SYNTHESIS OF OPTIMAL FEEDBACKS FOR LINEAR SYSTEMS UNDER STATE CONSTRAINTS

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Abstract: A method of on-line constructing an optimal feedback control for linear systems under state and control constraints is suggested. At first, a fast algorithm of constructing an open-loop control is worked out. It is based on successive correction of the solution to the optimal control problem with intermediate state constraints by changing the location of constraints and introducing additional constraints for identifying contact points and boundary arcs. Then, the scheme of constructing open-loop solutions is modified in order to generate a realization of optimal closed-loop control in any control process.

Keywords: optimal control, linear systems, constraints, algorithms

1. INTRODUCTION

In realistic control problems state constraints are as typical as control constraints. By now for such problems the theory of necessary optimality conditions of the Maximum Principle type in the class of measurable functions has been developed quite completely (Hartl *et al.*, 1995). But the structure of the optimal control (OC) in the class of measurable functions can be so complicated that it could not be implemented with the use of up-to-date technical facilities. So, it is an actual problem to compute an optimal open-loop control in such a class of functions that are practically implementable and allow to get solutions as close as is wished to solutions in the class of measurable functions with respect to performance.

In the paper a method of constructing open-loop and closed-loop solutions to linear OC problems with state constraints is presented. It is based on the fast algorithm of solving a linear OC problem with intermediate state constrained (Balashevich *et al.*, 2001). Methodologically, the suggested implementation of optimal closed-loop control develops the approach (Gabasov *et al.*, 1995; Gabasov *et al.*, 2000). According to this approach, before starting the actual control process preliminary work to form the framework of the solution is performed, and then in the course of the control process preliminary results are getting more precise on the basis of the realized states. The efforts on treating other admissible but not realized in the actual process states are not spent.

2. PROBLEM STATEMENT

On a fixed interval $T = [0, t^*]$ consider an OC problem for a linear system with a state constraint

$$c'x(t^*) \to \max, \quad \dot{x} = Ax + bu,\ x(0) = x_0, \quad (1)$$

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$$d'x(t) \geq \alpha, \quad |u(t)| \leq 1, \quad t \in T.$$

Here $x = x(t)$ is an n-vector of system state at an instant t; $u = u(t)$ is a value of a scalar control; $A \in \mathbb{R}^{n\times n}$; $b, c, d \in \mathbb{R}^n$; $\alpha \in \mathbb{R}$ are given.

An integer k is called an order of state constraint $d'x(t) \geq \alpha$, $t \in T$, if

$$d'A^i b = 0, \ i = \overline{0, k-2}, \quad d'A^{k-1}b \neq 0.$$

A piecewise constant function $u(t)$, $t \in T$, is said to be discrete (with a quantization step $h = t^*/N$) if it has the form $u(\vartheta) = u(t)$, $\vartheta \in [t, t+h[$, $t \in T_h = \{0, h, \ldots, t^* - h\}$, N is an integer.

A constrained piecewise continuous function $|u(t)| \leq 1$, $t \in T$, is called an admissible control if it together with a corresponding trajectory $x(t)$, $t \in T$, of (1) satisfies the relations:

1) an output signal $y(t) = d'x(t)$, $t \in T$, of system (1) satisfies the state constraint on the set T_h: $y(t) \geq \alpha$, $t \in T_h$;

2) the ends of boundary arcs of the output signal belong to T_h: $T(u) = \{t \in T : \ y(t) \equiv \alpha\} = \cup_{r=1}^{r^*} T^r$, $T^r = [\theta_r, \theta^r[$, θ_r, $\theta^r \in T_h$, $r = \overline{1, r^*}$; $\theta_1 < \theta^1 < \ldots < \theta_{r^*} < \theta^{r^*}$;

3) on boundary arcs $T(u)$ the control $u(t)$, $t \in T$, has the form $u(t) = u(x(t))$, $t \in T(u)$, where $u(x) = d'A^k x / d'A^{k-1}b$;

4) on the set $\bar{T}(u) = T \backslash T(u)$ the control $u(t)$, $t \in \bar{T}(u)$, is a discrete function.

An admissible control $u^0(t)$, $t \in T$, is called an optimal open-loop control of problem (1) if it provides the maximum value for performance index of problem (1).

To define optimal feedback control let us imbed problem (1) into the family

$$\begin{aligned} &c'x(t^*) \to \max, \quad \dot{x} = Ax + bu, \ x(\tau) = z, \quad (2)\\ &d'x(t) \geq \alpha, \quad |u(t)| \leq 1, \quad t \in T^\tau = [\tau, t^*], \end{aligned}$$

depending on $\tau \in T_h$ and $z \in \mathbb{R}^n$.

Let $u^0(t|\tau, z)$, $t \in T^\tau$, be an optimal open-loop control of problem (2) for a position (τ, z); $T(u|\tau, z) = \cup_{r=1}^{r^*(\tau,z)} [\theta_r(\tau, z), \theta^r(\tau, z)[$ be a set of boundary arcs corresponding $u^0(t|\tau, z)$, $t \in T^\tau$; X_τ be a set of $z \in \mathbb{R}^n$ for which there exists $u^0(t|\tau, z)$, $t \in T^\tau$, and $\theta_1(\tau, z) > \tau$.

A function

$$u^0(\tau, z) = u^0(\tau|\tau, z), \quad z \in X_\tau, \ \tau \in T_h, \qquad (3)$$

is called an optimal (discrete) feedback of (1).

As a trajectory of the system closed by optimal feedback (3) and affected by a piecewise continuous disturbance $w(t)$, $t \in T$,

$$\dot{x} = Ax + bu^0(t, x) + w(t), \quad x(0) = x_0, \qquad (4)$$

a continuous solution to the equation

$$\dot{x} = Ax + bu^*(t) + w(t), \quad x(0) = x_0,$$

with the control $u^*(\vartheta) = u^0(t, x(t))$, $\vartheta \in [t, t+h[$, $t \in T_h$, is accepted.

In a certain control process under a realized disturbance $w^*(t)$, $t \in T$, only the values

$$u^*(\tau) = u^0(\tau, x^*(\tau)), \quad \tau \in T_h, \qquad (5)$$

along a realized isolated continuous trajectory $x^*(t)$, $t \in T$, are used. Moreover, it is enough to know how to calculate the current value $u^*(\tau)$ at any current instant $\tau \in T_h$, on the base of the current state $x^*(\tau)$, in time not exceeding h, i.e. in real time.

Function (5) is said to be a realization of optimal feedback in a particular control process. Any device able to calculate values (5) for any particular control process in real time is an optimal controller.

Thus, the problem of synthesis of optimal feedback is reduced to constructing an algorithm of an optimal controller. An optimal controller performs fast corrections of optimal open-loop control of current problem (2) subject to small variations of initial state. The realization of the optimal closed-loop control fed to the system presents a set of the first signals of optimal open-loop controls for every realized position.

3. GENERAL SCHEME OF COMPUTING AN OPTIMAL OPEN-LOOP CONTROL

Below an iterative method of computing OC of problem (1) is suggested. Its iteration consists of three procedures: 1) formulation of simplified OC problem; 2) solution to the simplified problem (SP); 3) analysis of the solution of SP.

In SP, in contrast with the initial problem, state constraints are imposed not on the whole interval T, but only at isolated points. Similar problems with intermediate state constrains have been studied in detail in (Balashevich *et al.*, 2001), and the algorithm of solving such problems forms the basis of the suggested method.

On the first iteration, choose from $T_h \cup t^*$ an arbitrary set $S = \{s_1, \ldots, s_{j^*}\}$, $0 < s_1 < \ldots < s_{j^*} = t^*$, and form an initial SP

$$\begin{aligned} &c'x(t^*) \to \max, \quad \dot{x} = Ax + bu, \ x(0) = x_0, \ (6)\\ &d'x(s) \geq \alpha, \ s \in S; \quad |u(t)| \leq 1, \ t \in T. \end{aligned}$$

OC $\tilde{u}^0(t)$, $t \in T$, of problem (6) is computed by the method (Balashevich *et al.*, 2001) and the behaviour of the output signal $y(t) = d'x(t)$, $t \in T$, of system (1) under the control $\tilde{u}^0(t)$, $t \in T$, is analysed. Detect the first point $\varrho \in T_h$ of violation of the state constraint: $y(\varrho) < \alpha$, and pass on to the next iteration.

On the follow-up iterations the SPs are formed by changing the number and place of intermediate constraints and by revealing boundary arcs $T(u)$.

In general case, SP has the form:

$$\begin{gathered} c'x(t^*) \to \max, \\ \dot{x} = Ax + bu,\ x(0) = x_0,\ t \in \bar{T}(u), \\ \dot{x} = \bar{A}x,\ t \in T(u), \\ d'x(\theta_r) = \alpha, \quad d'A^i x(\theta_r) = 0,\ i = \overline{1, k-1}, \\ |d'A^k x(\theta_r)| \le |d'A^{k-1}b|,\ r = \overline{1, r^*}, \\ d'x(s) \ge \alpha,\ s \in S; \quad |u(t)| \le 1,\ t \in \bar{T}(u), \end{gathered} \tag{7}$$

where $\bar{A} = A - bd'A^k/d'A^{k-1}b$.

Problem (7), unlike (6), contains fixed intervals $T(u)$ of allocation of the output signal on the boundary of the state constraint. OC $\tilde{u}^0(t)$, $t \in \bar{T}(u)$, of problem (7) is computed and the behaviour of the output signal $y(t)$, $t \in T_h$, under the control

$$u(t) = \begin{cases} \tilde{u}^0(t), & t \in \bar{T}(u), \\ u(x(t)), & t \in T(u), \end{cases} \tag{8}$$

is analysed to detect the minimal $\varrho \in \bar{T}_h(u) = \bar{T}(u) \cap T_h$, wherein the state constraint is violated.

The method finishes when $y(t) \ge \alpha$, $t \in T_h$. Thus, as a result of transforming and solving SPs we form a set $T(u)$ and OC $\tilde{u}^0(t)$, $t \in \bar{T}(u)$, of (7), which determine OC (8) of problem (1).

4. SOLUTION TO SIMPLIFIED PROBLEM

Let us describe key elements of the method (Balashevich *et al.*, 2001) of constructing OC for problem (6).

Write down problem (6) in the functional form:

$$\sum_{t \in T_h} c(t)u(t) \to \max,$$

$$\alpha(s) \le \sum_{t \in T_h} d(s,t)u(t),\ s \in S;\ |u(t)| \le 1,\ t \in T_h.$$

Here $c(t) = \int_t^{t+h} \psi_c'(\vartheta)b(\vartheta)d\vartheta$,

$$d(s,t) = \begin{cases} \int\limits_t^{t+h} G(s,\vartheta)b(\vartheta)d\vartheta,\ s > t, \\ 0,\ s \le t, \end{cases}$$

$$\alpha(s) = \alpha - G(s,0)x_0,$$

$\psi_c(t)$, $t \in T$, is a solution to the adjoint equation

$$\dot{\psi} = -A'\psi, \quad \psi(t^*) = c; \tag{9}$$

$G(s,t)$, $t \in [0,s]$, $s \in S$, is a $1 \times n$-vector-function, a solution to the equation

$$\partial G(s,t)/\partial t = -G(s,t)A, \quad G(s,s) = d. \tag{10}$$

Following (Balashevich *et al.*, 2001), choose from sets S, T_h arbitrary subsets S_{sup}, T_{sup} such that $|T_{sup}| = |S_{sup}|$. Form the matrix

$$D_{sup} = \begin{pmatrix} d(s,t),\ t \in T_{sup} \\ s \in S_{sup} \end{pmatrix}.$$

A set $K_{sup} = \{S_{sup}, T_{sup}\}$ is called a support of problem (7) if $\det D_{sup} \ne 0$.

A support K_{sup} characterizes controllability of the output signal $y = d'x(s)$, $s \in S_{sup}$, by impulses applied at support instants $t \in T_{sup}$.

A support K_{sup} is accompanied by the following elements:
1) the Lagrange multipliers $\nu(s)$, $s \in S$. Nonsupport components of multipliers are assigned to be zero: $\nu_n = (\nu(s),\ s \in S_n = S \backslash S_{sup}) = 0$; support components $\nu_{sup} = (\nu(s),\ s \in S_{sup})$ are computed as a solution to the equation

$$\nu_{sup}' D_{sup} = c_{sup}',$$

where $c_{sup} = (c(t),\ t \in T_{sup})$;
2) a cotrajectory $\psi(t)$, $t \in T$, is a solution to the equation

$$\dot{\psi} = -A'\psi, \quad \psi(t^*) = c - H'\nu; \tag{11}$$

3) a cocontrol (switching function)

$$\Delta(t) = \int\limits_t^{t+h} \psi'(\vartheta)bd\vartheta, \quad t \in T_h;$$

4) a pseudocontrol $\omega(\vartheta) = \omega(t)$, $\vartheta \in [t, t+h[$, $t \in T_h$, and an output pseudosignal $\zeta(s)$, $s \in S$. At first, assign nonsupport values $\omega_n = (\omega(t),\ t \in T_n = T_h \backslash T_{sup})$, of a pseudocontrol:

$$\omega(t) = \operatorname{sign} \Delta(t),\ \text{if}\ \ \Delta(t) \ne 0;$$

$$\omega(t) \in [-1,1],\ \text{if}\ \ \Delta(t) = 0,\ t \in T_n,$$

and support values $\zeta_{sup} = (\zeta(s),\ s \in S_{sup})$ of an output pseudosignal:

$$\zeta(s) = \alpha,\ \text{if}\ \ \nu(s) < 0,\ s \in S_{sup};$$

$$\zeta(s) \ge \alpha,\ \text{if}\ \ \nu(s) \ge 0,\ s \in S_{sup}.$$

Support components $\omega_{sup} = (\omega(t), t \in T_{sup})$ are computed from the equation

$$D_{sup}\omega_{sup} = \begin{pmatrix} \zeta(s) - d'x_0(s), \\ s \in S_{sup} \end{pmatrix},$$

where $x_0(t)$, $t \in T$, is a trajectory of the system

$$\dot{x} = Ax + bu,\ x(0) = x_0,\ t \in T(u) \tag{12}$$

with a discrete control $u(t) = \begin{cases} \omega(t), & t \in T_n, \\ 0, & t \in T_{sup}. \end{cases}$

Nonsupport components $\zeta_n = (\zeta(s), s \in S_n)$ are computed as $\zeta(s) = d' x_\omega(s)$, $s \in S_n$, where $x_\omega(t)$, $t \in T$, is a trajectory of (12) with $u(t) = \omega(t)$, $t \in T$.

Theorem. For optimality of a support K_{sup} it is necessary and sufficient that the accompanying pseudocontrol $\omega(t)$, $t \in T_h$, and output pseudosignal $\zeta(s)$, $s \in S$, satisfy the inequalities $|\omega(t)| \le 1$, $t \in T_{sup}$, $\alpha(s) \le \zeta(s)$, $s \in S_n$.

A pseudocontrol $\omega(t)$, $t \in T_h$, accompanying the optimal support is OC of problem (6): $\tilde{u}^0(t) = \omega(t)$, $t \in T_h$.

In the course of iterations of the method transformations of support are performed to obtain an optimal support K^0_{sup}.

Thus, in accordance with algorithm (Balashevich *et al.*, 2001) an optimal support K^0_{sup} of problem (6) will be constructed and auxiliary data accompanying this support will be formed. OC $\tilde{u}^0(t)$, $t \in \bar{T}_h(u)$, of problem (6) is a pseudocontrol $\omega(t)$, $t \in \bar{T}_h(u)$, accompanying K^0_{sup}.

The amount of computation required for constructing an optimal support K^0_{sup} depends on proximity of the initial support to the optimal one. The most computational expenses are required for solving initial SP (6), as before its solution there is no any accompanying information. Further, for solving SPs an optimal support and accompanying data stored after the previous SP are used. This allows to construct the optimal support of the current problem by refining the optimal support of the previous problem.

5. ANALYSIS OF SOLUTION AND FORMULATION OF SIMPLIFIED PROBLEM

Having solved initial SP (6), we get OC $\tilde{u}^0(t)$, $t \in T$, and accompanying data. Compute the trajectory $\tilde{x}^0(t)$, $t \in T_h$, the values of the output signal $y(t) = d'\tilde{x}^0(t)$, $t \in T_h$, and its derivatives $y^{(i)}(t)$, $i = \overline{1,k}$, $t \in T_h$. As the values $y^{(i)}(t)$, $t \in T_h$, are computed, the sets

$$T^i(t) = \{\vartheta \in \{h, 2h, \ldots, t\} : \\ y^{(i)}(\vartheta - h) y^{(i)}(\vartheta) < 0\}, \quad i = \overline{1,k},$$

are formed and additional data accompanying points of $T^i(t)$, $i = \overline{1,k}$, is stored.

Let at any instant $\varrho \in T_h$ the state constraint is violated: $y(\varrho) < \alpha$. Below we distinguish a point $\varrho^* \in T_h$, until which the system has been integrated, and the minimal instant $\varrho \in T_h \cap [0, \varrho^*]$, wherein the state constraint is violated.

Put $\varrho^* = \varrho$. Store the sets $T^i(\varrho^*)$, $i = \overline{1,k}$, and verify the condition of activation of the state constraint at the point ϱ:

$$|d' A^i \tilde{x}^0(\varrho)| \le \varepsilon, \ i = \overline{1, k-1}; \\ |d' A^k \tilde{x}^0(\varrho)| \le |d' A^{k-1} b|, \tag{13}$$

where $\varepsilon > 0$ is a small number.

If condition (13) holds, put $\theta_1 = \varrho$, $\theta^1 = \varrho + h$ and get SP of type (7). Otherwise we include ϱ into S and get the problem of type (6). At the same time, from the set $S \backslash \{t^*, \varrho - h, \varrho + h)$we delete points, wherein intermediate constraints of solved problem (6) are passive.

As a new SP differs from the solved one only by constraints at the instant ϱ, the method (Balashevich *et al.*, 2001) uses the optimal support of the solved problem and stored additional data. The optimal support of the new SP is constructed by refining the available support.

Let the Kth SP have been formed. Before solving it, on the base of the solution of the $K-1$th SP the following elements are known: 1) point $\varrho^* \in T_h$; 2) sets $T^i(\varrho^*)$, $i = \overline{1,k}$; 3) accompanying data of $K-1$th problem.

Having solved the Kth SP,examine the interval $[0, \varrho^*]$ for violation of the state constraint. The state constraint can be violated at some point $\varrho \in \bar{T}_h(u)$ left to ϱ^* if:

1) $\varrho = s - h$ or $\varrho = s + h$ where $s \in S$ is a contact point;

2) $\varrho \in T^1(\varrho^*)$ or $\varrho + h \in T^1(\varrho^*)$;

3) $\varrho = \theta_r - h$ or $\varrho = \theta^r + h$ where $[\theta_r, \theta^r]$ is a boundary arc.

Thus, to detect a point ϱ it is enough to compute the values

$$y(\vartheta - h), y(\vartheta), \ \vartheta \in T^1(\varrho^*); \\ y(s-h), y(s+h), \ s \in S \cap [0, \varrho^*[, \\ y(\theta_r - h), y(\theta^r + h), \ \theta_r < \theta^r < \varrho^*, \ r = \overline{1, r^*}. \tag{14}$$

Values (14) are computed with the use of the stored auxiliary data without additional integration of the system.

In the cases 1), 2), we verify condition (13) and in the $K+1$th problem either include point ϱ into S, or form the boundary arc $[\varrho, \varrho + h]$. In the case 3), we extend the boundary arc putting $\theta_r := \theta_r - h$ or $\theta^r := \theta^r + h$. Delete from the set $S \backslash \{t^*, \varrho - h, \varrho + h\}$ points in which intermediate constraints of the Kth problem are passive.

If there are boundary arcs $[\theta_r, \theta^r]$, $r = \overline{1, r(\varrho^*)}$, on the interval $[0, \varrho^*]$ and after solving the Kth problem there are no violation points left to $[\theta_r, \theta^r]$, then in parallel with the $K+1$th problem over the interval $[\theta_r, \theta^r]$, problems for refining the ends of this interval are solved. In a SP instead of $[\theta_r, \theta^r]$ an interval $[\theta_r + h, \theta^r]$ or $[\theta_r, \theta^r - h]$ is considered. If as a result of solving mentioned problems there is no violation of the constraints at the point θ_r or θ^r, then the boundary arc is shortened respectively.

Thus, by solving a series of SPs we achieve fulfillment of the state constraint on the interval $[0, \varrho^*]$. After that, integrate the system from ϱ^* to the right with the use of OC of the last solved problem and detect a new point $\varrho^* = \varrho$ of violation. Repeat operation of the algorithm for the new ϱ^*.

If in the course of operating the algorithm we obtain problem (6), which doesn't have a solution, then there not exists the solution to initial problem (1).

In problem (7) the solution may not exist because of inconsistency between current boundary arcs and true boundary arcs of problem (1). In this case, we successively shorten boundary arcs, include released points of these arcs into the set S and solve new SPs. As a result, we either get a solvable problem and further deal with it, or get an unsolvable problem (6) and draw a conclusion about insolvability of initial problem (1).

6. REALIZATION OF OPTIMAL FEEDBACK

Before the beginning of the control process the controller constructs the optimal support $K^0_{sup}(0)$ by solving problem (6) without limitation on computer time and put $u^*(0) = u^0(0, x_0)$. At any current $\tau \in T_h$ the system reaches the state $x^*(\tau)$ from the state $x^*(\tau - h)$ under the action of the control $u^*(t) = u^*(\tau - h)$, $t \in [\tau - h, \tau[$, and the disturbance $w^*(t)$, $t \in [\tau - h, \tau[$. By assumption, at the previous instant $\tau - h$ the optimal controller solved problem (2) for the position $(\tau - h, x^*(\tau - h))$ and knows the pair $\{K^0_{sup}(\tau - h), T^{\tau - h}(u)\}$ of the optimal support and the set of boundary arcs. To compute $\{K^0_{sup}(\tau), T^{\tau}(u)\}$ the controller uses $\{K^0_{sup}(\tau - h), T^{\tau - h}(u)\}$ as an initial approximation and performs computations by the described above method with the use of accompanying data stored at the instant $\tau - h$. Due to the short quantization step h and a bounded disturbance $w^*(t)$, $t \in [\tau - h, \tau[$, the difference between the nominal state and the realized state $x^*(\tau)$ is small and to refine $\{K^0_{sup}(\tau - h), T^{\tau}(u - h)\}$ small amount of computation is required.

The main computational expenses while solving OC problem are caused by integration of primal and adjoint systems. So the effectiveness of the method is evaluated not by a number of iterations, but by the length of intervals used for integration of a primal or adjoint system to construct OC. Estimates of complexity of operating an optimal controller for OC problem with endpoint constraints are given in (Gabasov *et al.*, 2000).

A specific feature of a control system under a state constraint is that in the case of an active state constraint for a position $(\tau, x^*(\tau))$, the next position $(\tau + h, x^*(\tau + h))$ could be infeasible because of the action of disturbance $w^*(t)$, $t \in [\tau, \tau + h[$. To avoid activation of the state constraint in the course of control process, let us assume that at an instant $\tau \in T_h$ the controller knows bounds of disturbances $w(t) \in W = \{w \in R^n : w_* \le w \le w^*\}$, $t \in [\tau, \tau + h[$, one step ahead and instead of SPs of type (6), (7) strengthened SPs (6), (7) with additional constraint

$$d'x(\tau + h) \ge \alpha - \min_{w \in W} \int_0^h d'F(h - s)w\,ds,$$

where $\dot F = AF$, $F(0) = I$, are considered.

7. EXAMPLE

As an example, a problem of OC of a two-mass oscillating system is considered:

$$\begin{gathered} x_3(12) \longrightarrow \max, \quad \dot x_1 = x_3, \quad \dot x_2 = x_4, \\ \dot x_3 = -x_1 + x_2 - u, \ \dot x_4 = 0.1x_1 - x_2 + 0.1u, \\ x_1(0) = x_2(0) = 0, \ x_3(0) = 2, \ x_4(0) = 1, \quad (15) \\ y(t) = x_2(t) - x_1(t) \ge -0.5, \\ |u(t)| \le 1, \ t \in T = [0, 12]. \end{gathered}$$

Solve SP with intermediate constraints $y(s) \ge -0.5$ at points $s = 2i$, $i = \overline{1,6}$. The corresponding output signal $y(t)$, $t \in T$, is given in Fig. 1,a. The first point $\varrho \in T_h = \{kh, \ k = \overline{0, 299}\}$, of violation of the constraint $y(t) \ge -0.5$, $t \in T_h$, is $\varrho = 2.04$. Refine the solution to the SP replacing the constraint $y(2) \ge -0.5$ by $y(2.04) \ge -0.5$. After that, the first point of violation is $\varrho = 2.08$.

Repeat the procedure, moving the constraint of SP one step to the right and checking the place of the first point of violation of the state constraint. With the constraint $y(2.72) \ge -0.5$ the first point of violation is 2.64 and $\dot y(2.64) = 0.019$, which means the chance of placing the output signal $y(t)$, $t \in T$, on the boundary of the state constraint.

As an initial boundary arc we take an interval $[\theta_1, \theta^1]$ of the length h putting $\theta_1 = 2.64$. After procedure of refining we get a boundary arc $T_1(u) = [2.72, 3.04]$. The function $y(t)$, $t \in T$, is presented in Fig. 1, b.

Successively check the values $y(t)$ at points of T_h and detect the violation of the constraint $y(t) \ge$

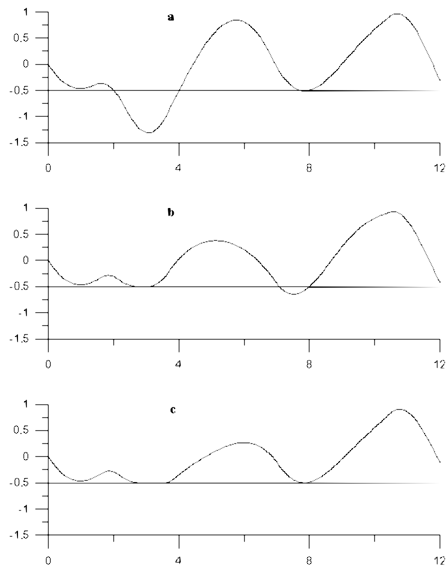

Fig. 1. Behaviour of the output signal

-0.5 at the point $\varrho = 7.08$. In 19 steps a contact point $s = 7.84$ is found and a boundary arc $[2.76, 3.56[$ is identified which provide the solution to problem (15). In Fig. 1, c the corresponding output signal $y(t)$, $t \in T$, is presented; in Fig. 2 OC is given. Thus, the solution to problem (15) is

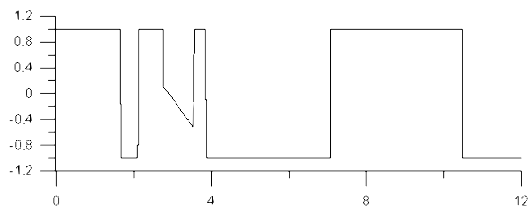

Fig. 2. Optimal open-loop control

equivalent to the solution of the SP

$$x_3(12) \longrightarrow \max,$$
$$\dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = -x_1 + x_2 - u,$$
$$\dot{x}_4 = 0.1x_1 - x_2 + 0.1u, \quad t \in [0, 2.76[\cup[3.56, 12],$$
$$x_1(0) = x_2(0) = 0, \; x_3(0) = 2, \; x_4(0) = 1,$$
$$-x_3(2.76) + x_4(2.76) = 0,$$
$$|-1.1x_1(2.76) - 2x_2(2.76)| \leq 1.1,$$
$$\dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = -0.9x_2/1.1,$$
$$\dot{x}_4 = -0.9x_2/1.1, \quad t \in [2.76, 3.56[,$$
$$x_2(7.84) - x_1(7.84) \geq -0.5,$$
$$|u(t)| \leq 1, \; t \in [0, 2.76[\cup[3.56, 12[.$$

Assume that in the course of control process the oscillating system is affected by disturbance presented in the fourth equation $\dot{x}_4 = 0.1x_1 - x_2 + 0.1u + w$ and the realizing disturbance has the form

$$w^*(t) = \begin{cases} |0.3 \sin 2t|, \; t \in [0, 3[\cup[6, 12[; \\ -|0.3 \sin 2t|, \; t \in [3, 6[. \end{cases}$$

This disturbance is unknown to the optimal controller but at any instant $\tau \in T_h$ it knows that $|w(t)| \leq 0.3$, $t \in [\tau, \tau + h[$, and a current state $x^*(\tau)$ is measured.

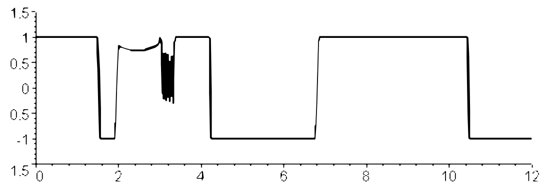

Fig. 3. Realization of the optimal feedback

The realization of the optimal feedback is presented in Fig. 3. The corresponding output signal is given in Fig. 4.

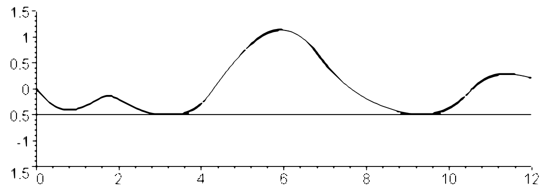

Fig. 4. Realized output signal

As a unit of complexity one complete integration of a primal or adjoint system on interval T is taken. The complexity of computing a value $u^*(\tau)$ at each instant $\tau \in T_h$ does not exceed 0.12.

Let η be the time required for a microprocessor to integrate a system on interval T. Then using these microprocessors one can realize the feedback if $0.12\eta \leq h$. Based on this inequality one can choose a quantization step h for a given microprocessor, and on the other hand, take a suitable microprocessor for a given h.

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