ATOMIC FACTORIZATION PROBLEM FOR BIVARIATE PARAUNITARY MATRICES AND CONSEQUENCES

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Abstract: The property of losslessness, embraced by systems characterized by paraunitary matrices is crucial in robust design and numerical analysis of such systems. Here the paraunitary constraint is imposed on the bivariate polynomial matrix factorization algorithm of Guiver and Bose (1982), and an atomic factorization problem is formulated to relate the existing results and get new results.

Keywords: Paraunitary matrices, Robust multidimension systems, bivariate polynomial matrix factorization.

1. INTRODUCTION

Passivity and losslessness properties of systems are central to state-space and transform domain realizations of control and signal processing systems. When endowed with such properties, the relevant system does not amplify noise introduced at any point in the system. Such systems (lossless, passive) can be made robust to parameter variations as well. The most elementary expression for losslessness is the rotation matrix, which is the building block for more general classes of matrices or operators called isometric, unitary and inner. In electrical circuit theory, Kirchhoff circuits consist of a variety of basic elements, which need not be linear or constant. If an element such as an inductance is nonlinear and explicitly time-dependent and is to be characterized as being lossless, its defining relationship must have a specific form. Representational forms occur in generalizations to multiport coupled inductances and the the conditions for equivalence of such representations have been recently investigated (Bose and Fettweis, 2004). Kirchhoff circuits can serve as reference circuits for wave digital filters, numerical integration of ordinary as well as partial differential equations (imposition of passivity and, specifically, losslessness constraints are important here) that characterize a wide spectrum of physical phenomena (Bilbao, 2004).

2. PARAUNITARINESS

The term paraunitary (and the associated term para-Hermitian) occurred first in the realizability theory of continuous time electrical networks (Koga, 1968). The restrictions of passivity and/or losslessness on linear, lumped, time-invariant, solvable multiports are known to yield useful properties of the characterizing matrices whose elements are real rational functions in the complex variable p. Denoting the Hurwitz conjugate of the $(m \times m)$ matrix $H(p)$ by $H(-p)$, $H(p)$ is paraunitary provided its inverse is the Hurwitz conjugate transpose i.e. $H(p)H^T(-p) = I_m$. Clearly, a paraunitary matrix is unitary for $p = j\omega$, i.e. when p has a zero real-valued component. The result extends to multivariate networks when the vector variable $\mathbf{p} \triangleq (p_1, p_2, \ldots, p_n)$ replaces p. Further, if a real rational matrix $W(\mathbf{p})$ satisfies $W(\mathbf{p}) + W^T(-\mathbf{p}) = O_m$, then $W(\mathbf{p})$ is called a reactance matrix. Therefore, if $W(\mathbf{p})$ is a reactance matrix, then its Cayley transform (Bose and Fettweis, 2004),

$$\begin{aligned} H(\mathbf{p}) &= (I_m - W(\mathbf{p}))(I_m + W(\mathbf{p}))^{-1} \\ &= (I_m + W(\mathbf{p}))^{-1}(I_m - W(\mathbf{p})) \end{aligned}$$

is a paraunitary matrix i.e.

$$H(\mathbf{p})H^T(-\mathbf{p}) = H^T(-\mathbf{p})H(\mathbf{p}) = I_m$$

which characterizes a lossless multidimensional multiport. It is noted that a nonrational matrix can be paraunitary e.g. a matrix whose diagonal entries are $e^{-1/p}$. Note that the inverse of $(I_m + W(\mathbf{p}))$ exists for the hypothesized networks.

The discrete-time counterparts of a $(m \times m)$ real rational paraunitary matrix $A(z)$ and a lossless matrix $B(z)$ are, respectively,

$$\begin{aligned} A(z)A^T(z^{-1}) = A^T(z^{-1})A(z) = I_m, \\ B(z^{-1}) = -B^T(z) \end{aligned}$$

When the coefficients are complex-valued, the transpose superscript is replaced by the hermitian conjugate or complex conjugate transpose. The multidimensional generalizations for the preceding two expressions are, respectively,

$$\begin{aligned} A(\mathbf{z})A^T(\mathbf{z}^{-1}) = A^T(\mathbf{z}^{-1})A(\mathbf{z}) = I_m, \\ B(\mathbf{z}^{-1}) = -B^T(\mathbf{z}) \end{aligned}$$

where the complex vector variable

$$\begin{aligned} \mathbf{z} \triangleq (z_1, z_2, \ldots, z_n) \text{ and} \\ \mathbf{z}^{-1} \triangleq (z_1^{-1}, z_2^{-1}, \ldots, z_n^{-1}). \end{aligned}$$

Again, the Cayley transform of such a discrete domain reactance matrix $B(z)$ is a discrete domain paraunitary matrix. Here, only polynomial matrices are of interest. Such a matrix whose highest degree in indeterminate z_i (or z_i in ring $K[z_1^{-1}, z_2^{-1}, \ldots, z_n^{-1}]$) is d_i, $i = 1, 2, \ldots, n$ has a McMillan degree $(d_1, d_2, \ldots, d_n)$.

3. MULTIDIMENSIONAL FILTER BANK REALIZATION

Any univariate $(m \times m)$ FIR paraunitary matrix can be factored into a product of (McMillan) degree one $(m \times m)$ paraunitary building blocks and an additional unitary matrix (Gao *et al.*, 2001). The situation is different in the multivariate case. A bivariate (2×2) FIR paraunitary matrix , of degree (2,2), for example may not be factorable as a product of lower degree paraunitary matrices (Park, 2001). These and higher order paraunitary matrix results are viewed in the setting of general linear group $GL_m(R)$ over the ring R.

The basic structure used in the realization of biorthogonal and the more restricted orthonormal filter banks are, respectively, the ladder and the lattice structures. The biorthogonal filter bank realization problem involves the study of $GE_m(R)$, the group generated by $(m \times m)$ elementary matrices. Cohn (Cohn, 1966) studied, in particular, rings R for which $GE_2(R) = GL_2(R)$ and called such a ring GE_2 for which every (2×2) invertible matrix is a product of elementary (2×2) matrices. He further showed that the ring of polynomials $K[z_1, z_2, \ldots, z_n]$ in the indeterminates $z_1, \ldots, z_n$ with coefficients in a field K is not a GE_2 ring for $n \geq 2$ (Cohn, 1966).
The basic building blocks in the two-channel filter bank ladder topology are

$$\begin{bmatrix} 1 & r_{12} \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ r_{21} & 1 \end{bmatrix}$$

where $r_{12}, r_{21} \in R$, and diagonal matrices containing units of R. In the case of ladder structure of an n-channel filter bank, the factorization of an analysis polyphase matrix as a finite product of elementary matrices $H_{ij}(r) \in GL_m(R)$ (where entries of $H_{ij}(r)$ are identical to those in the identity matrix I_m except for the off-diagonal entry $r \in R$ in the i^{th} row and j^{th} column) is needed. By Cohn's counterexample, this is not possible in general, for the two-channel case, but by Suslin's later result (Bose, 2003)[p. 177] is always possible in the m-channel $(m > 2)$ case when $R = K[z_1, z_2, \ldots, z_n]$

Orthonormal filter banks, characterized by paraunitary matrices, are special cases of biorthogonal filter banks. In the 2-channel case, they are realizable as a cascade of basic building blocks. When $R = K[z_1, z_2]$, atomic factorizations of lowest order may not be possible (Park, 2001). Note that a filter bank that is realizable with the ladder topology is realizable with the lattice topology, when $m = 2$, but not vice-versa. Orthonormal, like biorthogonal filter banks, are realizable and implementable when $R = K[z_1, z_2, \ldots, z_n]$, though a canonic configuration may not be feasible for the m-channel case, $m > 2$.

4. PROBLEM

Only for the sake of brevity, attention is restricted to the field $\mathbb{R}$ of reals. A square matrix $A(z_1, z_2) \in \mathbb{R}^{m \times m}[z_1, z_2]$ is paraunitary provided

$$A(z_1, z_2)A^T(z_1^{-1}, z_2^{-1}) = I_m$$

where I_m is the identity matrix of order m. Therefore,

$$\det A(z_1, z_2) = \pm z_1^{-l_1} z_2^{-l_2}$$

where l_1, l_2 are integers. The problems are:

P1 to decompose $A(z_1, z_2)$ as a product of lower degree "atomic" paraunitary matrices, where an atomic paraunitary matrix cannot be decomposed any further and

P2 parameterize, if possible, the atomic matrices in such a way that filter bank realization is facilitated (when $m = 2$, for example, it is well known that a univariate paraunitary matrix can be factored as a cascade of degree one lattices).

According to the procedure of Guiver and Bose (Guiver and Bose, 1982), a nonsingular matrix $F(z_1, z_2) \in Q^{m\times m}[z_1, z_2]$, where Q is an arbitrary but fixed field, whose determinant has the irreducible factorization

$$\det\ F(z_1, z_2) = \prod_{i=1}^{k} f_i(z_1, z_2)$$

can constructively be factored as

$$F(z_1, z_2) = \prod_{i=1}^{k} F_i(z_1, z_2),$$
$$\det\ F_i(z_1, z_2) = f_i(z_1, z_2),$$

through, importantly, computations only in the base field Q,where $F_i(z_1, z_2) \in Q^{m\times m}[z_1, z_2]$ is an atomic factor of $F(z_1, z_2)$. This factorization is unique upto unimodular orthogonal matrices. With the imposition of the paraunitary constraint on $F(z_1, z_2)$, the application of the Guiver-Bose algorithm may not yield all paraunitary factors $F_i(z_1, z_2)$, for $i = 1, 2, \ldots, k$. The question then becomes if an atomic factorization involving only paraunitary factors is obtainable from the initial factorization.

It is well known that in the univariate case, the parametrization,

$$\prod_i (I - \mathbf{v_i}\mathbf{v_i^T}(1 - z^{-1}))$$

for the unit norm vectors v_i is complete. The parameterized family of counterexample in (Park, 2001) implies that the set of products of non-commutative degree 1 atomic factors in $\prod_i (I - \mathbf{v_i}\mathbf{v_i^T}(1 - z_1^{-1}))$ and $\prod_i (I - \mathbf{u_i}\mathbf{u_i^T}(1 - z_2^{-1}))$ (where v_i's and u_i's are unit norm vectors) is not complete.

5. NEW RESULTS

Lannes and Bose (Lannes and Bose, 2005) have arrived at a constructive factorization procedure for a polynomial paraunitary matrix as a product of atomic paraunitary matrices, each of lowest degree. The generic form of these paraunitary factors results from the use of projection matrices. In the univariate case each factor is of degree 1. In the multivariate case the generic factor is constructed based on the assumption of its existence .

5.1 Projection operator on the image of a singular matrix

$M_n(K)$ denotes the space of square matrices of size m taking its coefficients over a field K. Likewise, $M_{nm}(K)$ is the space of $n \times m$ rectangular matrices. Here the fileld K will denote either R or C. Let M be a singular matrix of $M_n(K)$(or not of full rank in $M_{nm}(K)$) of rank $r < n$. The image of M forms a subspace W of K^n. There is a bounded linear map P satisfying $P^2 = P$ from K^n onto W and P is called a projection operator (Goldberg, 1966).The columns of P are the projections of the standard basis vectors and W is the image of P. Form $T = MM^*$, where M^* is the hermitian conjugate of M. Note that T is a self-adjoint operator and $Im(T) = Im(M)$. Now, define the characteristic polynomial associated with T, as $\chi_T(x)$. M, and hence T, being of rank r implies hat $\chi_T(x) = x^{(n-r)}Q(x)$, where $Q(0) \neq 0$. Then $\frac{Q(T)}{Q(0)}$ is the projection operator on T's kernel and $P = I - \frac{Q(T)}{Q(0)}$ is the projection operator on T's image. A square matrix P is an orthogonal projection matrix iff $P^2 = P = P^*$, where P^* denotes the adjoint of the matrix P. In particular, for all matrices A and B such that $Im(A) \subseteq Im(M)$ and $Im(B) \subseteq Im(M)^\perp$

$$PA = A \quad , \quad PB = 0$$

5.2 Complete solution in the multichannel univariate case

Theorem 1. If P is the projection matrix associated with A_0, then $P+(I-P)z^{-1}$is a paraunitary left factor of the paraunitary matrix $A(z^{-1})$.

The proof of this theorem can be found in (Lannes, 2004), (Lannes and Bose, 2005). A similar type of projection operator has been independently used about the same time in (Hardin *et al.*, 2004)[p. 379] . This leads to a very simple and powerful factorization procedure for all 1D paraunitary matrices, by iteratively repeating this factor extraction step, until the remaining matrix is a polynomial paraunitary matrix of degree one.

5.3 Extension to the multivariate case

The factorization algorithm presented in the previous section can be extended to the factorization of non-separable multivariate matrices when such a factorization exists. Indeed, as shown by Park (Park, 2001) the complete factorization into paraunitary matrices is not always possible, even if a general factorization exist (Guiver and Bose, 1982), (Bose, 2003) . If the paraunitary factorization exists, it can be obtained from Guiver and Bose's factorization algorithm (Guiver and Bose, 1982), (Bose, 2003) by the introduction of well-chosen unitary matrices in between the factors. But one can also again directly apply the projection operator factorization. The condition for which the factorization is possible emerges rather quickly from the algorithm and it is interesting to notice that it comes back to the same condition necessary to convert the Guiver and Bose's factorization into a paraunitary factorization.For details and examples, see (Lannes, 2004).

6. CONCLUSIONS

A projection operator provides a mechanism for extracting atomic paraunitary polynomial matrix factors from a polynomial paraunitary matirix. In the univariate case, this procedure is simpler than existing procedures (P.P.Vaidyanathan, 1993), where the corresponding projection matrix is restricted to be of rank 1 and is difficult to construct. It is conjectured that there exists at least one bivariate (and multivariate) polynomial paraunitary matrix of arbitrary but fixed McMillan degree that is an atomic (irreducible) factor. Therefore, there cannot be a canonic realization structures like the cascade of lattices (in the 1-D two channel case) for the 2-D two channel case. The procedure here is quite different from other attempts made to factor multivariate paraunitary matrices including the recent one in (Delgosha and Fekri, 2004), where the rational matrix case (IIR) was also considered. Due to the use of nonsquare bivariate polynomial matrices in signal processing and control, the role of paraunitariness in such a setting needs to be investigated.

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