameters. Only single input - single output system is considered.

Parameter estimation problem from given set of data $\mathcal{D}^t = \{y(t), y(t-1), \ldots, u(t), u(t-1), \ldots\}$ leads to minimization of error $\varepsilon = [e(t), e(t-1), \ldots]^T$.

So we are looking for the parameter vector
$x = [a_1, a_2, \ldots, b_0, b_1, \ldots]^T$ that ensures the best approximation of the output vector
$b = [y(t), y(t-1), \ldots, y(t-m-1)]^T$ by the vector Ax, where

$$A = \begin{bmatrix} y(t-1) & y(t-2) & \ldots & u(t) & \ldots \\ y(t-2) & y(t-3) & \ldots & u(t-1) & \ldots \\ \vdots & & \vdots & \vdots & \vdots \end{bmatrix} \qquad (2)$$

Both the vector b and the matrix A are formed from the measured data $\mathcal{D}^t$. The problem is the minimization of the norm of error vector

$$\min_x \|Ax - b\|_p \ . \qquad (3)$$

If $p = 2$ the solution is known as Least Squares (LS) (Björck, 1996) (Boyd *et al.*, 2002). In reality the noise of output measurement is also in the elements of data matrix A. Solution of this problem leads to the Total Least Squares (TLS) (Björck, 1996), (Huffel *et al.*, 1991). If the measurement of the input is noise free, the problem can be solved by Mixed LS and TLS (Björck, 1996), (Huffel *et al.*, 1991). If $1 \leq p \leq 2$ the new method of solution is developed and this is the contribution of the article.

3. P-NORM MINIMIZATION

Solution of minimization problem (3) is recapitulated in this section. If the p norm is restricted by $1 < p < 2$ the minimization problem is convex and the solution is unique. This problem can be solved by iterative algorithm which is known as Iteratively Reweighted Least Squares (IRLS) (Björck, 1996).

Iteratively Reweighted Least Squares

Let us solve the approximation problem

$$\min_x \left\{ \Psi(x) = \|Ax - b\|_p^p \right\} \qquad 1 < p < 2 \qquad (4)$$

Consider, that all coordinates of the residuum
$\varepsilon(x) = b - Ax$ are nonzero. Then the function $\Psi(x)$ can be defined as

$$\Psi(x) = \sum_{i=1}^{m} |\varepsilon_i(x)|^p = \sum_{i=1}^{m} |\varepsilon_i(x)|^{p-2} \varepsilon_i(x)^2 \qquad (5)$$

Previous problem is weighted Least Squares:

$$\min_x \left\| D(\varepsilon)^{\frac{p-2}{2}} (b - Ax) \right\|_2 , \qquad (6)$$

where $D(\varepsilon) = \operatorname{diag}(|\varepsilon|)$. Because of dependency of diagonal weighting matrix $D(\varepsilon)$ on unknown solution x the problem must be solved by iterative algorithm. The input of k-th iteration is

$$\varepsilon^{(k)} = b - Ax^{(k)}, \quad D^{(k)} = \operatorname{diag}\left(\left| \varepsilon^{(k)} \right|^{\frac{p-2}{2}} \right) \qquad (7)$$

Utilizing weighted LS algorithm $\delta x^{(k)}$ is obtained by solving

$$\delta x^{(k)} = \arg\min_{\delta\mathbf{x}} \left\| D^{(k)} \left(\varepsilon^{(k)} - A\delta x \right) \right\|_2 \qquad (8)$$

The next iteration $\mathbf{x}^{(k+1)}$ is obtained as

$$x^{(k+1)} = x^{(k)} + \delta x^{(k)} \ . \qquad (9)$$

For minimization of the norm $p = 1$ linear programming (LP) can be used. Two following problems are equivalent

$$\min_x \|Ax - b\|_1 \iff \qquad (10)$$
$$\min_y \left\{ 1^T \ y \ : \ Ax - b \le y, \ Ax - b \ge -y \right\}$$

Introducing augmented vector $z = \begin{bmatrix} x \\ y \end{bmatrix}$, standard form of LP problem is obtained

$$\min_z \left\{ c^T z \ : \ \bar{A}z \le \bar{b} \right\}, \qquad (11)$$
$$c^T = \left[\, 0, \ \ldots \ 0, \ 1, \ \ldots \ 1 \, \right]$$
$$\bar{A} = \begin{bmatrix} A & -I \\ -A & -I \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} b \\ -b \end{bmatrix}$$

The only drawback of such computation is that LP problem can have more than only one solution.

4. TLS SOLUTION BASED ON THE ROTATING SYSTEM OF COORDINATES

Let us first suppose the simple linear dependence between the data $y_i = au_i + b$, where y is the output (response) of the system and u is the input (stimulus). Our aim is to estimate parameters a, b from the measured data y_i, u_i.

In least squares solution it is supposed that errors are in output y_i (in equation) only, so the model has the form

$$y_i + \varepsilon_i = au_i + b \qquad (12)$$

and the LS solution is

$$\begin{aligned}(a, b) &= \arg\min \sum_i (\varepsilon_i)^2 \\ &= \arg\min \sum_i (y_i - au_i - b)^2\end{aligned}$$

In TLS solution it is supposed that the errors are also in stimulus, so the model is

$$y_i + \varepsilon_{yi} = a\left(u_i + \varepsilon_{ui}\right) + b(1 + \varepsilon_{bi}) \qquad (13)$$

The principal idea is that LS and TLS solution is the same when $a = 0$ (the possibility that data fix

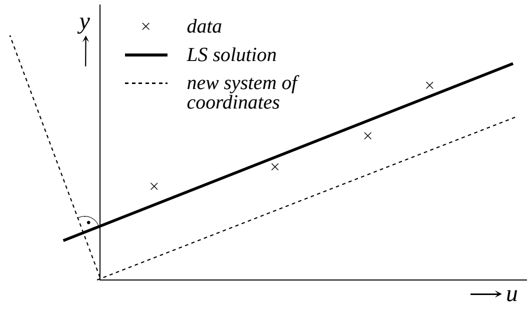

Fig. 1. Coordinate rotation in the method for TLS solution

exactly is omitted). So TLS problem is solved by LS by iterative way and in each iteration the coordinate system is rotated as in fig 1., where dashed lines form the new system of coordinates and solid line is LS solution.

But it is natural that coefficient 1 which forms part of the data is without error. So the problem leads to mixed LS and TLS solution and the model is in such case

$$y_i + \varepsilon_{yi} = a\left(u_i + \varepsilon_{ui}\right) + b \qquad (14)$$

Let us return to our model of discrete time linear dynamic system

$$y_k = a_1 y_{k-1} + \ldots + a_n y_{k-n} + b_0 u_k + \ldots \qquad (15)$$

The problem is the estimation of parameter vector x

$$x = [a_1, \ldots, a_n, b_0, \ldots, b_n]^T \qquad (16)$$

from output measurement vector $y = [y_1, y_2, \ldots, y_\nu]$ and input measurements u_t. So it is necessary to solve linear system $Ax = y$ where matrix A is formed from the data

$$A = \begin{bmatrix} y_0 & y_{-1} & \ldots & y_{1-n} & u_1 & \ldots & u_{1-n} \\ \vdots & & & & & & \\ y_{\nu-1} & y_{\nu-2} & \ldots & y_{\nu-n} & u_\nu & \ldots & u_{\nu-n} \end{bmatrix} \qquad (17)$$

The whole data matrix equals $D = [A, \ -y]$ and equation error vector is

$$\varepsilon = D \begin{bmatrix} x \\ 1 \end{bmatrix}. \qquad (18)$$

Note that the parameter vector is in such case augmented by last coordinate which equals 1.
LS solution of linear system is $x_{LS} = \arg\min \varepsilon^T \varepsilon$.

For the simplicity denote the whole data vector d (one row of the matrix D) of the dimension $\mu = 2n + 2$

$$\begin{aligned}d &= [d_1, d_2, \ldots, d_\mu] \\ &= [y_{t-1}, \ldots, y_{t-n}, u_t, \ldots, u_{t-n}, -y_t]\end{aligned} \qquad (19)$$

The affine dependence among the data is

$$d.x = d_1 x_1 + \ldots + d_{\mu-1} x_{\mu-1} + d_\mu x_\mu \qquad (20)$$

In this way in (μ) dimensional data space the linear subspace is defined. Such subspace is defined by the set of following $(\mu-1)$ row vectors (intersection of linear subspace with coordinate system planes)

$$\begin{bmatrix} 1 \; 0 \; \ldots \; 0 & -\dfrac{x_1}{x_\mu} \\ 0 \; 1 \; \ldots \; 0 & -\dfrac{x_2}{x_\mu} \\ \vdots & \\ 0 \; 0 \; \ldots \; 1 & -\dfrac{x_{\mu-1}}{x_\mu} \end{bmatrix} = P \in \mathcal{R}^{(\mu-1)\times(\mu)} \qquad (21)$$

Coordinate rotation means that the new system of coordinates is formed from such set of $(\mu-1)$ vectors together with one vector which is perpendicular to the rest. In this way the new system of coordinates is formed and QR factorization of matrix P^T is used to create it.

$$P^T = QR, \qquad \Longrightarrow \qquad P = R^T Q^T$$

where $Q \in \mathcal{R}^{(\mu)\times(\mu)}$ is orthogonal $(QQ^T = I)$ and matrix $R \in \mathcal{R}^{(\mu)\times(\mu-1)}$ is upper triangular, so

$$PQ = R^T \in \mathcal{R}^{(\mu-1)\times(\mu)} \qquad (22)$$

matrix R^T is lower triangular and due to its dimension the last (μ) column equals zero.

So the matrix Q forms the new system of orthonormal coordinates. Iterative LS solution of the TLS problem proceeds in the iterative way and in each iteration the coordinate rotation is used. Algorithm of the solution is the following:

- At the beginning LS problem

  $$\min_x \| \; \varepsilon = Ax - y \; \|^2 \qquad (23)$$

  is solved.
- From matrix P, see (21), in each iteration (i) by QR decomposition new system of coordinates is computed, then

  $$\varepsilon = D^{(i)} Q^{(i)} \; (Q^{(i)})^T x^{(i)} \qquad (24)$$

  So from the new data matrix $D^{(i+1)} = D^{(i)} Q^{(i)}$ by LS new vector $(x^{(i+1)})_{LS}$ is computed.
- From this follows that the original parameter vector $x^{(i+1)}$ equals

  $$(Q^{(i)})^T x^{(i+1)} = (x_{LS})^{(i+1)}$$

  and so

  $$x^{(i+1)} = Q^{(i)} \; (x_{LS})^{(i+1)} \qquad (25)$$

  and the parameter vector $x^{(i+1)}$is normalized $x^{(i+1)} := x^{(i+1)}/x^{(i+1)}_{/mu}$ in order that the last coordinate of vector x equals 1, see (18).
- TLS parameter vector solution x equals

  $$x^{(i+1)} = Q^{(1)} Q^{(2)} \ldots Q^{(i)} \; x^{(i+1)}_{LS} \qquad (26)$$

- The solution is terminated when

  $$\|x^{(i+1)} - x^{(i)}\|^2 < \gamma \qquad (27)$$

  or

  $$\max_k |x_k^{(i+1)} - x_k^{(i)}| < \gamma \qquad (28)$$

  where γ is the chosen accuracy of the solution.

4.1 Mixed LS and TLS solution based on the rotating system of coordinates

Let now suppose that inputs u_i in the data matrix are measured without errors. Such approach leads to mixed LS and TLS solution. The affine dependence among the data is the same as in (20). But inputs u_i which are the part of the data vector $(d_{n+1}, \ldots, d_{\mu-1})$ are measured without errors and so they are omitted when the system of coordinates is rotated. Omitting the data without errors the linear dependence among data is (compare with (20))

$$d_1 x_1 + \ldots + d_n x_n + d_\mu x_\mu \qquad (29)$$

In this way only in $(\mu - n - 1 = n + 1)$ dimensional data space the linear subspace is defined. Such subspace is defined by the set of only $(\mu - n - 2 = n)$ row vectors (which again show intersection of linear subspace with coordinate system planes)

$$\begin{bmatrix} 1 \; 0 \; \ldots \; 0 & -\dfrac{x_1}{x_\mu} \\ 0 \; 1 \; \ldots \; 0 & -\dfrac{x_2}{x_\mu} \\ \vdots & \\ 0 \; 0 \; \ldots \; 1 & -\dfrac{x_n}{x_\mu} \end{bmatrix} = P \in \mathcal{R}^{(n)\times(n+1)} \qquad (30)$$

The part of the new coordinate system is formed from such set of n vectors together with one vector which must be perpendicular to such set of vectors. Again the part of the new system of coordinates is obtained by QR factorization of matrix P^T as in the previous case. Then

$$PQ = R^T \in \mathcal{R}^{(n)\times(n+1)}$$

and the matrix R^T is lower triangular and due to its dimension the last $(n+1)$ column equals zero.

Matrix Q forms part of the new system of orthonormal coordinates. Total transformation matrix Q_c must be formed with respect to the data measured without errors in the following way. Matrix Q is divided to

$$Q = \begin{bmatrix} \bar{Q}_{(n\times n)} & q_1 \\ q_2 & q_{22} \end{bmatrix} \qquad (31)$$

where q_1 is column vector, q_2 is row vector and q_{22} is scalar. The total rotation matrix Q_c is then

$$Q_c = \begin{bmatrix} \bar{Q}_{(n\times n)} & 0_{(n\times(n+1))} & q_1 \\ 0_{((n+1)\times n)} & I_{((n+1)\times(n+1))} & 0_{((n+1)\times 1)} \\ q_2 & 0_{(1\times(n+1))} & q_{22} \end{bmatrix} \quad (32)$$

The algorithm of the solution of mixed LS and TLS is then the same.

5. SIMULATION EXAMPLES

In this section two simulation examples are presented which show the advantage of the proposed method. All simulation are realized with discrete time system of the third order

$$y(k) = -\sum_{i=1}^{3} a_i y(k-i) + \sum_{i=0}^{3} b_i u(k-i) + s \quad (33)$$

where $a_i = \begin{bmatrix} 0.5 & 0.1 & 0.4 \end{bmatrix}$, $b_i = \begin{bmatrix} 0 & 1 & 2 & 1.2 \end{bmatrix}$ and $s = 4$; so the parameter vector is $x = \begin{bmatrix} a_i & b_i & s \end{bmatrix}^T$. Noises with different distribution are added to show how the method works and in the second example outliers are added to the data.

1000 samples of input and output data are generated and noise $\mathcal{N}(0, 0.1)$ is added to the output - it is shown in fig. 2.

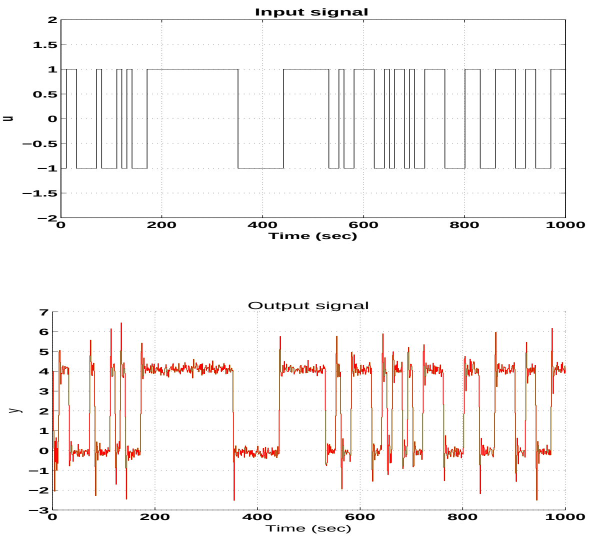

Fig. 2. Input and output of the system

Step responses of real and estimated system for $p = 1$ norm minimization are shown in fig. 3. Mixed total p-norm minimization method was used to solve the problem. It is due to the fact that input signal is measured without noise.

The norm of parameter estimation error can be used for the demonstration of estimation accuracy. The norm equals

$$J = \|x^* - x\|_p \quad (34)$$

where x^* is the parameter estimation vector and x is the vector of true parameters.

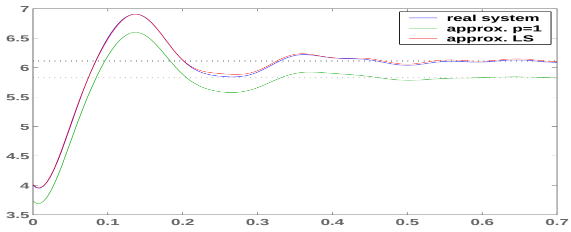

Fig. 3. Step responses of real and estimated system for $p = 1$ norm minimization

Fig. 4 shows how the norm of estimation error depends on the p-norm. If output of the system is without outliers (case with outliers is shown in the next example) the estimation error decreases if p-norm changes from $p = 1$ to $p = 2$. Mixed total p-norm minimization method was used to solve the problem.

If only total p-norm (not mixed p-norm) minimization is used, which solves the problem if noise is wrongly supposed to be in whole data matrix, the estimation error is larger and is shown in fig. 5. In both figures quadratic estimation error $\|x - x^*\|_2$ and p-norm estimation error $\|x - x^*\|_p$ are shown.

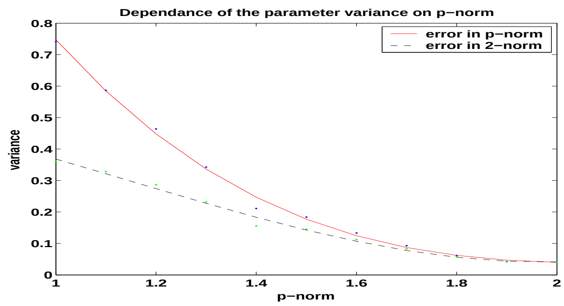

Fig. 4. Dependency of norm of estimation error on the used p-norm (mixed p-norm minimization is used)

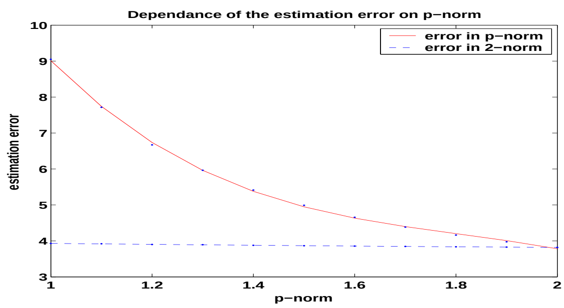

Fig. 5. Dependency of norm of estimation error on the used p-norm (total p-norm minimization is used)

In the next example outliers (wrong measurements) are added to the output of the system and also noise $\mathcal{N}(0, 1)$ is added to the output. There is 1000 samples of input and output signal and among them 15 outliers are added.

Output data are in this case shown in fig 6 (input is the same).

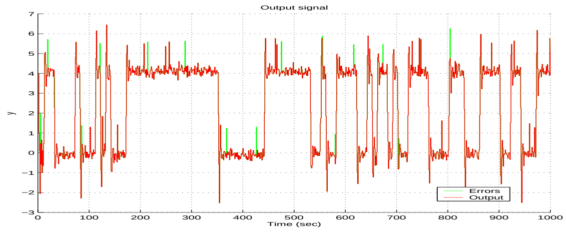

Fig. 6. System output data with outliers

Step responses of real and estimated system for $p = 1$ norm minimization is shown in fig. 7. Mixed total p-norm minimization method was used to solve the problem. It respects the fact that input signal is measured without noise.

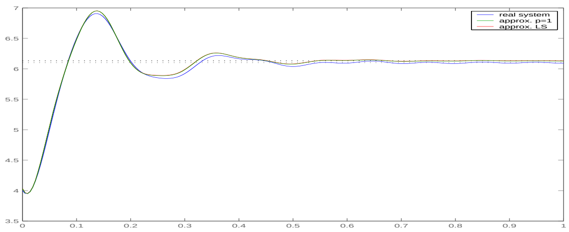

Fig. 7. Step responses of real end estimated system for $p = 1$ norm minimization

The Euclide norm of parameter estimation error is again used for the demonstration of estimation accuracy. Fig. 8 shows the dependency of norm of parameter estimation error on the p-norm.

From this simple simulation examples follow that the method is reliable, fast and using p-norm minimization results in better parameter estimation only when outliers in system output are presented. Many simulation was realized and similar results are obtained.

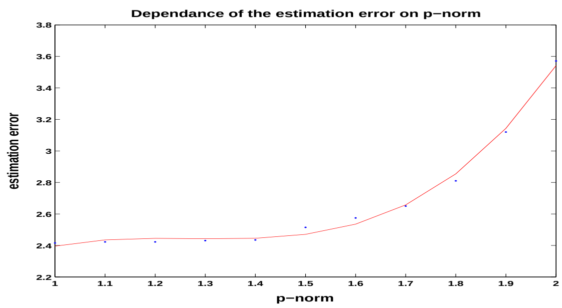

Fig. 8. Dependency of norm of estimation error on the used p-norm

6. CONCLUSIONS

In this paper new method of system parameter estimation is developed using different norms and output measurement errors also in the data matrix. It is the analog to well known Total Least Squares (TLS) and its variant Mixed Least Squares (LS) and TLS method. Such method is in the paper denoted as Total p-norm and Mixed total p-norm minimization method.

For the solution new original method for TLS solution is used which is based on the iterative LS solution in the rotating system of coordinates. The system of coordinates is rotated in such a way that LS and TLS solutions are identical. If instead of LS solution Iteratively Reweighted Least Squares are used the proposed method is obtained. Such approach opens new possibilities for solving different variants of TLS problem. For instance updating LS is simple and so updating Total p-norm method can be realized too. Simulation results show how the algorithm works and obtained results show the advantage of the method especially if outliers are presented.

7. ACKNOWLEDGEMENT

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