AUXILIARY SIGNAL DESIGN IN UNCERTAIN SYSTEMS WITH KNOWN INPUTS

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Abstract: A method of auxiliary signal design for active failure detection based on a multi-model formulation of normal and failed systems is extended to allow for a-priori information about initial conditions and the possibility of having a known additional input. Both theory and numerical algorithms are presented.

Keywords: Failure detection, model-based, fault detection, active, linear.

1. INTRODUCTION

This paper considers an extension of the problem of auxiliary signal design introduced in (Nikoukhah *et al.*, 2002) for which a complete solution have been given in (Campbell and Nikoukhah, 2004). In previous works, the models considered for normal and failed system were linear systems having for inputs the auxiliary signal and additional on-line measured inputs. In many cases, however, a-priori information exists about the initial condition and the models may be subject to a-priori known signals. In this paper it is shown that the design problem with a-priori information can be constructed along the same line as the one developed for zero initial condition and in the absence of a-priori information on input signals.

This extension is particularly important because it also allows the consideration of more general types of failures. For example, if a failure introduces a bias in the system (something which is encountered often in practice), it could not have been dealt with before. But now, the bias can be considered as a known constant input of the faulty model. Similarly, if the failure is modeled as a jump in the state of the system, this can now be modeled as a non zero initial condition at the start of the test period. Even though the mathematical results needed to obtain a solution in this more general case are somewhat different from those used before, the construction of the solution follows similar steps. In particular, the complexity of the solution is the same. Page limitations preclude a discussion of alternative approaches and applications.

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2. PROBLEM FORMULATION

The general models considered are of the form

$$\dot{\mathbf{x}}_i = \mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i\mathbf{v} + \mathbf{M}_i\boldsymbol{\nu}_i + \bar{\mathbf{M}}_i\mathbf{u}_i, \quad \text{(1a)}$$

$$\mathbf{E}_i\mathbf{y} = \mathbf{C}_i\mathbf{x}_i + \mathbf{D}_i\mathbf{v} + \mathbf{N}_\mathbf{i}\boldsymbol{\nu}_\mathbf{i} + \bar{\mathbf{N}}_\mathbf{i}\mathbf{u}_\mathbf{i} \quad \text{(1b)}$$

where $i = 0, 1$ correspond to normal and failed system models respectively. The $\mathbf{v}$ is the auxiliary signal which is computed prior to the test while $\mathbf{y}$ is the output that becomes known during the test. The inputs $\mathbf{u}_i$ are assumed to be known in advance.

Note that in contrast to $\mathbf{u}_i$, $\mathbf{y}$ cannot be used to design $\mathbf{v}$ since $\mathbf{v}$ is computed before the test. The only condition on the system matrices is that the $\mathbf{N}_i$'s have full row rank. The unknown initial conditions $\mathbf{x}_i(0)$ and noises $\boldsymbol{\nu}_i$ are assumed to satisfy the bounds

$$\mathcal{S}_i = (\mathbf{x}_i(0) - \bar{\mathbf{x}}_i)^T \mathbf{P}_{i0}^{-1} (\mathbf{x}_i(0) - \bar{\mathbf{x}}_i) + \int_0^s \boldsymbol{\nu}_i^T \mathbf{J}_i \boldsymbol{\nu}_i \, dt < 1, \ \forall s \in [0, T], \quad (2)$$

where the $\mathbf{J}_i$'s are signature matrices. In this formulation, $\bar{\mathbf{x}}_i$'s represent the a-priori information regarding the initial conditions $\mathbf{x}_i(0)$. Coefficients $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{M}, \mathbf{N}, \bar{\mathbf{M}}, \bar{\mathbf{N}}$ can be time varying. Bounds other than 1 are included by rescaling system coefficients.

This formulation includes a number of different problems. For example, it includes the case of purely additive noise where $\mathbf{J}_i = \mathbf{I}$. In that case we need only consider $s = T$ in (2) since the integrand is non-negative and the maximum value of the integral occurs at $s = T$.

But our problem formulation also includes problems with model uncertainty including some of those studied in (Petersen *et al.*, 2000; Petersen and Savkin, 1999). See (Campbell and Nikoukhah, 2004) for more details.

A signal $\mathbf{v}$ is proper if observation of $\mathbf{y}$ provides enough information to decide from which model y has been generated. That is, there exist no solution to (1) and (2) for $i = 0$ and 1 simultaneously. An optimal proper $\mathbf{v}$ is sought. The first step is to characterize the proper $\mathbf{v}$.

3. AUXILIARY SIGNAL DESIGN

Let

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{pmatrix}, \ \boldsymbol{\nu} = \begin{pmatrix} \boldsymbol{\nu}_0 \\ \boldsymbol{\nu}_1 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \end{pmatrix}, \ \bar{\mathbf{x}} = \begin{pmatrix} \bar{\mathbf{x}}_0 \\ \bar{\mathbf{x}}_1 \end{pmatrix}.$$

Since the $\mathbf{N}_i$'s are full row rank, for any L_2 functions $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{y}$, there exist L_2 functions $\boldsymbol{\nu}_i$ satisfying (1). So, the non-existence of a solution to (1) and (2) for $i = 0, 1$ is equivalent to:

$$\sigma(\mathbf{v}, s, \mathbf{u}, \bar{\mathbf{x}}) \geq 1 \quad (3)$$

where

$$\sigma(\mathbf{v}, s, \mathbf{u}, \bar{\mathbf{x}}) = \inf_{\substack{\boldsymbol{\nu}_0, \boldsymbol{\nu}_1, \mathbf{y} \\ \mathbf{x}_0, \mathbf{x}_1}} \max(\mathcal{S}_0, \mathcal{S}_1) \quad (4)$$

subject to (1), $i = 0, 1$. (4) is reformulated as:

$$\sigma(\mathbf{v}, s, \mathbf{u}, \bar{\mathbf{x}}) = \max_{\beta \in [0,1]} \Phi_\beta(\mathbf{v}, s, \mathbf{u}, \bar{\mathbf{x}}) \quad (5)$$

where

$$\Phi_\beta(\mathbf{v}, s) = \inf_{\substack{\boldsymbol{\nu}_0, \boldsymbol{\nu}_1, \mathbf{y} \\ \mathbf{x}_0, \mathbf{x}_1}} \beta \mathcal{S}_0 + (1 - \beta) \mathcal{S}_1 \quad (6)$$

subject to (1) for $i = 0, 1$. The interchange of $\inf$ and $\max$ is valid for the same reasons as given in (Campbell and Nikoukhah, 2004).

Note that $\Phi_\beta(\mathbf{v}, s)$ depends also on $\mathbf{u}$ and $\bar{\mathbf{x}}$, but since they are assumed fixed and known, to simplify the notations, they have been dropped from the list of arguments. Let

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_1 \end{pmatrix}, \ \mathbf{M} = \begin{pmatrix} \mathbf{M}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_1 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} \mathbf{B}_0 \\ \mathbf{B}_1 \end{pmatrix},$$

$$\mathbf{D} = \mathbf{F}_0 \mathbf{D}_0 + \mathbf{F}_1 \mathbf{D}_1, \ \bar{\mathbf{M}} = \mathbf{F}_0 \bar{\mathbf{M}}_0 + \mathbf{F}_1 \bar{\mathbf{M}}_1,$$

$$\mathbf{C} = \begin{pmatrix} \mathbf{F}_0 \mathbf{C}_0 & \mathbf{F}_1 \mathbf{C}_1 \end{pmatrix}, \mathbf{N} = \begin{pmatrix} \mathbf{F}_0 \mathbf{N}_0 & \mathbf{F}_1 \mathbf{N}_1 \end{pmatrix},$$

$$\bar{\mathbf{N}} = \begin{pmatrix} \mathbf{F}_0 \bar{\mathbf{N}}_0 & \mathbf{F}_1 \bar{\mathbf{N}}_1 \end{pmatrix}, \ \mathbf{P}_\beta^{-1} = \begin{pmatrix} \beta \mathbf{P}_{0,0}^{-1} & \mathbf{0} \\ \mathbf{0} & (1 - \beta) \mathbf{P}_{1,0}^{-1} \end{pmatrix},$$

$$\mathbf{J}_\beta = \begin{pmatrix} \beta \mathbf{J}_0 & \mathbf{0} \\ \mathbf{0} & (1 - \beta) \mathbf{J}_1 \end{pmatrix},$$

where $\mathbf{X}^\perp$ denotes a maximal row rank left annihilator of $\mathbf{X}$ and

$$\mathbf{F} = \begin{pmatrix} \mathbf{F}_0 & \mathbf{F}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{E}_1 \end{pmatrix}^\perp. \quad (7)$$

Then reformulate (6) as follows:

$$\Phi_\beta(\mathbf{v}, s) = \inf_{\boldsymbol{\nu}, \mathbf{x}} (\mathbf{x}(0) - \bar{\mathbf{x}})^T \mathbf{P}_\beta^{-1} (\mathbf{x}(0) - \bar{\mathbf{x}}) + \int_0^s \boldsymbol{\nu}^T \mathbf{J}_\beta \boldsymbol{\nu} \, dt \quad (8)$$

subject to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v} + \mathbf{M}\boldsymbol{\nu} + \bar{\mathbf{M}}\mathbf{u} \quad (9a)$$

$$\mathbf{0} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{v} + \mathbf{N}\boldsymbol{\nu} + \bar{\mathbf{N}}\mathbf{u}. \quad (9b)$$

Theorem 1. Let $\mathcal{B}$ be the set of all β such that, for all $s \leq T$, $\Phi_\beta(\mathbf{v}, s) > -\infty$. Suppose for some $\beta \in [0, 1]$, that ${\mathbf{N}_\perp}^T \mathbf{J}_\beta \mathbf{N}_\perp > \mathbf{0}, \quad \forall t \in [0, T]$ and the Riccati equation

$$\dot{\mathbf{P}} = (\mathbf{A} - \mathbf{S}_\beta \mathbf{R}_\beta^{-1} \mathbf{C}) \mathbf{P} + \mathbf{P} (\mathbf{A} - \mathbf{S}_\beta \mathbf{R}_\beta^{-1} \mathbf{C})^T - \mathbf{P} \mathbf{C}^T \mathbf{R}_\beta^{-1} \mathbf{C} \mathbf{P} + \mathbf{Q}_\beta - \mathbf{S}_\beta \mathbf{R}_\beta^{-1} \mathbf{S}_\beta^T, \quad \mathbf{P}(0) = \mathbf{P}_\beta \quad (10)$$

where

$$\begin{pmatrix} \mathbf{Q}_\beta & \mathbf{S}_\beta \\ \mathbf{S}_\beta^T & \mathbf{R}_\beta \end{pmatrix} = \begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix} \mathbf{J}_\beta^{-1} \begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix}^T \quad (11)$$

has a solution on $[0, T]$. Then $\beta \in \mathcal{B}$.

It is assumed from here on that the set of β satisfying the two conditions of Theorem 1 is not empty.

3.1 Construction of an optimal proper auxiliary signal

The problem to solve is:

$$\boldsymbol{\Gamma}_\beta = \min_{\mathbf{v}} \|\mathbf{v}\|^2, \quad \text{subject to} \ \max_{\substack{\beta \in [0,1] \\ s \in [0,T]}} \Phi_\beta(\mathbf{v}, s) \geq 1 \quad (12)$$

where $\|\mathbf{v}\|^2 = \int_0^T |\mathbf{v}|^2 \, dt$. The maximum value of $\Phi_\beta(\mathbf{v}, s)$ does not always occur at $s = T$, although it often does.

The Lagrangian for this problem is constructed as

$$\mathcal{L} = \Phi_\beta(\mathbf{v}, s) - \lambda_{\beta,s}\|\mathbf{v}\|^2, \tag{13}$$

so that it is necessary to solve

$$\mathbf{\Gamma}_\beta(s) = \max_{\mathbf{v}} \inf_{\boldsymbol{\nu},\mathbf{x}} \ (\mathbf{x}(0)-\bar{\mathbf{x}})^T\mathbf{P}_\beta^{-1}(\mathbf{x}(0)-\bar{\mathbf{x}}) + \int_0^s \boldsymbol{\nu}^T\mathbf{J}_\beta\boldsymbol{\nu} - \lambda|\mathbf{v}|^2\,dt \tag{14}$$

subject to (9a) and (9b).

If $\Phi_\beta(0,s) \geq 1$ for any s, then the optimal proper $\mathbf{v}$ is just zero.

Theorem 2. Suppose the two conditions of Theorem 1 are satisfied and let $\lambda^*_{\beta,s}$ be the infimum of the set of all λ for which the Riccati equation

$$\begin{aligned}\dot{\mathbf{P}} = (\mathbf{A}-\mathbf{S}_{\lambda,\beta}\mathbf{R}_{\lambda,\beta}^{-1}\mathbf{C})\mathbf{P} + \mathbf{P}(\mathbf{A}-\mathbf{S}_{\lambda,\beta}\mathbf{R}_{\lambda,\beta}^{-1}\mathbf{C})^T \\ -\mathbf{P}\mathbf{C}^T\mathbf{R}_{\lambda,\beta}^{-1}\mathbf{C}\mathbf{P} + \mathbf{Q}_{\lambda,\beta} - \mathbf{S}_{\lambda,\beta}\mathbf{R}_{\lambda,\beta}^{-1}\mathbf{S}_{\lambda,\beta}^T, \\ \mathbf{P}(0) = \mathbf{P}_\beta\end{aligned} \tag{15}$$

where

$$\begin{pmatrix}\mathbf{Q}_{\lambda,\beta} & \mathbf{S}_{\lambda,\beta}\\ \mathbf{S}_{\lambda,\beta}^T & \mathbf{R}_{\lambda,\beta}\end{pmatrix} = \begin{pmatrix}\mathbf{M} & \mathbf{B}\\ \mathbf{N} & \mathbf{D}\end{pmatrix}\begin{pmatrix}\mathbf{J}_\beta & \mathbf{0}\\ \mathbf{0} & -\lambda\mathbf{I}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{M} & \mathbf{B}\\ \mathbf{N} & \mathbf{D}\end{pmatrix}^T \tag{16}$$

has a solution on $[0,s]$. Then $\lambda_{\beta,s} > \lambda^*_{\beta,s}$where $\lambda_{\beta,s}$ satisfies the Lagrange conditions.

The proof of this result is the same as that for Theorem 3.3.2 in (Campbell and Nikoukhah, 2004) for the $\mathbf{u} = \mathbf{0}$ case. It implies, in particular, that for $\lambda_{\beta,s}$ satisfying the Lagrange conditions, Riccati equation (15) with $\lambda = \lambda_{\beta,s}$ has a solution on $[0,s)$.

The Riccati equation (15) is obtained by noting that the optimization problem (14) can be expressed as

$$\text{ext}_{\mathbf{x},(\boldsymbol{\nu},\mathbf{v})} \ (\mathbf{x}(0)-\bar{\mathbf{x}})^T\mathbf{P}_\beta^{-1}(\mathbf{x}(0)-\bar{\mathbf{x}}) + \int_0^s \begin{pmatrix}\boldsymbol{\nu}\\ \mathbf{v}\end{pmatrix}^T\begin{pmatrix}\mathbf{J}_\beta & \mathbf{0}\\ \mathbf{0} & -\lambda\mathbf{I}\end{pmatrix}\begin{pmatrix}\boldsymbol{\nu}\\ \mathbf{v}\end{pmatrix} dt \tag{17}$$

subject to

$$\begin{pmatrix}\dot{\mathbf{x}}\\ \mathbf{0}\end{pmatrix} = \begin{pmatrix}\mathbf{A}\\ \mathbf{C}\end{pmatrix} + \begin{pmatrix}\mathbf{M} & \mathbf{B}\\ \mathbf{N} & \mathbf{D}\end{pmatrix}\begin{pmatrix}\boldsymbol{\nu}\\ \mathbf{v}\end{pmatrix} + \begin{pmatrix}\bar{\mathbf{M}}\\ \bar{\mathbf{N}}\end{pmatrix}\mathbf{u}. \tag{18}$$

Here ext stands for extremum.

Note that for λ to equal $\lambda_{\beta,s}$ (satisfy Lagrange conditions), it is necessary that $\Phi_\beta(\mathbf{v},s) = 1$ where $\mathbf{v}$ is the optimal solution of the optimization problem (14). It is essential that this quantity be computed causally by a filter as s goes from 0 to T. The method which consists of finding the optimal $\mathbf{v}(t)$, $t \in [0,s]$, for every s, by solving a two point boundary value problem and using it to evaluate $\Phi_\beta(\mathbf{v},s)$ and in particular to test if it crosses 1, is extremely costly, inaccurate and almost impossible to implement.

It turns out that

$$\mathbf{\Gamma}_\beta(s) = \Phi_\beta(\mathbf{v},s) - \lambda_{\beta,s}\|\mathbf{v}\|^2, \tag{19}$$

where $\mathbf{v}$ is the optimal solution over $[0,s]$, can be evaluated causally because it is the cost of the linear quadratic optimization problem being solved (see (Campbell and Nikoukhah, 2004), Section 2.6.2).

Theorem 3. The solution $\mathbf{\Gamma}_\beta(s)$ of the optimization problem (14), satisfies

$$\dot{\mathbf{\Gamma}}_\beta = \boldsymbol{\mu}^T\mathbf{R}_{\lambda,\beta}^{-1}\boldsymbol{\mu}, \quad \mathbf{\Gamma}_\beta(0) = \mathbf{0}, \tag{20}$$

where $\boldsymbol{\mu}$ is given by

$$\boldsymbol{\mu} = \mathbf{C}\hat{\mathbf{x}} + \bar{\mathbf{N}}\mathbf{u} \tag{21a}$$

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} - \widetilde{\mathbf{S}}_{\lambda,\beta}\mathbf{R}_{\lambda,\beta}^{-1}\boldsymbol{\mu} + \bar{\mathbf{M}}\mathbf{u}, \ \hat{\mathbf{x}}(0) = \bar{\mathbf{x}} \tag{21b}$$

and $\widetilde{\mathbf{S}}_{\lambda,\beta} = \mathbf{S}_{\lambda,\beta} + \mathbf{P}\mathbf{C}^T$.

The problem, however, is that $\mathbf{\Gamma}_\beta$ is not what is needed, but rather Φ_β which can be expressed as

$$\Phi_\beta(\mathbf{v},s) = (\mathbf{x}(0)-\bar{\mathbf{x}})^T\mathbf{P}_\beta^{-1}(\mathbf{x}(0)-\bar{\mathbf{x}}) + \int_0^s \begin{pmatrix}\boldsymbol{\nu}\\ \mathbf{v}\end{pmatrix}^T\begin{pmatrix}\mathbf{J}_\beta & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{\nu}\\ \mathbf{v}\end{pmatrix} dt \tag{22}$$

where $\mathbf{x}$, $\boldsymbol{\nu}$ and $\mathbf{v}$ are optimal solutions of the optimization problem (14). It turns out that Φ_β can also be computed causally.

Theorem 4. Let $\hat{\mathbf{x}}$ and $\mathbf{P}$ be defined by (21b) and (15), and consider $\boldsymbol{\xi}$ and $\mathbf{\Psi}$ defined by the following equations:

$$\begin{aligned}\dot{\mathbf{\Psi}} = (\mathbf{A} - \widetilde{\mathbf{S}}_{\lambda,\beta}\mathbf{R}_{\lambda,\beta}^{-1}\mathbf{C})\mathbf{\Psi} + \bar{\mathbf{Q}} \\ + \mathbf{\Psi}(\mathbf{A} - \widetilde{\mathbf{S}}_{\lambda,\beta}\mathbf{R}_{\lambda,\beta}^{-1}\mathbf{C})^T, \mathbf{\Psi}(0) = \mathbf{P}_\beta\end{aligned} \tag{23}$$

$$\begin{aligned}\bar{\mathbf{Q}} = \mathbf{Q}_\beta - \widetilde{\mathbf{S}}_{\lambda,\beta}\mathbf{R}_{\lambda,\beta}^{-1}\mathbf{S}_\beta{}^T - \mathbf{S}_\beta\mathbf{R}_{\lambda,\beta}^{-1}\widetilde{\mathbf{S}}_{\lambda,\beta}^T \\ + \widetilde{\mathbf{S}}_{\lambda,\beta}\mathbf{R}_{\lambda,\beta}^{-1}\mathbf{R}_\beta\mathbf{R}_{\lambda,\beta}^{-1}\widetilde{\mathbf{S}}_{\lambda,\beta}^T\end{aligned} \tag{24}$$

where $\mathbf{Q}_\beta$, $\mathbf{S}_\beta$ and $\mathbf{R}_\beta$ are given in (11), and

$$\begin{aligned}\dot{\boldsymbol{\xi}} = -(\mathbf{A} - (\mathbf{S}_{\lambda,\beta} + \mathbf{P}\mathbf{C}^T)\mathbf{R}_{\lambda,\beta}^{-1}\mathbf{C} + \bar{\mathbf{Q}}\mathbf{\Psi}^{-1})^T\boldsymbol{\xi} \\ -(\mathbf{C}^T + \mathbf{\Psi}^{-1}\bar{\mathbf{Q}}\mathbf{D}^T)\mathbf{R}_{\lambda,\beta}^{-1}(\mathbf{C}\hat{\mathbf{x}} + \bar{\mathbf{N}}\mathbf{u}),\end{aligned} \tag{25}$$

with $\boldsymbol{\xi}(0) = \mathbf{0}$. Then

$$\Phi_\beta(\mathbf{v},s) = \boldsymbol{\xi}(s)^T\mathbf{\Psi}(s)\boldsymbol{\xi}(s) + \sigma(s) \tag{26}$$

where

$$\begin{aligned}\dot{\sigma} = \begin{pmatrix}\boldsymbol{\xi}\\ \mathbf{R}_{\lambda,\beta}^{-1}(\mathbf{C}\hat{\mathbf{x}} + \bar{\mathbf{N}}\mathbf{u}) - \widetilde{\mathbf{S}}_{\lambda,\beta}^T\boldsymbol{\xi}\end{pmatrix}^T\begin{pmatrix}\mathbf{Q}_\beta & \mathbf{S}_\beta\\ \mathbf{S}_\beta^T & \mathbf{R}_\beta\end{pmatrix} \\ \times\begin{pmatrix}\boldsymbol{\xi}\\ \mathbf{R}_{\lambda,\beta}^{-1}(\mathbf{C}\hat{\mathbf{x}} + \bar{\mathbf{N}}\mathbf{u}) - \widetilde{\mathbf{S}}_{\lambda,\beta}^T\boldsymbol{\xi}\end{pmatrix}, \sigma(0) = 0\end{aligned} \tag{27}$$

This result is obtained as a special case of the problem considered in the Appendix.

Computation of $\lambda_{\beta,s}$ The computation of the $\lambda_{\beta,s}$ can now be implemented using a lambda iteration algorithm. Suppose the problem is being solved over the time interval $[0,T)$. Then to compute the optimal $\lambda_{\beta,T}$, integrate the system of ordinary differential equations (ode) (21b), (15), (23), (25) and (27), with the zero crossing test $\boldsymbol{\xi}(t)^T\boldsymbol{\Psi}(t)\boldsymbol{\xi}(t)+\sigma(t)-1=0$. Theorem 8 is used if $\boldsymbol{\Psi}$ in (25) is not invertible. Many ode solvers, such as `lsodar`, have a built-in mechanism for halting the simulation when a specified quantity is zero. λ iterations are performed until the zero-crossing occurs at $t=T$. The corresponding λ is then $\lambda_{\beta,T}$.

In some cases, as λ varies, the zero-crossing may jump from an $s<T$ to a point beyond T. In such a case, the time from s to T is useless for reducing the cost of the test signal and the test interval can be shortened by setting $T=s$. In the sequel it is assumed that this adjustment is always done when applicable.

The above method very efficiently computes $\lambda_{\beta,T}$. Thanks to (19) and the fact that $\Phi_\beta(\mathbf{v},T) = 1$, it follows

$$\|\mathbf{v}\|^2 = (1-\boldsymbol{\Gamma}_\beta(T))/\lambda_{\beta,T} \tag{28}$$

Optimizing $\|\mathbf{v}\|^2$ over β is then done by simply discretizing the interval $[0,1]$, where β is confined to, and computing the cost for each discrete value of β. The optimal β is then simply the one corresponding to the lowest cost. Denote the optimal β and the corresponding $\lambda_{\beta,T}$, respectively β^* and λ^*.

Computation of the auxiliary signal Once β^* and λ^* are computed one can proceed with the computation of the optimal auxiliary signal. This is done by expressing the necessary conditions of optimality as a two-point boundary value system:

$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}\\ \boldsymbol{\zeta}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\Omega}_{11} & \boldsymbol{\Omega}_{12}\\ \boldsymbol{\Omega}_{21} & \boldsymbol{\Omega}_{22}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \boldsymbol{\zeta}\end{pmatrix} + \begin{pmatrix}\boldsymbol{\Omega}_{13}\\ \boldsymbol{\Omega}_{23}\end{pmatrix}\mathbf{u} \tag{29}$$

with boundary conditions:

$$\mathbf{x}(0) = \bar{\mathbf{x}} + \mathbf{P}_{\beta^*}\boldsymbol{\zeta}(0) \tag{30a}$$

$$\boldsymbol{\zeta}(\tau) = \mathbf{0}. \tag{30b}$$

where $\boldsymbol{\Omega}_{11} = -\boldsymbol{\Omega}_{22}^T = \mathbf{A} - \mathbf{S}_{\lambda^*,\beta^*}\mathbf{R}^{-1}_{\lambda^*,\beta^*}C$, $\boldsymbol{\Omega}_{12} = \mathbf{Q}_{\lambda^*,\beta^*} - \mathbf{S}_{\lambda^*,\beta^*}\mathbf{R}^{-1}_{\lambda^*,\beta^*}\mathbf{S}^T_{\lambda^*,\beta^*} = \boldsymbol{\Omega}_{21}^T$, $\boldsymbol{\Omega}_{22} = \mathbf{C}^T\mathbf{R}^{-1}_{\lambda^*,\beta^*}\mathbf{C}$, $\boldsymbol{\Omega}_{31} = \bar{\mathbf{M}} - \mathbf{S}_{\lambda^*,\beta^*}\mathbf{R}^{-1}_{\lambda^*,\beta^*}\bar{\mathbf{N}}$, $\boldsymbol{\Omega}_{32} = \mathbf{C}^T\mathbf{R}^{-1}_{\lambda^*,\beta^*}\bar{\mathbf{N}}$.

Theorem 5. The optimal auxiliary signal is

$$\mathbf{v}^* = ((\mathbf{B}^T - \mathbf{D}^T\mathbf{R}^{-1}_{\lambda^*,\beta^*}\mathbf{S}^T_{\lambda^*,\beta^*})\boldsymbol{\zeta} - \mathbf{D}^T\mathbf{R}^{-1}_{\lambda^*,\beta^*}(\mathbf{C}\mathbf{x}+\bar{\mathbf{N}}\mathbf{u}))/\lambda^* \tag{31}$$

where $\mathbf{x}$ and $\boldsymbol{\zeta}$ are solutions of (29), (30).

The solution of the boundary value problem can be constructed using various techniques. If the detection interval is short, straightforward techniques such as a shooting method can be used. On long intervals, the special properties of the $\boldsymbol{\Omega}$ matrix (Hamiltonian properties) can be used to construct numerically stable solutions. This problem which resembles the classical fixed-interval smoothing problem can be solved both by a two-filter solution or the extension of Tauch-Tung-Striebel method (Gelb, 1984).

The Tauch-Tung-Striebel method can be implemented by using System (21) which corresponds to the "forward Kalman filter" in this method. It is straightfoward to verify that the solution of the boundary value problem, $(\mathbf{x},\boldsymbol{\zeta})$, satisfy $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{P}\boldsymbol{\zeta}$ where $\mathbf{P}$ is the solution of the Riccati equation (15) with $\lambda=\lambda^*$ and $\beta=\beta^*$. This can be used to show that

$$\begin{aligned}\dot{\mathbf{x}} = {} & (\mathbf{A} - \mathbf{S}_{\lambda^*,\beta^*}\mathbf{R}^{-1}_{\lambda^*,\beta^*}\mathbf{C})\mathbf{x} \\ & +(\bar{\mathbf{M}} - \mathbf{S}_{\lambda^*,\beta^*}\mathbf{R}^{-1}_{\lambda^*,\beta^*}\bar{\mathbf{N}})\mathbf{u} + (\mathbf{Q}_{\lambda^*,\beta^*} - \\ & \mathbf{S}_{\lambda^*,\beta^*}\mathbf{R}^{-1}_{\lambda^*,\beta^*}\mathbf{S}^T_{\lambda^*,\beta^*})\mathbf{P}^{-1}(\mathbf{x}-\hat{\mathbf{x}}).\end{aligned} \tag{32}$$

(31) can also be rewritten as

$$\begin{aligned}\mathbf{v}^* = {} & ((\mathbf{B}^T - \mathbf{D}^T\mathbf{R}^{-1}_{\lambda^*,\beta^*}\mathbf{S}^T_{\lambda^*,\beta^*})\mathbf{P}^{-1}(\hat{\mathbf{x}}-\mathbf{x}) \\ & -\mathbf{D}^T\mathbf{R}^{-1}_{\lambda^*,\beta^*}(\mathbf{C}\mathbf{x}+\bar{\mathbf{N}}\mathbf{u}))/\lambda^*.\end{aligned} \tag{33}$$

System (32) can be integrated backward using the finale condition $\mathbf{x}(T) = \hat{\mathbf{x}}(T)$ and the optimal solution v is obtained from (33).

4. CONCLUSION

The methodology for constructing optimal robust auxiliary signals has been extended. In this new framework, the system under consideration can have nonzero initial condition and be driven by known input signals. This extension allows for the consideration of more general types of failures such as those which can be modeled by a bias.

The proposed method for obtaining the auxiliary signal is constructive and can easily be implemented in Scilab or Matlab.

The application of the auxiliary signal is performed in the manner as in Chapter 3 of (Campbell and Nikoukhah, 2004). Also, the presence of additional inputs and non zero initial condition does not affect the implementation of the detection filter obtained in Chapter 2 of (Campbell and Nikoukhah, 2004).

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5. APPENDIX

Consider the following optimization problem

$$J(s) = \text{ext}_{\mathbf{x}(0),\boldsymbol{\nu}} (\mathbf{x}(0) - vx_0)^T \mathbf{P}_0^{-1} (\mathbf{x}(0) - \mathbf{x}_0) + \int_0^s \boldsymbol{\nu}^T \boldsymbol{\Gamma} \boldsymbol{\nu}\; dt \tag{34}$$

subject to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\nu} + \mathbf{a} \tag{35a}$$

$$\mathbf{b} = \mathbf{C}\mathbf{x} + \mathbf{D}\boldsymbol{\nu}. \tag{35b}$$

Matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ have piecewise continuous time entries, $\mathbf{D}$ has full row rank, $\mathbf{P}_0 > \mathbf{0}$, and $\boldsymbol{\Gamma}$ is symmetric and invertible but not necessarily sign definite. Vectors $\mathbf{a}$ and $\mathbf{b}$ are known continuous functions of time. Note that this is not just one optimization problem but a family of optimization problems for different values of s.

Here ext stands for extremum which for the application in this paper stands for $\max_{\mathbf{v}} \min_{\mathbf{x},\nu}$ but the result stated here is more general.

The value of $J(s)$ can be computed causally for all s using a forward running filter. Now let

$$K(s) = (\mathbf{x}(0) - \mathbf{z}_0)^T \boldsymbol{\Pi}_0 (\mathbf{x}(0) - \mathbf{z}_0) + \int_0^s \boldsymbol{\nu}^T \boldsymbol{\Delta} \boldsymbol{\nu}\; dt \tag{36}$$

where $\mathbf{x}$ and $\boldsymbol{\nu}$ are optimal solutions to the optimization problem (34), (35). The problem we consider here is how can $K(s)$ be computed causally as well (not to fix s, run a backward filter to compute optimal $\mathbf{x}$ and $\boldsymbol{\nu}$ and then compute K). Here $\boldsymbol{\Pi}_0 > 0$ and $\boldsymbol{\Delta}$ is symmetric but not necessarily invertible or sign definite. Note that if $\mathbf{z}_0 = \mathbf{x}_0$, $\boldsymbol{\Pi}_0 = P_0^{-1}$ and $\boldsymbol{\Delta} = \boldsymbol{\Gamma}$, then $K(s) = J(s)$.

The solution to the optimization problem (34) , (35) is given by

$$J(s) = \int_0^s \boldsymbol{\mu}^T \mathbf{R}^{-1} \boldsymbol{\mu}\; dt \tag{37}$$

where $\boldsymbol{\mu} = \mathbf{C}\hat{\mathbf{x}} - \mathbf{b}$ and $\hat{\mathbf{x}}$ satisfies

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} - (\mathbf{S} + \mathbf{P}\mathbf{C}^T)\mathbf{R}^{-1}\boldsymbol{\mu} + \mathbf{a},\ \hat{\mathbf{x}}(0) = \mathbf{x}_0 \tag{38}$$

where $\mathbf{P}$ is the solution of the Riccati equation

$$\dot{\mathbf{P}} = (\mathbf{A} - \mathbf{S}\mathbf{R}^{-1}\mathbf{C})\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{S}\mathbf{R}^{-1}\mathbf{C})^T - \mathbf{P}\mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}\mathbf{P} + \mathbf{Q} - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T, \mathbf{P}(0) = \mathbf{P}_0 \tag{39}$$

with

$$\begin{pmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{pmatrix} = \begin{pmatrix} \mathbf{B} \\ \mathbf{D} \end{pmatrix} \boldsymbol{\Gamma}^{-1} \begin{pmatrix} \mathbf{B} \\ \mathbf{D} \end{pmatrix}^T. \tag{40}$$

This solution is obtained using the method of dynamic programming where we solve the optimization problem (34) by adding the constraint $\mathbf{x} = \mathbf{x}(s)$. The optimization problem over $[0, s]$ is solved assuming the value of $\mathbf{x}(s)$ is given and is equal to $\mathbf{x}$. This constrained cost is

$$J_{\mathbf{x}}(s) = (\mathbf{x} - \hat{\mathbf{x}}(s))^T \mathbf{P}^{-1}(s)(\mathbf{x} - \hat{\mathbf{x}}(s)) + \int_0^s \boldsymbol{\mu}^T \mathbf{R}^{-1} \boldsymbol{\mu}\, dt.$$

Optimizing it over $\mathbf{x}$ to find the optimal solution is trivial and gives $\mathbf{x} = \hat{\mathbf{x}}(s)$. The optimal value of $\boldsymbol{\nu}$ is obtained from

$$\min_{\boldsymbol{\nu}} (\boldsymbol{\nu} - \boldsymbol{\Gamma}^{-1}\mathbf{B}^T\mathbf{P}^{-1}(\mathbf{x} - \hat{x}))^T \boldsymbol{\Gamma} \times (\boldsymbol{\nu} - \boldsymbol{\Gamma}^{-1}\mathbf{B}^T\mathbf{P}^{-1}(\mathbf{x} - \hat{\mathbf{x}})) \tag{41}$$

subject to $\mathbf{b} - \mathbf{C}\mathbf{x} = \mathbf{D}\boldsymbol{\nu}$. See the proof of Theorem 2.6.2 in (Campbell and Nikoukhah, 2004). This means that the optimal $\boldsymbol{\nu}$ can be obtained using the following Lagrangian:

$$\mathcal{L} = (\boldsymbol{\nu} - \boldsymbol{\Gamma}^{-1}\mathbf{B}^T\mathbf{P}^{-1}(\mathbf{x} - \hat{\mathbf{x}}))^T \boldsymbol{\Gamma} \times (\boldsymbol{\nu} - \boldsymbol{\Gamma}^{-1}\mathbf{B}^T\mathbf{P}^{-1}(\mathbf{x} - \hat{\mathbf{x}})) + \boldsymbol{\lambda}^T(\mathbf{b} - \mathbf{C}\mathbf{x} - \mathbf{D}\mathbf{v}) \tag{42}$$

The optimality conditions yield

$$\boldsymbol{\Gamma}\boldsymbol{\nu} - \mathbf{D}^T\boldsymbol{\lambda} = \mathbf{B}^T\mathbf{P}^{-1}(\mathbf{x} - \hat{\mathbf{x}}) \tag{43}$$

$$\mathbf{D}\boldsymbol{\nu} = \mathbf{b} - \mathbf{C}\mathbf{x} \tag{44}$$

from which, thanks to (40), gives

$$\boldsymbol{\nu} = \boldsymbol{\Gamma}^{-1}(\boldsymbol{\Phi}(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{D}^T\mathbf{R}^{-1}(\mathbf{b} - \mathbf{C}\mathbf{x})) \tag{45}$$

where $\boldsymbol{\Phi} = (\mathbf{B} - \mathbf{S}\mathbf{R}^{-1}\mathbf{D})^T\mathbf{P}^{-1}$.

To construct a causal solution to K, consider the constrained problem with $\mathbf{x} = \mathbf{x}(s)$. Then make the assumption that with this constraint, the value of $K(s)$, which now is a function of $\mathbf{x}$, is quadratic in $\mathbf{x}$ and thus can be expressed as follows:

$$K(s) = (\mathbf{x} - \mathbf{z}(s))^T \boldsymbol{\Pi}(s)(\mathbf{x} - \mathbf{z}(s)) + \sigma(s) \tag{46}$$

where $\mathbf{z}(0) = \mathbf{z}_0$ and $\boldsymbol{\Pi}(0) = \boldsymbol{\Pi}_0$. Variables $\mathbf{z}(s)$, $\boldsymbol{\Pi}$, and σ are to be determined.

To verify the assumption and determine $\mathbf{z}(s)$, $\boldsymbol{\Pi}$ and σ, note that from (36) that

$$\dot{K} = \boldsymbol{\nu}^T \boldsymbol{\Delta} \boldsymbol{\nu} \tag{47}$$

where $\boldsymbol{\nu}$ is given in (45). On the other hand (46) provides that

$$\dot{K} = (\dot{\mathbf{x}} - \dot{\mathbf{z}})^T \mathbf{\Pi}(\mathbf{x} - \mathbf{z}) + (\mathbf{x} - \mathbf{z})^T \mathbf{\Pi}(\dot{\mathbf{x}} - \dot{\mathbf{z}}) + (\mathbf{x} - \mathbf{z})^T \dot{\mathbf{\Pi}}(\mathbf{x} - \mathbf{z}) + \dot{\sigma}. \quad (48)$$

Note that from (35a) and (45),

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{\Gamma}^{-1}(\mathbf{\Phi}(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{D}^T\mathbf{R}^{-1}(\mathbf{b} - \mathbf{C}\mathbf{x})) + \mathbf{a}$$

which can also be expressed as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \widetilde{\mathbf{Q}}\mathbf{P}^{-1}(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{S}\mathbf{R}^{-1}(\mathbf{b} - \mathbf{C}\mathbf{x})) + \mathbf{a}$$

where $\widetilde{\mathbf{Q}} = \mathbf{Q} - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T$. Setting (47) equal to (48) gives

$$(\mathbf{\Phi}(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{D}^T\mathbf{R}^{-1}(\mathbf{b} - \mathbf{C}\mathbf{x}))^T\mathbf{\Gamma}^{-1}\mathbf{\Delta}\mathbf{\Gamma}^{-1} \times (\mathbf{\Phi}(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{D}^T\mathbf{R}^{-1}(\mathbf{b} - \mathbf{C}\mathbf{x})) = ((\mathbf{A}\mathbf{x} + \widetilde{\mathbf{Q}}\mathbf{P}^{-1}(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{S}\mathbf{R}^{-1}(\mathbf{b} - \mathbf{C}\mathbf{x})) + \mathbf{a} - \dot{\mathbf{z}})^T\mathbf{\Pi}(\mathbf{x} - \mathbf{z}) + (\mathbf{x} - \mathbf{z})^T\mathbf{\Pi}((\mathbf{A}\mathbf{x} + \widetilde{\mathbf{Q}}\mathbf{P}^{-1}(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{S}\mathbf{R}^{-1}(\mathbf{b} - \mathbf{C}\mathbf{x})) + \mathbf{a} - \dot{\mathbf{z}}) + (\mathbf{x} - \mathbf{z})^T\dot{\mathbf{\Pi}}(\mathbf{x} - \mathbf{z}) + \dot{\sigma}. \quad (49)$$

Note that both sides of (49) are quadratic in $\mathbf{x}$.

Matching coefficients of second order terms of $\mathbf{x}$ gives the following Lyapunov equation for $\mathbf{\Pi}$,

$$(\mathbf{\Phi} - \mathbf{D}^T\mathbf{R}^{-1}\mathbf{C})^T\mathbf{\Gamma}^{-1}\mathbf{\Delta}\mathbf{\Gamma}^{-1}(\mathbf{\Phi} - \mathbf{D}^T\mathbf{R}^{-1}\mathbf{C}) = (\mathbf{A} + \widetilde{\mathbf{Q}}\mathbf{P}^{-1} - \mathbf{S}\mathbf{R}^{-1}\mathbf{C})^T\mathbf{\Pi} + \mathbf{\Pi}(\mathbf{A} + \widetilde{\mathbf{Q}}\mathbf{P}^{-1} - \mathbf{S}\mathbf{R}^{-1}\mathbf{C}) + \dot{\mathbf{\Pi}} \quad (50)$$

with $\mathbf{\Pi}(0) = \mathbf{\Pi}_0$.

Matching the linear terms in $\mathbf{x}$, gives

$$(\mathbf{\Phi} - \mathbf{D}^T\mathbf{R}^{-1}\mathbf{C})^T\mathbf{\Gamma}^{-1}\mathbf{\Delta}\mathbf{\Gamma}^{-1}(-\mathbf{\Phi}\hat{\mathbf{x}} + \mathbf{D}^T\mathbf{R}^{-1}\mathbf{b}) = \mathbf{\Pi}(\mathbf{S}\mathbf{R}^{-1}\mathbf{b} - \widetilde{\mathbf{Q}}\mathbf{P}^{-1}\hat{\mathbf{x}} + \mathbf{a} - \dot{\mathbf{z}}) - ((\mathbf{A} + \widetilde{\mathbf{Q}}\mathbf{P}^{-1} - \mathbf{S}\mathbf{R}^{-1}\mathbf{C})^T\mathbf{\Pi} + \dot{\mathbf{\Pi}})\mathbf{z} \quad (51)$$

which gives

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + (\widetilde{\mathbf{Q}}\mathbf{P}^{-1} - \mathbf{\Omega}\mathbf{\Phi})(\mathbf{z} - \hat{\mathbf{x}}) - (\mathbf{S} - \mathbf{\Omega}\mathbf{D}^T)\mathbf{R}^{-1}(\mathbf{C}\mathbf{z} - \mathbf{b}) + \mathbf{a} \quad (52)$$

where $\mathbf{\Omega} = \mathbf{\Pi}^{-1}(\mathbf{\Phi} - \mathbf{D}^T\mathbf{R}^{-1}\mathbf{C})^T\mathbf{\Gamma}^{-1}\mathbf{\Delta}\mathbf{\Gamma}^{-1}$.

Finally matching the terms which are independent of $\mathbf{x}$, after a long computation, yields

$$\dot{\sigma} = \bar{\boldsymbol{\mu}}^T\mathbf{\Gamma}^{-1}\mathbf{\Delta}\mathbf{\Gamma}^{-1}\bar{\boldsymbol{\mu}} \quad (53)$$

$$\bar{\boldsymbol{\mu}} = \mathbf{\Phi}(\mathbf{z} - \hat{\mathbf{x}}) + \mathbf{D}^T\mathbf{R}^{-1}(\mathbf{b} - \mathbf{C}\mathbf{z}). \quad (54)$$

Theorem 6. Let $\hat{\mathbf{x}}$ and $\mathbf{z}$ be defined by (38) and (52), where $\mathbf{P}$ is defined in (39) and $\mathbf{\Pi}$ is the solution of the Lyapunov equation (50). Then

$$K(s) = (\hat{\mathbf{x}}(s) - \mathbf{z}(s))^T\mathbf{\Pi}(s)(\hat{\mathbf{x}}(s) - \mathbf{z}(s)) + \int_0^s \bar{\boldsymbol{\mu}}^T\mathbf{\Gamma}^{-1}\mathbf{\Delta}\mathbf{\Gamma}^{-1}\bar{\boldsymbol{\mu}}\, dt \quad (55)$$

where $\bar{\boldsymbol{\mu}}$ is given in (54).

The result of Theorem 6 can gives $K(s)$ as long as $\mathbf{P}$ remains non-singular. This would be the case, in particular, if the optimization problem (34) were a pure min or pure max problem. In the application here though, it is a a max-min problem and the $\mathbf{P}$ matrix is not necessarily sign-definite and can become singular at a point on the interval. In this case, $\mathbf{\Pi}$ diverges at this point, even though K does not.

To avoid this problem, it is necessary to perform a change of variable in such a way as to avoid $\mathbf{P}^{-1}$ in all the equations of the system. This can be done as follows. Let $\mathbf{\Psi} = \mathbf{P}\mathbf{\Pi}\mathbf{P}$ and $\boldsymbol{\xi} = \mathbf{P}^{-1}(\hat{\mathbf{x}} - \mathbf{z})$. Note that $\dot{\mathbf{\Psi}} = \dot{\mathbf{P}}\mathbf{\Pi}\mathbf{P} + \mathbf{P}\dot{\mathbf{\Pi}}\mathbf{P} + \mathbf{P}\mathbf{\Pi}\dot{\mathbf{P}}$, which using (39) and (50) can be expressed as follows

$$\dot{\mathbf{\Psi}} = (\mathbf{A} - (\mathbf{S} + \mathbf{P}\mathbf{C}^T)\mathbf{R}^{-1}\mathbf{C})\mathbf{\Psi} + \mathbf{\Psi}(\mathbf{A} - (\mathbf{S} + \mathbf{P}\mathbf{C}^T)\mathbf{R}^{-1}\mathbf{C})^T + J\mathbf{\Gamma}^{-1}\mathbf{\Delta}\mathbf{\Gamma}^{-1}\mathbf{J}^T \quad (56)$$

$$\mathbf{J} = \mathbf{B} - (\mathbf{S} + \mathbf{P}\mathbf{C}^T)\mathbf{R}^{-1}\mathbf{D}. \quad (57)$$

Similarly, $\dot{\boldsymbol{\xi}} = \mathbf{P}^{-1}(\dot{\hat{\mathbf{x}}} - \dot{\mathbf{z}}) - \mathbf{P}^{-1}\dot{\mathbf{P}}\mathbf{P}^{-1}(\hat{\mathbf{x}} - \mathbf{z})$, which using in addition to (39) and (50), (38) and (52), can be expressed as follows

$$\dot{\boldsymbol{\xi}} = -(\mathbf{A} - (\mathbf{S} + \mathbf{P}\mathbf{C}^T)\mathbf{R}^{-1}\mathbf{C} + \mathbf{J}\mathbf{\Gamma}^{-1}\mathbf{\Delta}\mathbf{\Gamma}^{-1}\mathbf{J}^T\mathbf{\Psi}^{-1})^T\boldsymbol{\xi} - (\mathbf{C}^T + \mathbf{\Psi}^{-1}\mathbf{J}\mathbf{\Gamma}^{-1}\mathbf{\Delta}\mathbf{\Gamma}^{-1}\mathbf{D}^T) \times \mathbf{R}^{-1}(\mathbf{C}\hat{\mathbf{x}} - \mathbf{b}). \quad (58)$$

note that $\boldsymbol{\xi}(0) = 0$ and $\mathbf{\Psi}(0) = \mathbf{P}_0$.

Theorem 7. Let $\hat{\mathbf{x}}$ and $\boldsymbol{\xi}$ be defined by (38) and (58), where $\mathbf{P}$ is defined in (39) and $\mathbf{\Psi}$ is the solution of the Lyapunov equation (56). Then

$$K(s) = \boldsymbol{\xi}(s)^T\mathbf{\Psi}(s)\boldsymbol{\xi}(s) + \int_0^s \tilde{\boldsymbol{\mu}}^T\mathbf{\Gamma}^{-1}\mathbf{\Delta}\mathbf{\Gamma}^{-1}\tilde{\boldsymbol{\mu}}\, dt$$

where $\tilde{\boldsymbol{\mu}} = -\mathbf{J}^T\boldsymbol{\xi} + \mathbf{D}^T\mathbf{R}^{-1}(\mathbf{b} - \mathbf{C}\hat{\mathbf{x}})$.

However (58) still has a $\mathbf{\Psi}^{-1}$ in it. The actual integration that is implemented is given in the next theorem.

Theorem 8. $K(s)$ can be evaluated by integrating the system of differential equations in the variables $K, \mathbf{\Psi}, \boldsymbol{\theta} = \mathbf{\Psi}\boldsymbol{\xi}$ given by (56) and

$$\dot{\boldsymbol{\theta}} = (\mathbf{A} - (\mathbf{S} + \mathbf{P}\mathbf{C}^T)\mathbf{R}^{-1})\boldsymbol{\theta} - \mathbf{\Psi}(\mathbf{C}^T + \mathbf{J}\mathbf{\Gamma}^{-1}\mathbf{\Delta}\mathbf{\Gamma}^{-1}\mathbf{D}^T)\mathbf{R}^{-1}(\mathbf{C}\hat{\mathbf{x}} - \mathbf{b}) \quad (59)$$

$$\dot{K} = (\mathbf{b} - \mathbf{C}\hat{\mathbf{x}})^T\mathbf{R}^{-1}\mathbf{D}\mathbf{\Gamma}^{-1}\mathbf{\Delta}\mathbf{\Gamma}^{-1}\mathbf{D}\mathbf{R}^{-1}(\mathbf{b} - \mathbf{C}\hat{\mathbf{x}}) - (\mathbf{C}\mathbf{x} - \mathbf{b})^T\mathbf{R}^{-1}\mathbf{C}\boldsymbol{\theta} - \boldsymbol{\theta}^T\mathbf{C}^T\mathbf{R}^{-1}(\mathbf{C}\hat{\mathbf{x}} - \mathbf{b}). \quad (60)$$