.

DATA-BASED LQ SYNTHESIS – A MODIFICATION FOR ERROR-REDUCTION

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Abstract: The formulas for data-based LQ synthesis are modified to allow for the reduction in noise-induced design error. Modification in the design formulas includes the introduction of a linear filter that enhances the auto-regressive nature of the optimal control input solution.

Keywords: Data-based control synthesis, linear time invariant discrete system, linear quadratic regulator design, autoregressive process.

1. INTRODUCTION

Despite being a renowned method for modern control synthesis, the linear quadratic (LQ) design (Kalman, 1960; 1963) is known to suffer from two major problems: the need for a mostly unavailable information of the system states to implement the control law (Doyle and Stein, 1979; Maciejowski, 1989), and the need for an uncertain plant model for the synthesis of the control law (Gever, 1983).

Recently, Chan (2000) has proposed an output feedback design that duplicates the closed-loop response of a state feedback control law regardless of the initial state of the system. As a result, state information is no longer needed. In addition, the introduction of a data-based synthesis of this output feedback LQ regulator (Chan, 1999, 1996) also obviates the need for a parametric plant model.

However, any noise signal in the test data induces error into this data-based LQ (DBLQ) design. In general, the noise-induced error causes the data-based solution to deviate from an auto-regressive (AR) sequence with which a true LQ solution would comply. In order to suppress this design error, a linear filter that enhances the AR nature of the data-based solution is incorporated into the data-based design formulas. It will be shown that the modified DBLQ formulas enable significant reduction in the noise-induced designed errors.

2.THE BASIC DBLQ FORMULAS

Consider a linear discrete system with output y, input u, and

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + Bu(k), \quad y(k) = C\mathbf{x}(k) \qquad (1)$$

where $\mathbf{x} \in R^{n\times 1}$ is the state vector, and $A \in R^{n\times n}$, $B \in R^{n\times 1}$, and $C \in R^{1\times n}$ are constant matrices. The following performance index will be optimised for some integer $N > 0$:

$$J = \sum_{k=0}^{N}\left(\rho\, y(k)^2 + \sigma\, u(k)^2\right), \quad \rho \ge 0, \quad \sigma > 0. \qquad (2)$$

For a $N >> 1$, the following optimal control input solution will result:

$$u(k) = \bar{u}(k), \quad \bar{u}(k) = r(k) - K\,\mathbf{x}(k). \qquad (3)$$

where $K \in R^{1\times n}$ is a state feedback gain and r is a command.

A data-based version of (2) can be constructed, permitting data-based computation of $\bar{u}(k)$. Using the input data $q(k)$ and the output data $y(k)$ from the open-loop test, computation of $\bar{u}(k)$ can be performed as follows (Chan, 1996):

$$U = -\rho(\rho H'H + \sigma I)^{-1} H'Z \qquad (4)$$

where I denotes a $N \times N$ identity matrix,

$$U = \begin{bmatrix} \bar{u}(0) \\ \vdots \\ \bar{u}(N-1) \end{bmatrix}; \quad H = \begin{bmatrix} h(1) & & \\ \vdots & \ddots & \\ h(N) & \cdots & h(1) \end{bmatrix}, \quad Z = \begin{bmatrix} h(2) \\ \vdots \\ h(N+1) \end{bmatrix};$$

and $h(k)$, the kernel of (1), is obtained from

$$\begin{bmatrix} h(1) \\ \vdots \\ h(k) \end{bmatrix} = \begin{bmatrix} q(0) & & \\ \vdots & \ddots & \\ q(k-1) & \cdots & q(0) \end{bmatrix}^{-1} \begin{bmatrix} y(1) \\ \vdots \\ y(k) \end{bmatrix}. \qquad (5)$$

Note that the $\bar{u}(k)$ in (4) corresponds to r being an unit impulse command at $k = -1$; hence, $\bar{u}(-1) = 1$. In addition, data for the optimal system output, denoted as $\bar{y}$, can be computed from $\bar{u}(k)$ and $h(k)$ as follows

$$\bar{y}(k) = \sum_{l=0}^{k} \bar{u}(l-1) h(k-l+1) \qquad (6)$$

where it is noted that $\bar{u}(k) = 0$ for all $k < -1$.

An output feedback design to implement the optimal control law can then be synthesized using the computed data of $\bar{u}(k)$ and $\bar{y}(k)$ (Chan, 1996).

3. NOISE-INDUCED ERROR IN THE DESIGN

Any noise signal in $y(k)$ will be carried into $h(k)$ through (5). This induced noise in $h(k)$ then enters (4) and corrupts the computed data of $\bar{u}(k)$.

Because the noise enters (4) into the information matrix which is then inverted, the distortions in the computed data of $\bar{u}(k)$ will not be a simple additive induced error. In general, the sequence of $\bar{u}(k)$ computed from a noise-corrupted test data becomes erratic and no longer comply with an AR characteristic as would a true LQ solution. This observation suggests that a reduction in the noise-induced design error may be achieved by enhancing the AR nature of the solution for $\bar{u}(k)$. This design concept is put to work through the introduction of an auxiliary term in the DBLQ formulas.

4. MODIFIED FORMULA FOR NOISY DATA

4.1 The generalized AR sequence annihilator.

For any AR sequence, collectively denoted as $d(k)$, there exists appropriate complex numbers, $p_1, \cdots, p_t$ and $\xi_1, \cdots, \xi_t$ for some integer $t > 0$ such that.

$$d(k) = \xi_1 p_1^{\ k} + \cdots + \xi_t p_t^{\ k}, \quad k \geq 0. \qquad (7)$$

When filtered by a filter,

$$F(z) = f_0 + f_1 z^{-1} + \cdots + f_m z^{-m} = (z-1)^m, \qquad (8)$$

where m is some positive integer, it turns out that

$$\begin{aligned} F(z) \circ \{d(k)\} &= \sum_{l=0}^{m} f_l \, d(\ell + m - l) \\ &= \sum_{l=1}^{t} \xi_l \, p_l^{\ \ell+m} \times (1-p_l)^m, \quad \forall \ell \geq 0. \end{aligned} \qquad (9)$$

Note that the symbol, $F(z) \circ \{d(k)\}$, denotes the operation of $F(z)$ on $d(k)$.

In general, $p_1, \cdots, p_t$ represents the characteristic poles of $d(k)$. In this discussion, all involving data sequences are discrete output of some finite-speed continuous process. For sufficiently small sampling times, it is therefore reasonable to assume for all $i = 1, \cdots, t$ that $p_i = 1 - \varepsilon_i$ for some complex number ε_i satisfying $\|\varepsilon_i\| << 1$. Then, the fact that

$$\sum_{l=1}^{t} \xi_l \ p_l^{\ \ell+m} \times (1-p_l)^m \approx \sum_{l=1}^{t} \xi_l \times \varepsilon_l^{\ m}, \quad \forall \ell \geq 0 \qquad (10)$$

implies $F(z) \circ \{d(k)\} \to 0$ for a sufficiently large m.

This result suggests that it will be possible to annihilate $d(k)$ by filtering the sequence with $F(z)$ for some appropriate value of m. In Table 1, the signal annihilation power of $F(z)$ for various values of m is demonstrated on a linear and time invariant discrete process:.

4.2 The auxiliary homogeneous equation for AR sequences.

Because of (10), it becomes possible to enhance the AR characteristics of a data sequence, say $d(k)$, by introducing the following auxiliary homogeneous equation:

$$G \begin{bmatrix} d(1) \\ \vdots \\ d(k) \end{bmatrix} = 0 \qquad (11)$$

where

$$G = \begin{bmatrix} 0 & \cdots & 0 & f_m & \cdots & f_0 & & \\ \leftarrow & \hat{m} & \rightarrow & & \ddots & & \ddots & \\ 0 & \cdots & 0 & & & f_m & \cdots & f_0 \end{bmatrix}.$$

Note that we need $k \geq m$ in (11). Moreover, the $\hat{m}$ zero columns of G exclude the first $\hat{m}$ data of $d(k)$, which can not be annihilated when the sequence is not proper.

<u>Table 1 The signal annihilation power of *F*(*z*)</u>

$$\tilde{s}(\ell) = F(z) \circ \{d(k)\}, \ d(z) = \frac{(z-0.8)}{(z-0.95)(z-0.9)}$$

	$\|\tilde{s}(1)\|$	$\|\tilde{s}(2)\|$	$\|\tilde{s}(3)\|$	$\|\tilde{\mathbf{s}}\|$ †
$m=1$	5.00e-2	3.75e-2	2.66e-2	2.31e-2
$m=2$	1.25e-2	1.09e-2	9.43e-3	3.43e-3
$m=3$	1.63e-3	1.44e-3	1.28e-3	5.37e-4
$m=5$	1.91e-5	1.71e-5	1.54e-5	6.91e-6
$m=10$	2.00e-10	1.80e-10	1.62e-10	7.30e-10

† $\|\tilde{\mathbf{s}}\|$: The root-mean-square average of 20 data of $\tilde{s}$ starting from $\tilde{s}(4)$.

In this discussion, a data sequence is proper if its $z-$ transform contains less nonzero terms in its numerator than in its denominator. In general, a non-proper data sequence can be differentiated into a proper sequence following one or several consecutive impulse signals of appropriate amplitudes. When such a non-proper sequence is filtered with $F(z)$, the impulse signals survive because of their infinite convergent speed. As a result, the first several data of the filtered sequence may remain large for all values of m (Table 2).

Details on the selections of m and $\hat{m}$ for the design will be deferred until Sections 4.4 and 4.5.

Table 2 The annihilation of non-proper signal

$$\tilde{s}(\ell) = F(z) \circ \{d(k)\},\ d(z) = \frac{(z-0.8)(z-0.95)}{(z-0.9)}$$

	$\|\tilde{s}(1)\|$	$\|\tilde{s}(1)\|$	$\|\tilde{s}(1)\|$	$\|\tilde{\mathbf{s}}\|$
$m=2$	1.8500	8.45e-1	5.00e-4	2.29e-4
$m=4$	2.6950	8.45e-1	5.00e-5	2.29e-5
$m=6$	3.5395	8.45e-1	5.00e-6	2.29e-6
$m=8$	5.2284	8.44e-1	5.00e-8	2.29e-8
$m=10$	9.4506	8.44e-1	5.00e-13	2.29e-13

4.3 The modified DBLQ formula.

Because the auxiliary equation is homogeneous, the incorporation of (11) into the DBLQ formula only changes the information matrix of (4), as is shown in the following. Firstly, (4) implies that

$$(\rho H'H + \sigma I)U = -\rho H'Z \tag{12}$$

From (11) and keep in mind that $d(k)$ represents all AR sequences including $\bar{u}(k)$, we also have

$$G\ U = 0 \tag{13}$$

As a result, the following equation for U holds true:

$$\begin{aligned}(\rho H'H + \sigma I + G)U \\ = (\rho H'H + \sigma I)U = -\rho H'Z\end{aligned} \tag{14}$$

Then, the following modified formula of (4) results:

$$U = -\rho(\rho H'H + \sigma I + G'G)^{-1}H'Z \tag{15}$$

On the other hand, no change in (5) is necessary, as its information matrix is noise-free.

A value of m and a value of $\hat{m}$ are needed in (15) in order to prepare the $G'G$ matrix. A guideline to the selection of these two parameters are given next.

4.4 The selection of m.

In general, a large m enhances signal annihilation, hence the role of the $G'G$ matrix in the formula. On the other hand, a large value of m may causes the first several data of $\bar{u}(k)$ to be unregulated by the $G'G$ matrix, as the matrix is designed to work on the collective behaviour of $m+1$ data of $\bar{u}(k)$. Besides, it is also not justified to prepare a $G'G$ matrix which annihilates a signal to the degree that is below the noise level of the test data. Therefore, a smallest possible value of m is more preferable.

In practice, this value of m can be determined by considering the data noise level. From Table 1, it is seen that a data with a 10% noise-to-signal ratio do not need a value of m that exceeds unity.

4.5 The selection of $\hat{m}$.

In this design, the $z-$transform of the entire $\bar{u}(k)$ sequence, which includes $\bar{u}(-1)$, has a zero relative order (Chen, 1984), and hence is not proper. However, by excluding $\bar{u}(-1)$, which equals 1 and represents the impulse signal of the sequence, the remaining data of $\bar{u}(k)$, those appear in (4) as unknowns, form a proper sequence. As a result, a $\hat{m}=0$ can be a workable choice for the designs.

On the other hand, it is also known that the closed-loop poles of a LQ design approaches the zeros of the plant as a limit when the ratio of σ/ρ vanishes (Chan, 1986). For general discrete plants, where zeros are less than poles in number (Åström, et.al, 1984), more impulse signals may therefore appear in $\bar{u}(k)$ after $k \ge 0$, shall the ratio of σ/ρ gets very small. In this case, a $\hat{m} > 0$ shall be used.

Nevertheless, a smallest possible value of $\hat{m}$ should be used in order to minimize the number of data of $\bar{u}(k)$ that will be unregulated by the $G'G$ matrix. In general, a $\hat{m}=0$ can be used for a regulator design when the ratio of σ/ρ is large. Test computations of (15) using various values of $\hat{m}$ also reveal that a $\hat{m}=1$ or a $\hat{m}=2$ is normally suffice when the ratio of σ/ρ drops really small, say in the design for a LQ signal tracker (Chan, 1986).

5. A DESIGN EXAMPLE

The following plant is tested:

$$\frac{y(t)}{u(t)} = \frac{s^3 + s^2 + 0.18s + 0.016}{s^4 + 1.05s^3 + 0.145s^2 - 0.05s + 0.0075}.$$

The discrete system is formed with a sampling time of 0.05 second and the open-loop data of $y(k)$ is generated using a unit step test command.

For this test, additive pseudo random data noise of certain noise-to-signal (N/S) ratios are injected into $y(k)$, and the induced errors in the computed data of $\bar{u}(k)$ and $\bar{y}(k)$ are examined. Since the subsequent synthesis of the output feedback controller requires data of both $\bar{u}(k)$ and $\bar{y}(k)$, the root-mean-square errors in these sequences are multiplied together to form a combined error-to-noise (E/N) ratio. As the DBLQ synthesis is concerned, this ratio reflects the overall noise amplification effect of the formulas.

Three noise levels, $\mathrm{N/S} = 10^{-3}$, $\mathrm{N/S} = 10^{-2}$, and $\mathrm{N/S} = 10^{-1}$, are used to represent light, medium, and heavy corruption of the test data. In order to account for the different data noise levels, two values of m are also used: $m = 1$ for $\mathrm{N/S} = 10^{-1}$ and $\mathrm{N/S} = 10^{-2}$, and $m = 2$ for $\mathrm{N/S} = 10^{-3}$. For each noise level used, DBLQ synthesis is performed for six σ/ρ ratios, from $\sigma/\rho = 10^{0}$ all the way down to $\sigma/\rho = 10^{-5}$. These six values of the σ/ρ ratio are equally grouped into 3 categories: large, medium, and small.

The value of $\hat{m}$ used matches the σ/ρ ratios in terms of their categories: $\hat{m} = 0$ for the large values of σ/ρ, $\hat{m} = 1$ for the medium, and $\hat{m} = 1$ for the small. Then, results of the computation using (15) are plotted in Fig. 1 alongside the plots for the results obtained using (4). In all tests, a $N = 100$ is used.

It is seen that the E/N ratio increases as the σ/ρ ratio is decreased. Moreover, the E/N ratio is reduced on all tests with the inclusion of the $G'G$ matrix into the DBLQ formula.

For large values of σ/ρ, $\sigma/\rho \geq 0.01$, a low E/N ratio, $\mathrm{E/N} < 1$, results even when the $G'G$ matrix is not included. For these cases, the error reduction effect of the auxiliary term is mild.

However, as the noise-induced error increases at small values of the σ/ρ ratio, the error reduction effect of the $G'G$ matrix intensifies, when it is needed most. For the case with the heavily corrupted test data, the E/N ratio never exceeds unity when the auxiliary term is incorporated.

For the low $\mathrm{N/S}$ tests, the E/N ratios are large mainly because they are divided by small noise levels. The absolute induced errors may be mild.

6. CONCLUDING REMARK

An auxiliary term to reduce the noise-induced design error in the data-based LQ synthesis has been developed. The auxiliary term reduces the error in the synthesis by enhancing the auto-regressive nature of the optimal solution. Simulation tests show that significant reduction in the noise-induced error has been achieved when the auxiliary term is incorporated into the design formula.

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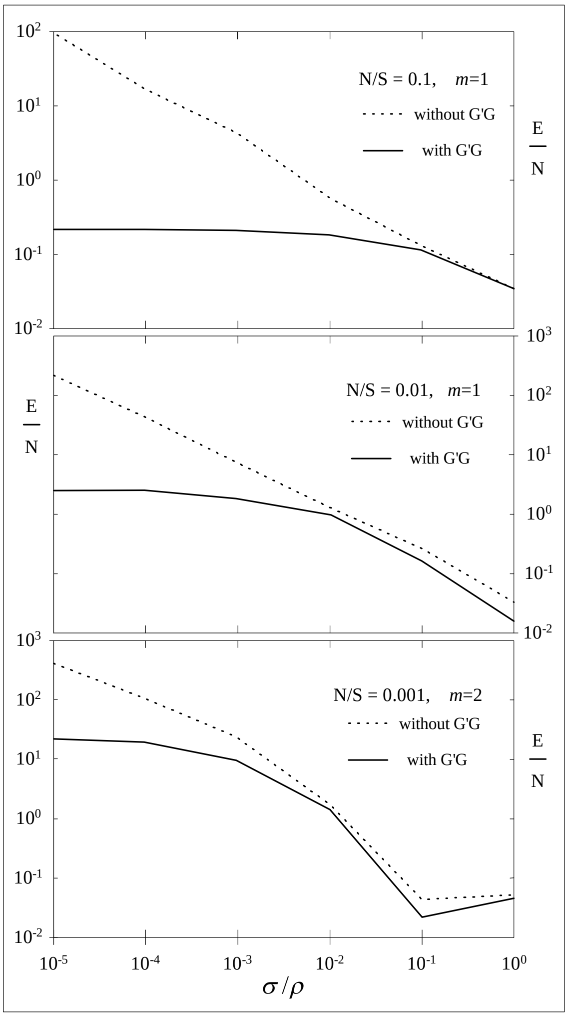

Fig. 1. Results of the noise reduction tests with the $G'G$ matrix.