A SEMI ANALYTIC APPROACH TO THE ROBUST STABILITY OF QUASIPOLYNOMIALS [1]

J. Santos [*,2] **S. Mondié** [*,3] **V. Kharitonov** [*]

[*] *CINVESTAV-IPN, Departamento de Control Automático, Av. IPN 2508, Col. Zacatenco, México D. F. A.P. 14-740, MEXICO C.P. 07360. E-mail: smondie@ctrl.cinvestav.mx*

Abstract: A semi-analytic tool for the study of the robust stability of time delay systems is presented. The approach is based on graphical methods. The conditions of the Finite Nyquist Theorem for quasipolynomials given in (Santos *et al.*, 2003b) are improved. The Finite Inclusions Theorem for polytopic and interval quasipolynomials families is derived. An algorithm describing how to apply this new technique to analyze the robust stability of families of quasipolynomials is presented. The effectivity of this algorithm, which is implemented in Matlab, is validated with the analysis of two examples.

Keywords: Timed-delay systems, Robust stability, Graphical test.

1. INTRODUCTION

The topic of this contribution is the robustness analysis of time delay systems of retarded type (Bellman and Cooke, 1963) of the form

$$\dot{x}(t) = \sum_{i=0}^{q} A_i x(t-\tau_i) \qquad (1)$$

where $A_i, i = 1, ..., q$ are real matrices of size $n \times n$ and $0 = \tau_0 < \tau_1 < ... < \tau_q$ are time delays.

The characteristic function of system (1) is a quasipolynomial which can be written as

$$f(s) = s^n + \sum_{k=0}^{n-1} \sum_{l=0}^{m} \alpha_{kl} s^k e^{-\beta_l s} \qquad (2)$$

where $0 = \beta_0 < \beta_1 < ... < \beta_m$ are linear combinations of delays τ_i, $i = 1, ..., q$ and α_{kl}, $k = 0, ..., n-1$, $l = 0, ..., m$ are multilinear combinations of the coefficients of matrices A_i, $i = 1, ..., q$.

The quasipolynomial $f(s)$ is of the retarded type and has a finite number of roots in any right half complex plane. In what follows we assume that $f(s)$ have no roots on the imaginary axis of the complex plane. It is well known that the stability of the quasipolynomial (2) implies that of the time delay system (1).

In this paper we consider the stability problem for a class of uncertain systems (1) where the characteristic function involves a polytope of quasipolynomials of retarded type.

Our approach is within the graphical framework. The main goal is to reduce the computational complexity associated with the evaluation of the value set of the family along the imaginary axis. The main ingredients of the substantial reduc-

[1] Supported by CONACyT 41276-Y
[2] Supported by CONACyT 130255
[3] On leave at Heudiasyc, Compiègne, France

tion we achieve is based on the following ideas: The first one is an extension of the Finite Inclusion/Exclusion Theorems obtained for polynomials in (Rantzer, 1989) and (Djaferis, 1995) to the case of quasipolynomials. Indeed, this allows to perform the stability analysis of the quasipolynomial family based on the location on the complex plane of the value set of the family computed for a finite number of frequencies only. The second one, is the well known fact that the convexity of the polytopic family is inherited by its value set, which allows to restrict our search to the set of extreme quasipolynomials of the family. The third one is that the stability of the extreme quasipolynomials of a family can be determined from that of the subset of so-called essential quasipolynomials given in (Kharitonov and Zhabko, 1994).

The note is organized as follows: In section 2 the argument principle, and related stability results for quasipolynomials are reminded, and in section 3 we prove new, easier to verify, stability conditions for the Finite Nyquist Theorem which plays a central role in our work. In section 4, we prove the Finite Inclusion Theorem for polytopic families of quasipolynomials and for interval families of quasipolynomials. In Section 5, we give an efficient algorithm for the practical utilization of the Finite Inclusions Theorem. Finally, in Section 6 we present some illustrative examples and the note ends with some concluding remarks.

2. STABILITY OF RETARDED QUASIPOLYNOMIALS WITH KNOWN COEFFICIENTS

In this section we recall that the numerical change of the argument of a quasipolynomial in a given interval on the imaginary axis allows to conclude on its stability. First, we begin with the following proposition.

Proposition 1. (Santos *et al.*, 2003a) Let $0 < \varepsilon \leq \frac{\pi}{2}$ be given and ξ the positive root of the polynomial equation $C_0\xi^n + C_1\xi^{n-1} \cdots + C_{n-1}\xi = \sin(\frac{\varepsilon}{2})$, where $0 \leq C_k = \sum_{l=0}^{m} |\alpha_{kl}|$, $k = 0, ..., n-1$, $l = 0, ..., m$. Then, the change of the argument of $f(j\omega)$ for $\omega \in [R, \infty)$, with $R \geq \frac{1}{\xi_\varepsilon}$, is bounded by ε.

The analysis of the contributions of a Nyquist contour to the argument of $f(j\omega)$ along with the Argument Principle leads to the following result.

Theorem 2. (Santos *et al.*, 2003b) Let $0 < \varepsilon \leq \frac{\pi}{2}$ be given and let $\Phi_{[0,jR]}$ be the net change of the argument of $f(j\omega)$ for $\omega \in [0, R]$, with R given in Proposition 1. Then, the unique integer $N \geq 0$ that satisfies the inequality

$$-\varepsilon < N\pi - (n\frac{\pi}{2} - \Phi_{[0,jR]}) < \varepsilon \tag{3}$$

is equal to the number of roots of $f(s)$ with $\mathrm{Re}(s) > 0$.

The above result gives the number of roots inside a Nyquist contour in the right half plane by computing the net change of the argument on a finite segment of the imaginary axis. For $N = 0$ and $0 < \varepsilon \leq \frac{\pi}{2}$, the following stability condition is obtained.

Corollary 3. (Santos *et al.*, 2003b) Let $0 < \varepsilon \leq \frac{\pi}{2}$ be given and let $\Phi_{[0,jR]}$ be the net change of the argument of $f(j\omega)$ for $\omega \in [0, R]$, where R is given by Proposition 1, then $f(s)$ is stable if and only if

$$n\frac{\pi}{2} - \frac{\pi}{2} < \Phi_{[0,jR]} < n\frac{\pi}{2} + \frac{\pi}{2}. \tag{4}$$

3. FINITE NYQUIST THEOREM FOR REAL QUASIPOLYNOMIALS

In this section, we introduce and prove new conditions for the Finite Nyquist Theorem for quasipolynomials. These conditions are equivalent to those provided in (Santos *et al.*, 2003b), but these are easier to verify because more information on how the first and last arguments must be chosen is given.

Theorem 4. The quasipolynomial $f(s)$ is stable if and only if there exists an integer $r \geq 1$, angles $\theta_i \in \mathbb{R}$ for $i \in \mathbb{N}$, $0 \leq i \leq r$ and real frequencies $0 = \omega_0 < \omega_1 < \cdots < \omega_r \leq R$, where R is defined in Proposition 1, such that

$$\theta_0 = 0, \tag{5}$$

$$n\frac{\pi}{2} - \frac{\pi}{2} < \theta_r < n\frac{\pi}{2} + \frac{\pi}{2}, \tag{6}$$

$$\forall\ 0 \leq i < r-1: \quad |\theta_{i+1} - \theta_i| \leq \pi \tag{7}$$

$$\forall\ 0 \leq i \leq r: \quad f(j\omega_i) \neq 0 \tag{8}$$

$$\forall\ 0 \leq i \leq r: \ \arg f(j\omega_i) \equiv \theta_i \pmod{2\pi}. \tag{9}$$

Proof.
Sufficiency:
Observe first that condition (3) of Theorem 2 implies that

$$n\frac{\pi}{2} - N\pi - \varepsilon \leq \Phi_{[0,jR]} \leq n\frac{\pi}{2} - N\pi + \varepsilon,\ \ 0 \leq \varepsilon < \frac{\pi}{2}.$$

Since N is a nonnegative integer, $\Phi_{[0,jR]}$ is such that

$$\Phi_{[0,jR]} < n\frac{\pi}{2} + \frac{\pi}{2}. \tag{10}$$

In the following, we prove that $n\frac{\pi}{2} - \frac{\pi}{2} < \Phi_{[0,jR]}$. Observe first that without any loss of generality we may assume that all differences $\theta_{j+1} - \theta_j$ have the same sign. If it is not the case and there exists $j < r-2$, such that $\theta_{j+2} - \theta_{j+1}$ and $\theta_{j+1} - \theta_j$ have opposite sign then either

$$0 \leq \theta_{j+2} - \theta_{j+1} < \pi \text{ and } -\pi < \theta_{j+1} - \theta_j \leq 0,$$

or

$$-\pi \le \theta_{j+2} - \theta_{j+1} < 0 \text{ and } 0 < \theta_{j+1} - \theta_j \le \pi.$$

In both cases $-\pi < \theta_{j+2} - \theta_j < \pi$ holds. Moreover,

$$\begin{aligned}\theta_r - \theta_0 &= \sum_{i=0}^{r-1}(\theta_{i+1} - \theta_i) \\ &= \sum_{i=0}^{j-1}(\theta_{i+1} - \theta_i) + \theta_{j+2} - \theta_{j+1} \\ &+ \theta_{j+1} - \theta_j + \sum_{i=j+2}^{r-1}(\theta_{i+1} - \theta_i) \\ &= \sum_{i=0}^{j-1}(\theta_{i+1} - \theta_i) + \theta_{j+2} - \theta_j \\ &+ \sum_{i=j+2}^{r-1}(\theta_{i+1} - \theta_i). \qquad (11)\end{aligned}$$

So, we can exclude θ_{j+1} and ω_{j+1} from our considerations. In this way, in a finite number of steps, we arrive at the situation when all differences $\theta_{i+1} - \theta_i$, have the same sign. Furthermore, it follows from (5) and (6) that these differences are positive.

Now, observe that

$$\Phi_{[0,jR]} \ge \sum_{i=0}^{r-1} |\theta_{i+1} + 2k_{i+1}\pi - \theta_i - 2k_i\pi| . \quad (12)$$

As $|X| \le |X + 2\pi k|$ holds for all integer k and all X such that $|X| \le \pi$, (12) it follows that

$$\Phi_{[0,jR]} \ge \sum_{i=0}^{r-1} |\theta_{i+1} - \theta_i| .$$

All the differences are positive,

$$\begin{aligned}\Phi_{[0,jR]} &\ge \sum_{i=0}^{r-1} |\theta_{i+1} - \theta_i| = \sum_{i=0}^{r-1} (\theta_{i+1} - \theta_i) \\ &= \theta_r - \theta_0, \qquad (13)\end{aligned}$$

then it follows from (6) that

$$\Phi_{[0,jR]} > n\frac{\pi}{2} - \frac{\pi}{2}. \quad (14)$$

Finally, from (13) and (14) and Corollary 3 we conclude that the quasipolynomial $f(s)$ is stable.

Necessity:

Now, we assume that the quasipolynomial $f(s)$ is stable. It follows from (2) and Lemma 2.14 in (Stépán, 1989) that $f(0)$ is a positive real number, therefore, we can chose $\omega_0 = 0$ and $\theta_0 = 0$. Next, according to Corollary 3, $n\frac{\pi}{2} - \frac{\pi}{2} < \Phi_{[0,jR]} < n\frac{\pi}{2} + \frac{\pi}{2}$ and we can define

$$\begin{aligned}\theta_i &= i\frac{\pi}{2},\ i = 0, ..., n-1, \\ \theta_n &= (n-1)\frac{\pi}{2} + \epsilon < \Phi_{[0,jR]},\ \ |\epsilon| \ll 1.\end{aligned}$$

Observe that (5), (6) and (7) are satisfied. Finally, let ω_i, $i = 0, ..., n$ be the smallest value of the frequency such that $\arg f(j\omega_i) = \theta_i$ with $\omega_i > \omega_{i-1}$, $i \ge 1$. The result is proved. ■

4. STABILITY OF QUASIPOLYNOMIALS WITH UNCERTAIN COEFFICIENTS

In what follows, we show that the Finite Nyquist Theorem can be employed to study the robust stability of a polytopic family of retarded quasipolynomials. This result is named the Finite Inclusions Theorem (*F.I.T.*).

4.1 Robust stability of a polytopic family of quasipolynomials

Let us consider a polytopic family of quasipolynomials described by

$$F(s) = \left\{ \sum_{l=1}^{T} \mu_l f_l(s) |\ \mu_l \ge 0, \sum_{l=1}^{T} \mu_l = 1 \right\}, \quad (15)$$

where $f_l(s)$, $l = 1, ..., T$ are quasipolynomials of the form (2). We know from the Edge Theorem (Fu *et al.*, 1989) that the family $F(s)$ is robustly stable if and only if all the edges of $F(s)$ are stable. Then a reduction of the number of one parameter families analyzed can be achieved by using the Zero Exclusion Principle with the help of the concept of value set that follows.

For a given a frequency $\omega \in \mathbb{R}$, the value set of $F(s)$ at this frequency is defined as

$$V_F(\omega) = \left\{ \sum_{l=1}^{T} \mu_l f_l(j\omega) |\ \mu_l \ge 0, \sum_{l=1}^{T} \mu_l = 1 \right\}.$$

The Zero Exclusion Principle states that one can conclude on the stability of the family $F(s)$ by observing the graphical behavior in the complex plane of the value set $V_F(\omega)$ for all frequencies $\omega \ge 0$. The main inconvenient of this technique is that involves many computations. Therefore it would be desirable to have a similar graphical result with less computations.

We propose a graphical result which also uses the value set $V_F(\omega)$ of the family $F(s)$ for a finite number of frequencies to determine its robust stability. If the value set lies in appropriately defined sectors for this set of frequencies, then the family $F(s)$ is stable. Due to the fact that the convexity of the polytopic family is inherited by the value set, for each $\omega \in \mathbb{R}$ the value set $V_F(\omega)$ is the convex hull of the complex numbers $f_l(j\omega)$, $l = 1, ..., T$. As a consequence, we just need to know the vertices of $V_F(\omega)$ in order to determine the sector inclusions. In what follows, we see that The Finite Nyquist Theorem can be extended to robust stability analysis purposes.

This result is actually a corollary of the Finite Nyquist Theorem.

Theorem 5. The polytopic family defined in (15) is robustly stable if there exists $r \geq 1$ sectors $S_i = \{\rho e^{j\theta} \,|\, \rho > 0, \quad a_i < \theta < b_i\}$ for $0 \leq i \leq r$ and real frequencies $0 = \omega_0 < \omega_1 < \cdots < \omega_r$ such that

$$a_0 = b_0 = 0, \tag{16}$$

$$n\frac{\pi}{2} - \frac{\pi}{2} \leq a_r < b_r \leq n\frac{\pi}{2} + \frac{\pi}{2}, \tag{17}$$

$\forall \ 0 \leq i < r-1:$

$$\max\{b_{i+1} - a_i, b_i - a_{i+1}\} \leq \pi, \tag{18}$$

$\forall \ 1 \leq l \leq T, \quad \forall \, 0 \leq i \leq r:$

$$f_l(j\omega_i) \subset S_i = \{\rho e^{j\theta} \,|\, \rho > 0, a_i < \theta < b_i\}. \tag{19}$$

Proof. For $l = 1, ..., T$ condition (19) implies that for all i, $f_l(j\omega_i) \neq 0$ and there exists an argument $\theta \in (a_i, b_i)$ such that $\arg f_l(j\omega_i) \equiv \theta \pmod{2\pi}$. Then conditions (16) and (17) imply that θ satisfies the conditions (5), (6), (8) and (9) of Theorem 4. Furthermore, from the inequality

$$|\theta - \theta'| < \sup_{\substack{\theta \in (a,b) \\ \theta' \in (c,d)}} |\theta - \theta'| = \max\{d - a, b - c\},$$

it follows from (18) that condition (7) is also satisfied, hence every $f_l(s)$, $l = 1, ..., T$ is stable. The conditions (19) imply that $f_l(j\omega_i)$, $l = 1, ..., T$ belong to the sectors S_i. Since the value set $V_F(\omega_i)$, $i = 0, ..., r$ is the convex hull of the complex numbers $f_l(j\omega_i)$, $l = 1, ..., T$, condition (19) implies that the value set $V_F(\omega)$ also belongs to the sector S_i, then all the elements of this family satisfy Theorem 4 hence they are stable and we conclude that the family $F(s)$ is robustly stable. ■

4.2 Robust stability of an interval quasipolynomial family

A particular example of polytopic families is an interval family of quasipolynomials of the form (2) described by

$$I = \left\{ \begin{array}{c} s^n + \sum_{k=0}^{n-1} \left(\sum_{l=0}^{m} \alpha_{kl} s^k \right) e^{-\beta_l s}, \\ \alpha_{kl} \in [\underline{\alpha}_{kl}, \overline{\alpha}_{kl}], \\ k = 0, ..., n-1, l = 0, ..., m. \end{array} \right\} \tag{20}$$

The total number of vertex quasipolynomials generated by this class of uncertainty is $2^{n(m+1)}$ because each coefficient a_{kl}, $k = 0, ..., n-1$, $l = 0, ..., m$ has two extreme points.

According to the explanation of the previous subsection, we can apply the Finite Inclusion Theorem to study the robust stability of (20) considering the family I as a polytopic family of $2^{n(m+1)}$ vertex elements. Nevertheless, the main inconvenient of this approach is the large number of elements that we have to manage. We can achieve a further reduction of the number of vertexes because, as shown in (Kharitonov and Zhabko, 1994), the set of essential vertexes of the whole family I is at most 4^{m+1}. It is given by the set

$$U_I = \left\{ \sum_{l=0}^{m} p_l^{(z_l)}(s) e^{\beta_l s}, \quad \begin{array}{l} z_l \in \{1, 2, 3, 4\}, \\ l = 0, ..., m, \end{array} \right\} \tag{21}$$

where $p_l^{(z_l)}(s)$ are the polynomials

$$p_l^{(1)}(s) = \underline{\alpha}_{0l} + \underline{\alpha}_{1l} s + \overline{\alpha}_{2l} s^2 + \overline{\alpha}_{3l} s^3 + \cdots ;$$
$$p_l^{(2)}(s) = \overline{\alpha}_{0l} + \underline{\alpha}_{1l} s + \underline{\alpha}_{2l} s^2 + \overline{\alpha}_{3l} s^3 + \cdots ;$$
$$p_l^{(3)}(s) = \overline{\alpha}_{0l} + \overline{\alpha}_{1l} s + \underline{\alpha}_{2l} s^2 + \underline{\alpha}_{3l} s^3 + \cdots ;$$
$$p_l^{(4)}(s) = \underline{\alpha}_{0l} + \overline{\alpha}_{1l} s + \overline{\alpha}_{2l} s^2 + \underline{\alpha}_{3l} s^3 + \cdots .$$

Now, according to Theorem 5, we only have to verify the inclusions of the elements of the set U_I into each sector S_i. Then we can state the following result for the robust stability analysis of the interval quasipolynomial I.

Corollary 6. The interval quasipolynomial family I defined in (20) is robustly stable if there exist $r \geq 1$ sectors $S_i = \{\rho e^{j\theta} \,|\, \rho > 0, \quad a_i < \theta < b_i\}$ for $0 \leq i \leq r$ and real frequencies
$0 = \omega_0 < \omega_1 < \cdots < \omega_r$ such that the conditions

$$a_0 = b_0 = 0,$$

$$n\frac{\pi}{2} - \frac{\pi}{2} \leq a_r < b_r \leq n\frac{\pi}{2} + \frac{\pi}{2},$$

$\forall \ 0 \leq i < r-1:$

$$\max\{b_{i+1} - a_i, b_i - a_{i+1}\} \leq \pi,$$

$0 \leq i \leq r:$

$$U_I(j\omega_i) \subset S_i = \{\rho e^{j\theta} \,|\, \rho > 0, a_i < \theta < b_i\}.$$

with $U_I(s)$ defined in (21), are satisfied.

5. PRACTICAL APPLICATION OF F.I.T.

Our proposal for the practical utilization of the Finite Inclusions Theorem to determine the robust stability of a family $F(s)$ of quasipolynomials is based on the following general strategy. We first need to determine if the corresponding central quasipolynomial $f_0(s)$ is stable. If so, then we determine a set of frequencies $W = \{\omega_0, \omega_1, \ldots, \omega_r\}$ in order to verify all the conditions of Theorem 5. Let $A(\omega)$ and $B(\omega)$ be the real and imaginary parts of $f_0(j\omega)$. Plotting $A(\omega)$ vs $B(\omega)$ in the complex plane for $0 \leq \omega < \infty$ gives a curve that goes from the real positive axis to the left of the complex plane passing at least through n quadrants because $f_0(s)$ is stable. For each frequency ω for which this curve crosses the imaginary or real axis we can assure that the value set $V_F(\omega)$

also crosses the same axis. These frequencies are a good initial choice for the frequencies of the set W. If these frequencies do not allow to conclude on the stability, additional frequencies are added to the set W.

The above describe general strategy is implemented with the following algorithm.

5.1 F.I.T. Algorithm

Step 1. Given the polytopic family $F(s)$ described by (15) use Corollary 3 to determine the stability of the central quasipolynomial $f_0(s) = \frac{1}{T}\sum_{l=1}^{T} f_l(s)$. If $f_0(s)$ is stable then follow with Step 2, otherwise the algorithm stops and the family $F(s)$ is not robustly stable.

Step 2. Determine a set of frequencies $W = \{\omega_0, \omega_1, \ldots, \omega_r\}$ to evaluate the conditions of the Finite Inclusions Theorem and set $i = 0$.

Step 3. For the frequency ω_i of the set W, determine the sector $S_i \doteq (a_i, b_i)$ such that $F(j\omega_i) \subset S_i$.

Step 4. For the frequency ω_{i+1} of the set W, determine the sector $S_{i+1} \doteq (a_{i+1}, b_{i+1})$ such that $F(j\omega_{i+1}) \subset S_{i+1}$.

Step 5. Compute $\gamma = \max\{b_{i+1} - a_i, b_i - a_{i+1}\}$.

Step 6. If $\gamma \leq \pi$, then $i = i + 1$ and go to Step 3. If $\gamma \leq \pi$ and $i = r - 1$ go to Step 8.

Step 7. If $\gamma > \pi$, set $\eta = 0$ and do the following:

a).- determine a frequency $\omega_* \in (\omega_i, \omega_{i+1})$ and a sector $S_* \doteq (a_*, b_*)$ such that $F(j\omega_*) \subset S_*$.

b).- compute $\gamma_* = \max\{b_* - a_i, b_i - a_*\}$ where (a_i, b_i) was determined in Step 3.

c).- If $\gamma_* \leq \pi$, then set $\omega_i = \omega_*$ and go to Step 4.

d).- If $\gamma_* > \pi$ and $\eta \leq 4$, then $\eta = \eta + 1$ and return to point a).

e).- If $\gamma_* > \pi$ and $\eta > 4$ the algorithm stops, and we can not say wether $F(s)$ **is or not robustly stable**.

Step 8. If the conditions $-\frac{\pi}{2} \leq a_0 < b_0 \leq \frac{\pi}{2}$ and $n\frac{\pi}{2} - \frac{\pi}{2} \leq a_r < b_r \leq n\frac{\pi}{2} + \frac{\pi}{2}$ are satisfied then $F(s)$ **is robustly stable**, otherwise add a new frequency $\omega_{r+1} = R$ and determine the sector S_{r+1} such that $F(j\omega_{r+1}) \subset S_{r+1}$.

Step 9. Compute $\gamma = \max\{b_{r+1} - a_r, b_r - a_{r+1}\}$. If $\gamma \leq \pi$ verify that $-\frac{\pi}{2} \leq a_0 < b_0 \leq \frac{\pi}{2}$ and $n\frac{\pi}{2} - \frac{\pi}{2} \leq a_{r+1} < b_{r+1} \leq n\frac{\pi}{2} + \frac{\pi}{2}$ hold. If both are satisfied then $F(s)$ **is robustly stable**, otherwise we can not say wether $F(s)$ **is or not robustly stable**.

5.2 Some comments on the FIT Algorithm

Now, we give some important comments on the algorithm presented.

(1) For the numerical computation of $\Phi_{[0,jR]}$ in Proposition 1, an appropriate discretization δ step of $[0, jR]$ must be chosen in order to get a correct computation of $\Phi_{[0,jR]}$.
(2) To get W in Step 2, determine all the frequencies $\omega \geq 0$ for which $\operatorname{Re} f_0(j\omega)$ or $\operatorname{Im} f_0(j\omega)$ are zero and select only the frequencies for which there is a true change of $\arg f_0(j\omega)$.
(3) Step 7 permits to add a new frequency $\omega_* \in (\omega_i, \omega_{i+1})$ when two consecutive sectors (S_i, S_{i+1}) are not in the same semi-plane, i.e. $\gamma > \pi$. The parameter η sets how many times this procedure is done. If this limit is reached the algorithm stops because it is impossible to satisfy the condition (18) of the Theorem 5, therefore we can not say anything about the stability or instability of $F(s)$.
(4) To determine $S_i \doteq (a_i, b_i)$ such that $F(j\omega_i) \subset S_i$, compute the set $\Phi_i = \left[\theta_i^1, ..., \theta_i^T\right]$, here θ_i^l, $l = 1, ..., T$, is the completed argument of $f_l(j\omega_i) \in \mathbb{C}$, $l = 1, ..., T$, for ω_i. Then $a_i = \min(\Phi_i) - \epsilon$, $b_i = \max(\Phi_i) + \epsilon$, $\epsilon > 0$ is chosen by the user.

An important result of our research is the implementation of this algorithm in MatLab. A detailed description of the algorithm and the corresponding programs is available in (Santos, 2004).

6. ILLUSTRATIVE EXAMPLES

In what follows we present two examples that were analyzed using the above mentioned toolbox. All the sectors are given in radians.

Example 7. Consider the following polytopic combination of quasipolynomials corresponding to 4 extreme points of the stability chart of an inverted pendulum studied in (Stépán and L., 2000):

$$F(s) = \left\{ \sum_{l=1}^{4} \mu_l f_l(s), \mu_l \geq 0, \sum_{l=1}^{4} \mu_l = 1 \right\} \quad (22)$$

where

$$f_1(s) = s^2 - 58.86 + (60.699375 + 8.09325s)e^{-0.1s},$$
$$f_2(s) = s^2 - 58.86 + (73.575 + 8.09325s)e^{-0.1s},$$
$$f_3(s) = s^2 - 58.86 + (73.575 + 14.344182s)e^{-0.1s},$$
$$f_4(s) = s^2 - 58.86 + (60.699375 + 14.344182s)e^{-0.1s}.$$

The corresponding frequencies and sectors that satisfy the F.I.T. conditions computed using **FIT_POLITOPICA** are shown in Table 1. The conclusion is that the family (22) is robustly stable.

Table 1. Frequencies and sectors for Example 7.

i	ω_i	$S_i:$	a_i	b_i
0	0	$S_0:$	-0.0175	0.0175
1	1.7426	$S_1:$	0.0694	1.4909
2	3.3908	$S_2:$	0.1534	2.1994
3	4.5240	$S_3:$	0.2460	2.5266
4	5.6097	$S_4:$	1.0819	3.3377
5	10.5468	$S_5:$	1.7573	3.5110

The graphical behavior of the value sets of the family (22) for the frequencies of Table 1 is depicted on Figure 1.

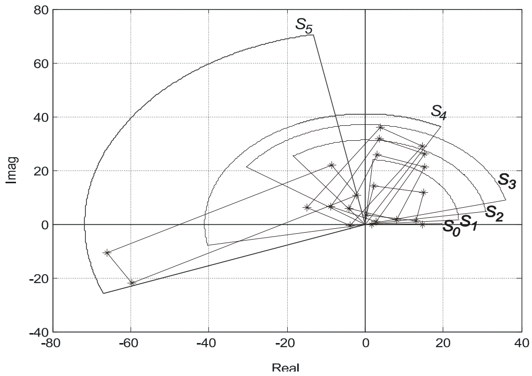

Fig. 1. Frequencies and sectors for Example 7.

Example 8. Consider the polytopic family of quasipolynomials

$$Q = s^3 + a_{20}s^2 + a_{10}s + a_{00} + (a_{21}s^2 + a_{11}s + a_{01})e^{-3s} \quad (23)$$

where

$$a_{20} \in [14.5, 17],\ a_{10} \in [11.5, 12.5],\ a_{00} \in [15.5, 16.5],$$
$$a_{21} \in [0.8, 2.9],\ a_{11} \in [2, 3],\ a_{01} \in [1, 2.5].$$

The total number of vertexes of this family are $2^6 = 64$ but, according to (Kharitonov and Zhabko, 1994) the essential vertexes are only $4^2 = 16$. The corresponding frequencies and sectors that satisfy the F.I.T. conditions computed using **FIT_INTERVALO** are shown in Table 2. The conclusion is that the family (23) is robustly stable.

Table 2. Frequencies and sectors for Example 8.

i	ω_i	$S_i:$	a_i	b_i
0	0	$S_0:$	-0.0175	0.0175
1	1.0199	$S_1:$	1.1119	2.0361
2	3.1249	$S_2:$	3.0865	3.1898
3	27.0000	$S_3:$	4.0960	4.2190

The value sets of (23) for the frequencies of the Table 2 are depicted with an appropriate scaling on Figure 2.

7. CONCLUDING REMARKS

The conditions of the Finite Nyquist Theorem are improved. The Finite Inclusions Theorem for

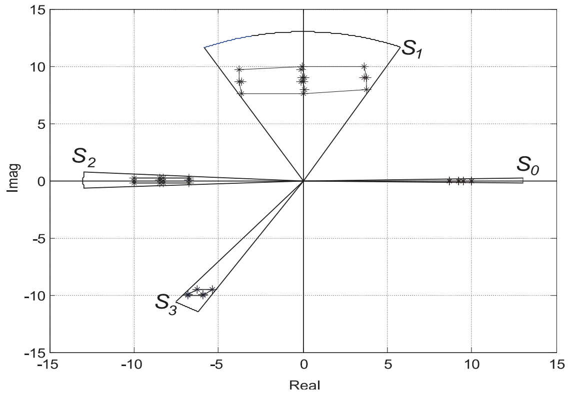

Fig. 2. Frequencies and sectors for Example 8.

robust stability of quasipolynomials is proved. An algorithm for the practical utilization of this result is also given. The effectivity of this algorithm which was implemented in Matlab is illustrated with two examples.

REFERENCES

Bellman, R. and K. L. Cooke (1963). *Differential-Difference Equations.* Academic Press. New York.

Djaferis, T. (1995). *Robust Control Design, a Polynomial Approach.* Kluwer Academic Publishers, Amherts Massachusetts.

Fu, M., A. W. Olbrot and M. P. Polis (1989). Robust stability for time-delay systems: The edge theorem and graphical test. *IEEE Trans. on Automatic Control* **34**, 813-819.

Kharitonov, V. L. and A. Zhabko (1994). Robust stability for time delay systems. *IEEE Trans. on Automatic Control* **39**, 2388-2397.

Rantzer A. (1989). A finite zero exclusion principle, In: *Robust Control of Linear Systems and Nonlinear Control, Proceedings of the International Symposium MTNS-89.* Vol. 2. pp. 239-245. Birkhäuser Boston. U. S.

Santos, J. (1994) *Método para el análisis de la estabilidad robusta de sistemas con retardo.* Ph. D. Thesis, CINVESTAV-IPN, México, (In Spanish).

Santos, J., S. Mondié and V. L. Kharitonov (2003*a*). Finite inclusions theorem based algorithm for robust stability of time delay systems. In: *AMCA*. Ensenada B. C. México.

Santos, J., S. Mondié and V. L. Kharitonov (2003*b*). Robust stability of time delay systems and the finite inclusions theorem. *4th IFAC Workshop on TDS, Rocquencourt, France.*

Stépán, G. (1989). *Retarded Dynamical Systems: Stability and Characteristic Function*, Longman Scientific and Technical. U. K.

Stépán, G. and Kollár L. (2000). Balancing with reflex delay. *Mathematical and Computer Modelling.* **31**, 199-205.