RATIONAL IMPLEMENTATION OF DISTRIBUTED DELAY USING EXTENDED BILINEAR TRANSFORMATIONS

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Abstract Based on an extension of the bilinear transformation, a rational implementation for distributed delay in linear control laws is proposed. This implementation converges much faster than the rational implementation inspired from the δ-operator. The implementation has an elegant structure of chained bi-proper nodes cascaded with a strictly proper node. The stability of each node is determined by the choice of the total number N of the nodes. The H^∞-norm of the implementation error approaches 0 when N goes to ∞ and hence the stability of the closed-loop system can be guaranteed. In addition, the steady-state performance of the system is retained. Simulation examples are given to verify the results and to show comparative study with other implementations.

Keywords: distributed delay, finite-spectrum assignment, modified Smith predictor, dead-time compensator, γ-operator, bilinear transformation, rational implementation, implementation error

1. INTRODUCTION

A distributed delay is a finite integral over the time, e.g,

$$v(t) = \int_0^h e^{A\zeta} B u(t-\zeta) d\zeta, \qquad (h > 0). \qquad (1)$$

The s-domain equivalent, i.e., the transfer function from u to v, is:

$$Z(s) = (I - e^{-(sI-A)h})(sI-A)^{-1}B. \qquad (2)$$

This is a finite-impulse-response (FIR) block because all the poles are canceled by the zeros. Distributed delays often appear as a part of dead-time compensators for processes with dead time, in particular, for unstable processes as a part of the finite-spectrum-assignment control law (Manitius and Olbrot, 1979; Watanabe, 1986; Wang *et al.*, 1999) or in the form of a modified Smith predictor (Watanabe and Ito, 1981; Palmor, 1996). Distributed delays also appear in H^∞ control of (even, stable) dead-time systems (Zhong, 2003*e*; Meinsma and Zwart, 2000; Zhong, 2003*b*; Zhong, 2003*c*; Mirkin, 2003*b*; Zhong, 2003*d*) and continuous-time deadbeat control (Zhong, 2003*a*). Due to the requirement of internal stability, such an FIR block has to be approximately implemented as a stable block without hidden unstable poles.

A common way to do this is to replace the distributed delay by the sum of a series of discrete (often commensurate) delays (Manitius and Olbrot, 1979; Watanabe and Ito, 1981; Palmor, 1996) (other interesting implementations using resetting mechanism can be found in (Tam and Moore, 1974) and (Mondié *et al.*, 2001)). There have been some arguments about the possibility of causing instability by doing this. This has attracted a lot of attention from the research community; see (Mondié *et al.*, 2001; Van Assche *et al.*, 1999; Santos and Mondié, 2000; Mondié and Santos, 2001; Van Assche *et al.*, 2001; Engelborghs *et al.*, 2001; Mirkin, 2003*a*; Michiels *et al.*, 2003; Fattouh *et al.*, 2001; Mondié and Michiels, 2003). It was proposed as an open problem in the survey paper (Richard, 2003). Recently, it has been proved that the implementation using quadrature approximations does not cause instability (Zhong, 2004). Moreover, the steady-state performance of the system can be retained by using an improved implementation. Recent research (Zhong, 2005) shows that the implementation can be done using rational transfer functions. This makes the implementation much easier than that involving discrete delays because a rational transfer function is easier to implement. However, as mentioned in (Zhong, 2005), the convergence is not fast enough and a better approach is needed.

Further to the work in (Zhong, 2005), this paper proposes a rational implementation with much faster convergence. Some of the reasoning developed in (Zhong, 2005) will be adopted here. However, the implementation proposed here is not a straightforward extension. In particular, the proof of the convergence is not trivial. The proposed implementation meets all the five key points for implementation of distributed delay summarized in (Zhong, 2005).

Following the structure developed in (Zhong, 2004; Zhong, 2005), the implementation is regarded as a pure approximation/implementation problem of distributed delay in the frequency domain. It does not matter whether the system delay exists in the input, the measurement or the state or what the control law is as long as there is a distributed delay in the control law. Although these papers focus on the implementation of distributed delay in control laws, the approaches proposed are also useful for approximating systems involving a distributed delay, even a discrete delay. See (Partington, 2004; Partington, 1991) for more details on this topic.

Due to the page limit, the relevant background information is kept to a minimum. See (Zhong, 2004; Zhong, 2005) for more details.

2. PRELIMINARY: BILINEAR TRANSFORMATIONS

The well-known γ-operator in digital and sampled-data control circles is defined as

$$\gamma = \frac{2}{\tau} \cdot \frac{q-1}{q+1},$$

where q is the shift operator and τ is the sampling period (Świder, 1998; Åström and Wittenmark, 1989; Franklin *et al.*, 1990). It is often used to digitizing a continuous-time transfer function. The transformation defined by the γ-operator is also called the bilinear transformation, or the Tustin's transformation. It actually corresponds to the trapezoidal rule for numerical integration. It also connects to the (lower) linear fractional transformation $\mathcal{F}_l$ and the (right) homographic transformation $\mathcal{H}_r$, which are frequently used in H^∞control, with $\gamma = \frac{2}{\tau}\mathcal{F}_l(\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}, q) = \frac{2}{\tau}\mathcal{H}_r(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, q)$. See (Zhong, 2003*e*; Zhong, 2003*d*) for the definition of notations used here. In this paper, the term "bilinear transformation" is preferred due to the extension introduced in the next section.

The shift operator q can be solved as

$$q = \mathcal{H}_r(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}, \frac{\tau}{2}\gamma) = \frac{1+\frac{\tau}{2}\gamma}{1-\frac{\tau}{2}\gamma}.$$

Since $q \rightarrow e^{\tau s}$ when $\tau \rightarrow 0$ (Kannai and Weiss, 1993), we can approximate $e^{-\tau s}$ as

$$e^{-\tau s} \approx q^{-1} = \frac{1-\frac{\tau}{2}\gamma}{1+\frac{\tau}{2}\gamma}.$$

Furthermore, γ holds the following *limiting property*:

$$\lim_{\tau\rightarrow 0} \gamma = \lim_{\tau\rightarrow 0} \frac{2}{\tau}\frac{e^{\tau s}-1}{e^{\tau s}+1} = s.$$

This means γ-operator is an approximation of the differential operator $p = \frac{\mathrm{d}}{\mathrm{dt}}$. Using the approximation $\gamma \approx s$, then $e^{-\tau s} \approx \frac{1-\frac{\tau}{2}s}{1+\frac{\tau}{2}s}$. This actually recovers the first-order Padé approximation of $e^{-\tau s}$.

Since the γ-operator offers better approximation than the δ-operator (which corresponds to the forward rectangular rule) (Świder, 1998; Åström and Wittenmark, 1989; Franklin *et al.*, 1990), the γ-operator (i.e., the bilinear transformation) is exploited to implement the distributed delay. The framework developed in (Zhong, 2005) will be followed; the major difficulty lies in the proof of the convergence.

3. IMPLEMENTATION OF DISTRIBUTED DELAY

For a natural number N and the delay $h > 0$, the function Φ of matrix A is defined as

$$\Phi = (\int_0^{\frac{h}{N}} e^{-A\zeta} d\zeta)^{-1}(e^{-A\frac{h}{N}} + I),$$

which is independent of s. Furthermore, a bilinear transformation Γ is defined as

$$\Gamma = (e^{\tau(sI-A)} - I)(e^{\tau(sI-A)} + I)^{-1}\Phi, \qquad (3)$$

with $\tau = \frac{h}{N}$. This can be regarded as the extension of the bilinear transformation to the matrix case. Γ holds:

(i) the *limiting property*

$$sI - A = \lim_{\tau\rightarrow 0} \Gamma, \qquad (4)$$

(ii) the *static property*

$$sI - A|_{s=0} = \Gamma|_{s=0} = -A,$$

(iii) the *cancellation property*

$$\Gamma|_{sI-A=0} = 0.$$

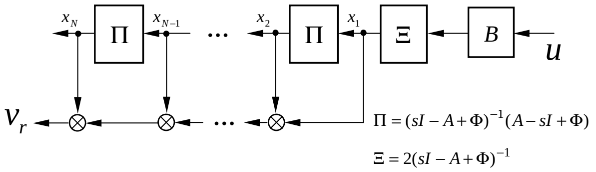

Figure 1. Rational implementation of distributed delay: $Z_r = \Sigma_{k=0}^{N-1}\Pi^k \Xi B$

According to the mechanism developed in (Zhong, 2005), this Γ is able to bring a rational implementation to guarantee the stability of the closed-loop system and the steady-state performance.

From (3), we have

$$e^{-(sI-A)\frac{h}{N}} = (\Phi - \Gamma)(\Phi + \Gamma)^{-1}.$$

Substitute this into (2), then

$$Z(s) = (I - (\Phi - \Gamma)^N (\Phi + \Gamma)^{-N})(sI - A)^{-1}B. \quad (5)$$

Due to the limiting property (4), we have $\Gamma \approx sI - A$. Substitute this into (5), then Z can be approximated as Z_r given below:

$$\begin{aligned} & Z_r(s) \\ &= (I - (\Phi - sI + A)^N (sI - A + \Phi)^{-N})(sI - A)^{-1}B \quad (6) \\ &= (I - (\Phi - sI + A)(sI - A + \Phi)^{-1}) \\ &\quad \cdot \Sigma_{k=0}^{N-1}(\Phi - sI + A)^k (sI - A + \Phi)^{-k}(sI - A)^{-1}B \\ &= 2(sI - A + \Phi)^{-1}\Sigma_{k=0}^{N-1}(\Phi - sI + A)^k (sI - A + \Phi)^{-k}B \\ &= \Sigma_{k=0}^{N-1}\Pi^k \Xi B, \quad (7) \end{aligned}$$

with

$$\Pi = (\Phi - sI + A)(sI - A + \Phi)^{-1}, \qquad \Xi = 2(sI - A + \Phi)^{-1}.$$

The hidden, possibly unstable, poles in Z have disappeared from Z_r. This approximation converges to Z when $N \to +\infty$, as will be proved in Section 4.

Π is a bi-proper rational transfer function while Ξ is strictly proper. They share the same denominator and thus the same stability property. Z_r can be easily implemented as a chain of rational transfer functions shown in Figure 1. Since Ξ is strictly proper, so is Z_r. This indicates there always exists a large enough N such that the implementation does not affect the stability of the closed-loop system, provided that each node is stable. See Section 4 for more details.

The stability of each node Π or Ξ is determined by the number N. This is governed by the theorem below. Denote *an* eigenvalue[1] of A as $\bar{\sigma} + j\bar{\omega}$. Then the corresponding eigenvalue of $\frac{h}{N}A$ is $\sigma + j\omega$ with $\sigma = \frac{h}{N}\bar{\sigma}$ and $\omega = \frac{h}{N}\bar{\omega}$.

Theorem 1. The following conditions are equivalent:

(i) Z_r, Π, Ξ or $A - \Phi$ is stable;
(ii) $\int_0^{\frac{h}{N}} e^{A\zeta}d\zeta$ is antistable;
(iii) $\sigma\cos\omega + \omega\sin\omega - \sigma e^{-\sigma} > 0$, ignoring the case when $\sigma = 0$ and $\omega = 0$.

PROOF. (i)⇔(ii): The A-matrix of each node is

$$\begin{aligned} A - \Phi &= A - \left(\int_0^{\frac{h}{N}} e^{-A\zeta}d\zeta\right)^{-1}(e^{-A\frac{h}{N}} + I) \\ &= A + A(e^{-A\frac{h}{N}} - I)^{-1}(e^{-A\frac{h}{N}} + I) \\ &= 2A(e^{-A\frac{h}{N}} - I)^{-1}e^{-A\frac{h}{N}} \\ &= -2A(e^{A\frac{h}{N}} - I)^{-1} \\ &= -2\left(\int_0^{\frac{h}{N}} e^{A\zeta}d\zeta\right)^{-1}. \end{aligned} \quad (8)$$

It was assumed that A is nonsingular here. Actually, the final equality holds for a singular A as well. Since the inverse operation does not change the sign of the real part of an eigenvalue, the signs of the real part of the eigenvalues of $A - \Phi$ are opposite from those of $\int_0^{\frac{h}{N}} e^{A\zeta}d\zeta$. Hence, Z_r (Π and Ξ) is stable if and only if $\int_0^{\frac{h}{N}} e^{A\zeta}d\zeta$ is antistable.

(ii)⇔(iii): see (Zhong, 2005). This completes the proof.

Corollary 2. If all the eigenvalues of A are real, then each node Π or Ξ is stable for any natural number N.

Surprisingly, the node or the implementation shares the same stability with the node or the implementation derived using the δ-operator in (Zhong, 2005). Hence, some of the results there can be applied to this implementation as well. For example, the three-dimensional surface of $f(\sigma, \omega) = \sigma\cos\omega + \omega\sin\omega - \sigma e^{-\sigma}$, a sufficient condition to guarantee the stability of the node or the implementation, and the contour of $f(\sigma, \omega)$ at level 0 can be found there. Figure 2 shows a part of the contour focusing on the circle around the origin with a radius of 2.8. All the eigenvalues of $A\frac{h}{N}$ fall into this circle when $N > \underline{N}$ with

$$\underline{N} = \left\lceil 0.357h \cdot \max_i |\lambda_i(A)| \right\rceil, \quad (9)$$

where $\lceil \cdot \rceil$ is the ceiling function. When this condition holds, all the conditions in Theorem 1 are satisfied. In particular, each node is stable. See (Zhong, 2005) for more details. This result is needed in the next section.

4. CONVERGENCE OF THE IMPLEMENTATION

The lemma below is crucial to prove the convergence of the implementation discussed later.

[1] The stability analysis in this paper only involves functions of matrix A. Hence, the eigenvalues of the functions are the scalar functions of the eigenvalues of A. In other words, the eigenvalues of the functions can be obtained one by one. See (Dorny, 1975, Section 4.6) for more details.

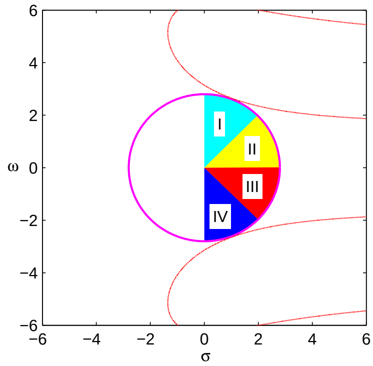

Figure 2. The circle into which all eigenvalues of $A\frac{h}{N}$ fall when $N > \underline{N}$

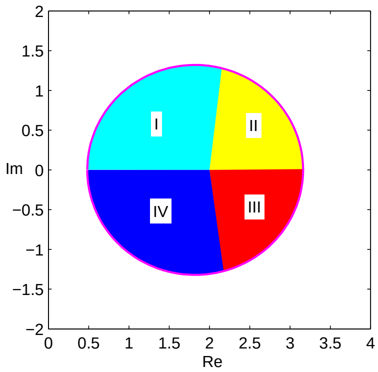

Figure 3. The area mapped from the right-half circle in Figure 2 via $\phi = c\frac{1+e^{-c}}{1-e^{-c}}$

Lemma 3. Φ is antistable if $N > \underline{N}$ for $\underline{N}$ in (9).

PROOF. Temporarily assume that A does not have an eigenvalue 0. According to (8), we have

$$\Phi\frac{h}{N} = A\frac{h}{N}(I - e^{-A\frac{h}{N}})^{-1}(e^{-A\frac{h}{N}} + I). \qquad (10)$$

As mentioned before, if $N > \underline{N}$ then all the eigenvalues of $A\frac{h}{N}$ fall into the circle around the origin with a radius of 2.8 shown in Figure 2. The map $\phi = c\frac{1+e^{-c}}{1-e^{-c}}$ maps all the points inside this circle into the shaded area shown in Figure 3, where only the mapped area for the right-half circle is shown. The mapped area for the left-half circle overlaps with that of the right-half circle because the map is symmetric with respect to c. The mapped area is on the open right half plane. Since the eigenvalues of $\Phi\frac{h}{N}$ are mapped from the eigenvalues of $A\frac{h}{N}$ via this map, as can be seen from (10), the real part of the eigenvalues of $\Phi\frac{h}{N}$, and hence of Φ, is always positive, i.e., Φ is antistable.

This is also true when A has an eigenvalue of 0 (the corresponding eigenvalue of $\Phi\frac{h}{N}$ is 2) because the singularity $c = 0$ in the map ϕ is removable and the origin is mapped to the point $(2, 0)$. This completes the proof.

Theorem 4. Denote the approximation error of Z_r as $E_r = Z - Z_r$. Then
$$\lim_{N\to+\infty} \|E_r(s)\|_\infty = 0.$$

PROOF. According to (2) and (6), the implementation error E_r is

$$\begin{aligned}
E_r &= (I - e^{-(sI-A)h})(sI-A)^{-1}B - (I - (\Phi - sI + A)^N(sI - A + \Phi)^{-N})(sI-A)^{-1}B \\
&= \int_0^h e^{-(sI-A)\zeta} d\zeta \cdot B - 2N\Phi\int_0^1 (-I + 2\Phi(\Phi - (sI-A)\zeta)^{-1})^{-N-1}(\Phi - (sI-A)\zeta)^{-2} d\zeta \cdot B \\
&= \int_0^{\frac{h}{N}} e^{-(sI-A)N\zeta} dN\zeta \cdot B - N\int_0^{\frac{h}{N}} 2\Phi(-I + 2\Phi(\Phi - (sI-A)\frac{N}{h}\zeta)^{-1})^{-N-1}(\Phi - (sI-A)\frac{N}{h}\zeta)^{-2} d\frac{N}{h}\zeta \cdot B \\
&= N\int_0^{\frac{h}{N}} \left(e^{-(sI-A)N\zeta} - \frac{2N}{h}\Phi(-I + 2\Phi(\Phi - (sI-A)\frac{N}{h}\zeta)^{-1})^{-N-1}(\Phi - (sI-A)\frac{N}{h}\zeta)^{-2}\right) d\zeta \cdot B \\
&= N\int_0^{\frac{h}{N}} \left(e^{-(sI-A)N\zeta} - \frac{2N}{h}\Phi(\Phi + (sI-A)\frac{N}{h}\zeta)^{-N-1}(\Phi - (sI-A)\frac{N}{h}\zeta)^{N-1}\right) d\zeta \cdot B.
\end{aligned}$$

The integrand can be expanded into a series of ζ as

$$I - \frac{2N}{h}\Phi^{-1} + \zeta\cdot N\left((\frac{2N}{h}\Phi^{-1})^2 - I\right)(sI-A)(I + O(\zeta)).$$

When $N \to +\infty$, the terms of ζ^0 and ζ^1 disappear because $\frac{2N}{h}\Phi^{-1} \to I$ and $N\left((\frac{2N}{h}\Phi^{-1})^2 - I\right) \to 0$. This means the convergence is much faster than the case in (Zhong, 2005), where the term of ζ^1 does not disappear when $N \to +\infty$.

Now, consider the stability of the matrix

$$A - \Phi\frac{h}{N}\zeta^{-1} = A - \Phi - \Phi((\frac{h}{N}\zeta)^{-1} - 1)$$

for $\zeta \in (0, \frac{h}{N}]$. If $N > \underline{N}$, then $A - \Phi$ is stable, as mentioned at the end of the previous section, and $-\Phi$

is stable as well according to Lemma 3. $(\frac{h}{N}\zeta)^{-1} - 1 \geq 0$ when $\zeta \in (0, \frac{h}{N}]$. Hence, $(\Phi + (sI - A)\frac{N}{h}\zeta)^{-N-1}(\Phi - (sI - A)\frac{N}{h}\zeta)^{N-1}$ is stable for $\zeta \in (0, \frac{h}{N}]$ when $N > \underline{N}$. This means that the integrand is bounded on the closed right half plane when $N > \underline{N}$ for $\zeta \in [0, \frac{h}{N}]$ and so is $N\left((\frac{2N}{h}\Phi^{-1})^2 - I\right)(sI - A)(I + O(\zeta))$. Assume that, for $N > \underline{N}$,

$$\left\| N\left((\frac{2N}{h}\Phi^{-1})^2 - I\right)(sI - A)(I + O(\zeta)) \cdot B \right\| < M,$$

then, for $N > \underline{N}$,

$$\|E_r(s)\|_\infty \leq \left\| (I - \frac{2N}{h}\Phi^{-1})B \right\| h + N \int_0^{\frac{h}{N}} M\zeta d\zeta. \quad (11)$$

The two terms on the right-hand side all approach 0 when $N \to +\infty$. This completes the proof.

This theorem indicates that there alway exists a number N such that the implementation is stable and, furthermore, the H^∞-norm of the implementation error is less than a given positive value. According to the well-known small-gain theorem, the stability of the closed-loop system can always be guaranteed.

5. NUMERICAL EXAMPLE

Consider the simple plant $\dot{x}(t) = x(t) + u(t-1)$ with

$$u(t) = -(1 + \lambda_d)\left(e^1 \cdot x(t) + \int_0^1 e^\zeta u(t-\zeta)d\zeta\right) + r(t), \quad (12)$$

where $r(t)$ is the reference signal. This example has been widely studied in the literature; see e.g. (Engelborghs *et al.*, 2001; Van Assche *et al.*, 1999; Santos and Mondié, 2000; Fattouh *et al.*, 2001). Here, $A = 1$, $B = 1$ and $h = 1$. The closed-loop system has only one pole at $s = -\lambda_d$, which is stable when $\lambda_d > 0$. The distributed delay in (12) is

$$v(t) = \int_0^1 e^\zeta u(t-\zeta)d\zeta, \quad (13)$$

and the s-domain equivalent is $Z(s) = \frac{1-e^{1-s}}{s-1}$. The proposed rational implementation Z_r is

$$Z_r(s) = \Sigma_{k=0}^{N-1}\left(\frac{2-\epsilon s}{2-2\epsilon+\epsilon s}\right)^k \frac{2\epsilon}{2-2\epsilon+\epsilon s}$$

with $\epsilon = 1 - e^{-\frac{1}{N}}$. Since A has no non-real eigenvalues, Z_r is always stable, even for $N = 1$.

The approximation error for different N is shown in Figure 4. The static error is zero and the approximation error approaches 0 at both high and low frequencies. The approximation error decreases when the number N of the nodes increases. The convergence is fast, in particular, for low frequencies.

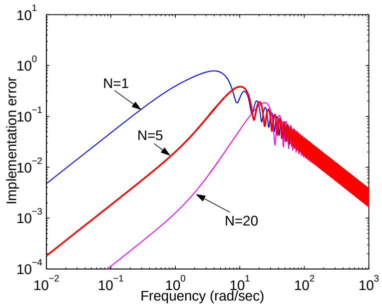

Figure 4. The implementation error of Z_r for different N

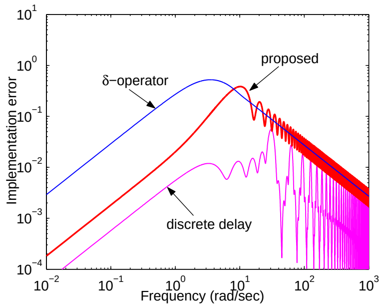

Figure 5. Comparison of different implementations ($N = 5$)

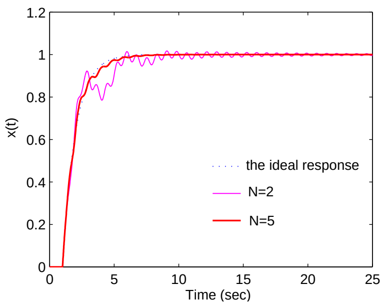

Figure 6. System responses when $r(t) = 1(t)$

Figure 5 shows the implementation error of different implementations for $N = 5$. The discrete-delay implementation proposed in (Zhong, 2004) is denoted as "discrete delay" in the figure and the rational implementation proposed in (Zhong, 2005), which was derived from the δ-operator, is denoted as "δ-operator" in the figure. The proposed implementation is much better than the one derived using the δ-operator. Although it is still worse than the the discrete-delay implementation, it has the advantage of easy implementation. Actually, it is good enough, as can be seen from Figure 6, where the step response when $N = 5$ is very close to the ideal response.

The unit-step responses of the system, as shown in Figure 6, are obtained using (12) with $\lambda_d = 1$, i.e.,

$$u = -(1 + \lambda_d)\left(e^1 \cdot x + v\right) + r, \; v = Z_r \cdot u$$

in the s-domain for different N (note that no change is made to the control law). When $N = 1$, the system is unstable because of the large approximation error. When $N = 2$, the system is stable though slightly oscillatory. When $N = 5$, the response is very close to the ideal response. All the stable responses guarantee the steady-state performance.

6. CONCLUSIONS

Based on an extension of the bilinear transformation, an approach has been proposed to implement distributed delay using rational transfer functions. The implementation consists of a series of bi-proper nodes cascaded with a low-pass node. The implementation converges much faster than the one proposed in (Zhong, 2005). Surprisingly, each node in the implementation shares the same stability as that in (Zhong, 2005). The H^∞-norm of the implementation error approaches 0 when the number N of nodes goes to ∞. Hence, there always exists a number N to guarantee the stability of the closed-loop system. In addition, the steady-state performance of the system is also guaranteed. In addition to the easy implementation, the proposed rational implementation does not involve any extra parameter to choose apart from the number N of the nodes. In particular, no parameter for a low-pass filter is needed to choose, which is an essential part in the literature, e.g., (Mirkin, 2003*a*; Michiels *et al.*, 2003; Mondié and Michiels, 2003; Zhong, 2004). Simulation examples are given to verify the results and to compare different implementations.

ACKNOWLEDGEMENT

This work was partially supported by EPSRC under grant No. EP/C005953/1.

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