# A METHOD FOR OBTAINING CONTINUOUS SOLUTIONS TO MULTIPARAMETRIC LINEAR PROGRAMS

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Abstract: A modification of the geometric algorithm for solving multiparametric linear programs (mp-LP) is presented. The modification preserves the simplicity of the algorithm and ensures that the optimal, piecewise affine, mapping from parameter to solution space is continuous. When the mp-LP has non-unique solutions, the optimizer with the least Euclidian norm is selected. 

Keywords: Multiparametric programming, constrained optimal control, linear programming, quadratic programming, explicit model predictive control.

## 1. INTRODUCTION

The multiparametric linear programming problem (mp-LP) is as follows:

$$z^*(\theta) \triangleq \min_{x\in\mathbb{R}^n} \left\{ c^T x \,|Ax \leq b + S\theta \right\}, \qquad (1)$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{q\times n}$, $b \in \mathbb{R}^{q\times 1}$, $S \in \mathbb{R}^{q\times p}$, and the vector $x \in \mathbb{R}^n$ is to be optimized for all values of the parameter vector $\theta \in \Theta$, where $\Theta \subseteq \mathbb{R}^p$ is some polyhedral set. In other words, we seek the optimal solution $x^* : \Theta \mapsto \mathbb{R}^n$.

Gal and Nedoma (Gal and Nedoma 1972) presented the first algorithm for solving multiparametric linear programs. The approach is based on visiting the optimal bases of the associated simplex tableau. Subsequently, algorithms similar to that of Gal et al. have been developed (van der Panne 1975, Yu and Zeleny 1976). The value function is piecewise affine on a number of polyhedra in the parameter space, but can also, by using the dual solution, be represented as the maximum of a finite number of affine functions. The relationship between these representations can be utilized to solve mp-LP programs (Schechter 1987).

Only recently, (Borrelli *et al.* 2003) proposed a fundamentally different algorithm for mp-LPs where the geometric properties of the problem is used to explore the parameter space directly. The direct exploration of the parameter space was first presented for multiparametric mixed integer linear programs (Acevedo and Pistikopoulos 1997) and later utilized in an algorithm for multiparametric quadratic programs (Bemporad *et al.* 2002*b*).

Extensive work has been done on the properties of the value- and optimizer functions for different types of multiparametric problems. The optimal value as a function of the parameters has been shown to be continuous for strictly convex mp-QP (Bemporad *et al.* 2002*b*, Fiacco 1983, Bank *et al.* 1983), for mp-LP (Gal and Nedoma 1972, Dinkelbach 1969), for convex mp-QP (Best and Ding 1972) and mp-NLPs (Fiacco 1983, Bank *et al.* 1983) that satisfies certain conditions.

The optimizer function will under some assumptions admit a continuous selection for a general class of multiparametric optimization problems (Michaels 1956). When the optimizer is unique, conditions for the optimizer function to be continuous is given in (Dantzig *et al.* 1967), and these concepts are further developed for cases where the optimizer may be non-unique in (Bank *et al.* 1983, Zhao 1997).

An algorithm that obtains a continuous optimizer function for (single) parametric LPs is presented in (Zhang and Liu 1990) while (Bohm 1975) indicates how to construct a continuous solution for mp-LPs. The latter, however, is not a complete algorithm for obtaining solutions to mp-LP programs. No region of optimality is constructed for the associated element of the continuous selection. Difficulties that arise when both the primal and dual solution of the mp-LP are non-unique, are not discussed.

The geometric algorithm for mp-LP (Borrelli *et al.* 2003) has the advantage of simplicity compared to the algorithm by (Gal and Nedoma 1972), especially when the primal solution is non-unique. A modification to the geometric algorithm that preserves the simplicity and yields a continuous solution function is therefore presented. Instead of choosing an arbitrary optimal basis to construct the region of optimality for an element of the selection, a quadratic criterion is minimized subject to mp-LP optimality. Using results from strictly convex multiparametric quadratic programming the minimizer function is characterized (Bemporad *et al.* 2002*b*, Tøndel *et al.* 2003*a*) and based on theory on minimization of strictly convex functions over continuous point to set maps (Berge 1963, Dantzig *et al.* 1967, Bank *et al.* 1983), global continuity of the optimizer function is proven.

When the algorithm is used to obtain an explicit solution to a model predictive control problem (Bemporad *et al.* 2002*a*), a continuous mapping is an advantage since a perturbation in the state, will not lead to discontinuous changes in the control input.

## 2. MULTIPARAMETRIC LINEAR PROGRAMMING

### 2.1 Preliminaries

If $A$ is a matrix, then $A_i$ denotes the $i^{\text{th}}$ row of $A$ and $A_{\mathcal{J}}$ denotes the sub-matrix consisting of the rows of $A$ corresponding to the index set $\mathcal{J}$.

Recall that the set of affine combinations of points in a set $S \subset \mathbb{R}^n$ is called the *affine hull* of $S$. The *dimension of a set* $S \subset \mathbb{R}^n$, denoted $\dim(S)$; is the dimension of the affine hull of $S$. If the dimension of $S$ is $n$, then $S$ is said to be full-dimensional.

Let the set of parameters for which the minimum in (1) exists be denoted $\Theta^*$ and let $X^*(\theta)$ be the set of optimizers to (1) for a given $\theta \in \Theta^*$. It is assumed that $\Theta^*$ is full-dimensional, see (Borrelli *et al.* 2003) for details.
The following definitions are taken from (Borrelli *et al.* 2003) and (Tøndel *et al.* 2003*b*).

*Definition 1.* (Active set). Let $x$ be a feasible solution to (1) for a given $\theta$. We define the *active constraints* as the set of constraints which fulfill $A_i x - b_i - S_i\theta = 0$, and *inactive constraints* as the set which fulfills $A_i x - b_i - S_i\theta < 0$. The *active set* $\mathcal{A}(x,\theta)$ is the set of indices of the active constraints, that is,

$$\mathcal{A}(x,\theta) \triangleq \{i \in \{1,\ldots,q\} | A_i x - b_i - S_i\theta = 0\}.$$

Moreover, let $\mathcal{N}(x,\theta)$ denote the set of inactive constraints, that is, $\mathcal{N}(x,\theta) \triangleq \{1,\ldots,q\}\backslash\mathcal{A}(x,\theta)$.

*Definition 2.* (Optimal active set). Let $\theta$ be given. Let the *optimal active set* $\mathcal{A}^*(\theta)$ be the set of constraints which are active for all $x \in X^*(\theta)$, that is

$$\mathcal{A}^*(\theta) \triangleq \{i | i \in \mathcal{A}(x,\theta), \forall x \in X^*(\theta)\} = \bigcap_{x \in X^*(\theta)} \mathcal{A}(x,\theta).$$

Let $\mathcal{N}^*(\theta) \triangleq \{1,\ldots,q\}\backslash\mathcal{A}^*(\theta)$.

*Definition 3.* (LICQ). For an active set $\mathcal{A}$, we say that the *linear independence constraint qualification (LICQ)* holds if the set of active constraint gradients are linearly independent, i.e., $A_{\mathcal{A}}$ has full row rank.

*Definition 4.* (Critical region). Given an optimal active set $\mathcal{A}^*$ we define the *critical region* as the set of parameters for which the optimal active set remains unchanged, that is,

$$\Theta_{\mathcal{A}^*} = \{\theta \in \Theta | \mathcal{A}^*(\theta) = \mathcal{A}^*\}. \tag{2}$$

It should be noted that critical regions are convex and that their closures are polyhedral. Since the optimal active set is unique for all $\theta \in \Theta^*$, critical regions cannot intersect, however, the intersection of their closures may be non-empty. Since $\Theta^*$ is assumed to be full-dimensional and the number of optimal active sets is finite, there exists a finite number of full-dimensional critical regions such that the union of their closures is equal to $\Theta^*$. The goal is to find a representation of the optimal mapping $x^* : \Theta^* \mapsto \mathbb{R}^n$ over a finite set of closed, full-dimensional, polyhedra $\mathcal{R} \triangleq \{R_{\mathcal{A}} | \mathcal{A} \in \mathcal{I}\}$(Borrelli *et al.* 2003) where $\cup_{\mathcal{A}\in\mathcal{I}} R_{\mathcal{A}} = \Theta^*$, $\mathcal{I}$ contains a subset of all possible active sets $\{\mathcal{A}(x^*(\theta),\theta) | \theta \in \Theta^*\}$, and each polyhedron is associated with an affine function $x^*_{\mathcal{A}}(\theta)$ that is optimal for $\theta \in R_{\mathcal{A}}$. Given a $\theta \in \Theta^*$ and the associated optimal active set $\bar{\mathcal{A}}^*$ such that the critical region $\Theta_{\bar{\mathcal{A}}^*}$ is full-dimensional. If $X^*(\theta)$ is a singleton for all $\theta \in \Theta_{\bar{\mathcal{A}}^*}$, then then $R_{\bar{\mathcal{A}}^*} \triangleq \mathrm{cl}(\Theta_{\bar{\mathcal{A}}^*})$. On the other hand, if $X^*(\theta)$ is not a singleton set, $\Theta_{\bar{\mathcal{A}}^*}$ is divided into a set of closed, full-dimensional, polyhedra $\{R_{\mathcal{A}_j},\ldots,R_{\mathcal{A}_k}\}$, $\{\mathcal{A}_j,\ldots,\mathcal{A}_k\} \in \mathcal{I}$, whose union is equal to $\mathrm{cl}(\Theta_{\bar{\mathcal{A}}^*})$, each associated with only one affine function. We refer to these polyhedra as *sub-regions*. The optimal solution function

$$x^*(\theta) = x^*_{\mathcal{A}}(\theta) \ \text{ if } \ \theta \in R_{\mathcal{A}},$$

is single valued, since if a given $\theta$ is in more than one $R_{\mathcal{A}}$, $x^*_{\mathcal{A}}$ is chosen according to some predetermined ordering of the sets in $\mathcal{R}$. If for every pair $(\mathcal{A}_i,\mathcal{A}_j) \in \mathcal{I}\times\mathcal{I}$:

$$x^*_{\mathcal{A}_i}(\theta) \neq x^*_{\mathcal{A}_j}(\theta) \Rightarrow \dim(R_{\mathcal{A}_i} \cap R_{\mathcal{A}_j}) \leq p-1, i \neq j,$$

then a given $\theta$ may only be in more than one $R_{\mathcal{A}}$ for lower dimensional subsets of $\Theta^*$.

Note that *closure of a full-dimensional critical region* is abbreviated *critical region* from this point on.

The dual of (1) can be written as (Borrelli *et al.* 2003)

$$v^*(\theta) \triangleq \min_{\pi \in \mathbb{R}^q} \left\{ (b + S\theta)^T \pi \,\middle|\, A^T \pi = c, \pi \leq 0 \right\}. \tag{3}$$

The primal feasibility, dual feasibility and the complementary slackness conditions for problems (1) and (3) are

$$Ax \leq b + S\theta, \tag{4a}$$

$$A^T \pi = c, \quad \pi \leq 0, \tag{4b}$$

$$(A_i x - b_i - S_i \theta)\pi_i = 0, \quad \forall i \in \{1, \ldots, q\}, \tag{4c}$$

respectively.

When it is clear form the context, the argument $\theta$ (or $\theta_0$) will be omitted when referring to an optimal active or inactive set.

*2.2 Summary of the geometric approach*

For convenience the geometric algorithm is summarized in the following four points (i)-(iv), see (Borrelli *et al.* 2003) for details.

**i)** Unique primal and dual solution: When both the primal and dual solution to (1) are unique for $\theta = \theta_0$, the value function, the optimizer function and the critical region corresponding to the active set $\mathcal{A}^*(\theta_0)$, are uniquely given by

$$z^*_{\mathcal{A}^*}(\theta) = (b + S\theta)^T \pi^*(\theta_0), \tag{5a}$$

$$x^*_{\mathcal{A}^*}(\theta) = A^{-1}_{\mathcal{A}^*} S_{\mathcal{A}^*}\theta + A^{-1}_{\mathcal{A}^*} b_{\mathcal{A}^*}, \tag{5b}$$

$$R_{\mathcal{A}^*} = \{\theta \in \Theta \,|\, A_{\mathcal{N}^*} x^*(\theta) \leq b_{\mathcal{N}^*} + S_{\mathcal{N}^*}\theta\} \tag{5c}$$

respectively, where $\pi^*(\theta_0)$ is the optimal dual solution.

**ii)** Non-unique dual solution: When the dual solution to (1) is non-unique for $\theta = \theta_0$, the optimizer function and critical region are found by applying Gauss reduction to the system of equalities, $A_{\mathcal{A}^*} x = b_{\mathcal{A}^*} + S_{\mathcal{A}^*}\theta$.

**iii)** Non-unique primal solution: Let $\theta = \theta_0$. Whenever the primal solution to (1) is non-unique, the optimizer and critical region can not be characterized by (5b) and (5c). This problem is solved by choosing a vertex of the feasible set of (1) for which $x^*(\theta_0) \in X^*(\theta_0)$ and using $\mathcal{A}(x^*(\theta_0), \theta_0)$ instead of $\mathcal{A}^*(\theta_0)$ in (5b) and (5c).

**iv)** Non-unique primal and dual solution: One of the optimizers $x^*(\theta_0)$ is chosen as described under point (iii). Since LICQ is violated at $x^*(\theta_0)$, Gauss reduction is to applied the system of equalities

$$A_{\mathcal{A}(x^*(\theta_0),\theta_0)} x = b_{\mathcal{A}(x^*(\theta_0),\theta_0)} + S_{\mathcal{A}(x^*(\theta_0),\theta_0)}\theta \tag{6}$$

to find $x^*(\theta)$ and the associated sub-region.

Note that when the primal solution is non-unique, the region obtained by following the procedure under point (iii) or (iv) is not a critical region in the sense of Definition 4, but a sub-region.

**Algorithm 1** Geometric algorithm for mp-LP

1: Let $Y \subseteq \Theta$ be the current region to be explored and let $\theta_0$ be in the interior of $Y$.
2: Solve the LP (1) for $\theta = \theta_0$.
3: Determine which of the four cases (i)-(iv) that applies and find the optimizer function and the associated critical(sub)- region as described under the respective point.
4: Partition the rest of the region into convex polyhedra according to the procedure given in (Borrelli *et al.* 2003) and for each nonempty element repeat steps 1-4.

## 3. OBTAINING CONTINUOUS SOLUTIONS TO MP-LP PROBLEMS

By arbitrarily choosing one of the optimal bases in $X^*(\theta)$ when (1) has multiple primal solutions, as suggested in Algorithm 1(Borrelli *et al.* 2003), the mapping from parameter to solution space may become discontinuous. The idea of the present paper is to replace the mp-LP with a strictly convex mp-QP that has been constructed such that its unique and continuous optimizer function $x^*_{qp}(\theta)$ is an optimal function for (1). The following local mp-QP minimizes the norm of the optimizer for $\theta \in \Theta_{\mathcal{A}^*}$:

$$y^*(\theta) \triangleq \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T x, \tag{7a}$$

$$Ax \leq b + S\theta, \tag{7b}$$

$$c^T x = z^*_{\mathcal{A}^*(\theta_0)}(\theta), \tag{7c}$$

where $z^*_{\mathcal{A}^*(\theta_0)}(\theta)$ is the optimal value function found from (5a), hence (1) must be solved for $\theta = \theta_0$ to obtain (7c). Index the equality constraint by $q+1$. For $\theta = \theta_0$ denote the optimizer as $x^*_{qp}(\theta_0)$ and define the index set $\mathcal{A}_{qp}(\theta_0)$ as the indices of (7b) that are active at the QP optimum, that is

$$\mathcal{A}_{qp}(\theta_0) \triangleq \{i \in \{1, \ldots, q\} | A_i x^*_{qp}(\theta_0) = b_i + S_i\theta_0\}. \tag{8}$$

Denote the set of inactive constraints as $\mathcal{N}_{qp}(\theta_0) \triangleq \{1, \ldots, q\} \backslash \mathcal{A}_{qp}(\theta_0)$.

*Proposition 5.* $x^*_{qp}(\theta) \in X^*(\theta)$ and $\mathcal{A}^*(\theta) \subseteq \mathcal{A}_{qp}(\theta)$, $\forall \theta \in \Theta^*$.

**PROOF.** Follows directly from (7c) and Definition 2.

To reduce the number of optimal active sets for (7) that violate LICQ, the mp-QP is replaced with another mp-QP:

*Lemma 6.* LICQ is violated for all optimal active sets for (7). Moreover, the following mp-QP is equivalent to (7), $\forall \theta \in \Theta_{\mathcal{A}^*}$:

$$y^*(\theta) = \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T x, \tag{9a}$$

$$A_i x = b_i + S_i\theta, \quad i \in \mathcal{A}^*, \tag{9b}$$

$$A_i x \leq b_i + S_i\theta, \quad i \in \mathcal{N}^*, \tag{9c}$$

for which LICQ is violated only if it is violated for $\mathcal{A}_{qp}$.

**PROOF.** First it is shown that (7b)-(7c) and (9b)-(9c) define the same set. This holds trivially if $c = 0$.

Let $c \neq 0$ and define $\mathcal{P}(\theta) \triangleq \{x \,|Ax \leq b + S\theta\}$. It is clear that $\mathcal{F} \triangleq \mathcal{P}(\theta) \cap \{x \,|c^T x = z^*_{\mathcal{A}^*}(\theta)\}$ is a face of $\mathcal{P}(\theta)$. The constraints fulfilled with equality $\forall x \in \mathcal{F}$ are exactly the constraints whose indices are in $\mathcal{A}^*$. From (Jones *et al.* 2004, Definition 8 and Theorem 12) we have that it is a one to one mapping from these constraints to the faces of $\mathcal{P}(\theta)$, and that $\mathcal{F} = \{x \,|A_{\mathcal{A}^*} x = b_{\mathcal{A}^*} + S_{\mathcal{A}^*}\theta\} \cap \mathcal{P}(\theta)$. Since the sets defined by (7b)-(7c) and (9b)-(9c) are equal, the LICQ assertions hold trivially.

The optimizer function $x^*_{qp}(\theta)$ associated with the active set $\mathcal{A}^*$ is found by (Bemporad *et al.* 2002*b*)

$$x^*_{qp}(\theta) = A^T_{\mathcal{A}_{qp}}(A_{\mathcal{A}_{qp}} A^T_{\mathcal{A}_{qp}})^{-1}(b_{\mathcal{A}_{qp}} + S_{\mathcal{A}_{qp}}\theta), \quad (10)$$

and the critical region (for the mp-QP, and a sub-region for (1)) for which $x^*_{qp}(\theta)$ is optimal is given by:

$$A_{\mathcal{N}_{qp}} A^T_{\mathcal{A}_{qp}}(A_{\mathcal{A}_{qp}} A^T_{\mathcal{A}_{qp}})^{-1}(b_{\mathcal{A}_{qp}} + S_{\mathcal{A}_{qp}}\theta) \leq b_{\mathcal{N}_{qp}} + S_{\mathcal{N}_{qp}}\theta, \quad (11a)$$

$$\lambda_i(\theta) \geq 0, \quad \forall i \in \mathcal{A}_{qp} \backslash \mathcal{A}^*, \quad (11b)$$

where

$$\lambda_{\mathcal{A}_{qp}}(\theta) = -(A_{\mathcal{A}_{qp}} A^T_{\mathcal{A}_{qp}})^{-1}(b_{\mathcal{A}_{qp}} + S_{\mathcal{A}_{qp}}\theta), \quad (12)$$

where $\lambda_{\mathcal{A}_{qp}}$ denotes the components of $\lambda$ corresponding to $\mathcal{A}_{qp}$. When the LP has multiple dual solutions, $A_{\mathcal{A}_{qp}}$ may not have full rank. However, it is still possible to characterize the optimizer function and an associated sub-region with a reduced active set, see (Bemporad *et al.* 2002*b*). Note that if (1) has non-unique primal solutions, the polyhedron defined by (11a)-(11b) is not a critical region in the sense of Definition 4, but a sub-region. Clearly

$$x^*_{\mathcal{A}_i}(\theta) \neq x^*_{\mathcal{A}_j}(\theta) \Rightarrow \dim(R_{\mathcal{A}_i} \cap R_{\mathcal{A}_j}) \leq p-1, i \neq j,$$

is satisfied by uniqueness of $x^*_{qp}(\theta)$ and $\mathcal{A}^*$, $\forall \theta \in \Theta^*$.

Let the primal solution to (1) with $\theta = \theta_0$ be non-unique. Algorithm 2 is proposed to replace the arbitrary selection of a optimizer function in the geometric algorithm (Borrelli *et al.* 2003).

**Algorithm 2** Proposed method

1: Identify $\mathcal{A}^*$ for $\theta = \theta_0$.
2: Minimize the Euclidian norm of the the solution by solving the QP obtained by fixing $\theta = \theta_0$ in (9) and identify $\mathcal{A}_{qp}$.
3: If LICQ holds at $x^*_{qp}(\theta_0)$, compute the optimizer function $x^*_{qp}(\theta)$ from (10) and the sub-region from (11a)-(11b). If LICQ is violated at $x^*_{qp}(\theta_0)$ find $x^*_{qp}(\theta)$ and the associated sub-region with a reduced active set.

*Lemma 7.* When the solution $\pi^*(\theta)$ to (3) is unique for a given $\theta \in \Theta^*$, then $\mathcal{A}^*(\theta)$ is uniquely given by

$$\mathcal{A}^*(\theta) = \{i \in \{1, \ldots, q\} \,|\pi^*_i(\theta) < 0\}. \quad (13)$$

**PROOF.** Let $x$ be an optimal solution to (1). Define the sets $\mathcal{K} = \{i \in \{1, \ldots, q\} \,|i \in \mathcal{A}(x, \theta), \pi^*_i < 0\}$ and $\mathcal{J} = \{i \in \{1, \ldots, q\} \,|i \in \mathcal{A}(x, \theta), \pi^*_i = 0\}$. It is obvious that $i \in \mathcal{K} \Rightarrow i \in \mathcal{A}^*$ since the complementarity condition holds for all optimal $x$. From (Mangasarian 1979) we have that $\pi^*$ is unique if and only if LICQ holds for $A_{\mathcal{K}}$ and there is at least one feasible solution $d$ to the system

$$A_{\mathcal{K}} d = 0, A_{\mathcal{J}} d < 0, \quad (14)$$

Assume first that the primal solution is non-unique. The set of feasible directions at $x$ is given by $\{r \,|A_{\mathcal{A}(x,\theta)} r \leq 0\}$ and consequently $\bar{x} = x + \alpha d$ is feasible for sufficiently small scalar $\alpha > 0$. It is clear that for $\alpha > 0$ the constraints in $\mathcal{J}$ are inactive, so it suffices to show that $\bar{x}$ is optimal:

$$c^T(x + \alpha d) = \pi^{*T} A(x + \alpha d) = \pi^{*T} Ax = c^T x, \quad (15)$$

where we have used (4b), (4c), and (14). This implies $i \in \mathcal{J} \Rightarrow i \notin \mathcal{A}^*$. If the primal solution is unique, then we have from (Mangasarian 1979) that

$$A_{\mathcal{K}} d = 0, A_{\mathcal{J}} d \leq 0, \quad (16)$$

has no solution $d \neq 0$, hence, the solution to (14) is $d = 0$, and consequently $\mathcal{J} = \emptyset$.

**Algorithm 3** Modified geometric algorithm

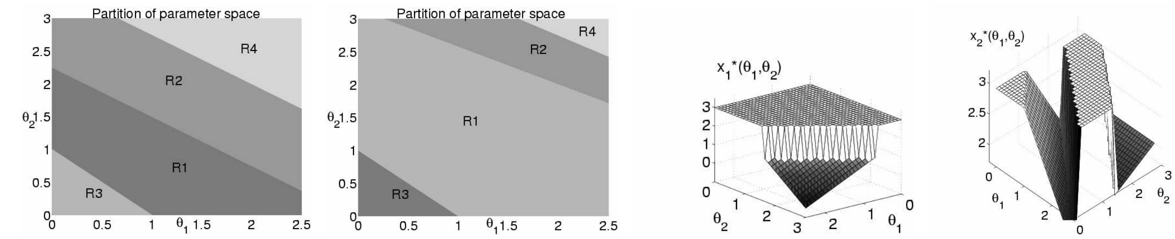
1: Let $Y \subseteq \Theta$ be the current region to be explored and let $\theta_0$ be in the interior of $Y$.
2: Solve the LP (1) for $\theta = \theta_0$.
3: Determine which of the four cases (i)-(iv) described in section 2.2 that applies.
4: **if** the primal solution is non-unique **then**
5: Execute Algorithm 2.
6: **else**
7: Find $x^*_{\mathcal{A}^*}(\theta)$ and the associated critical region as described in section 2.2.
8: **end if**
9: Partition the rest of the region into convex polyhedra according to the procedure given in (Borrelli *et al.* 2003) and for each nonempty element repeat steps 1-9.

*Remark 8.* When both the primal and dual solution are non-unique, $\mathcal{A}^*$ can be identified by using an interior point method to solve (1)(Güler and Ye 1993).

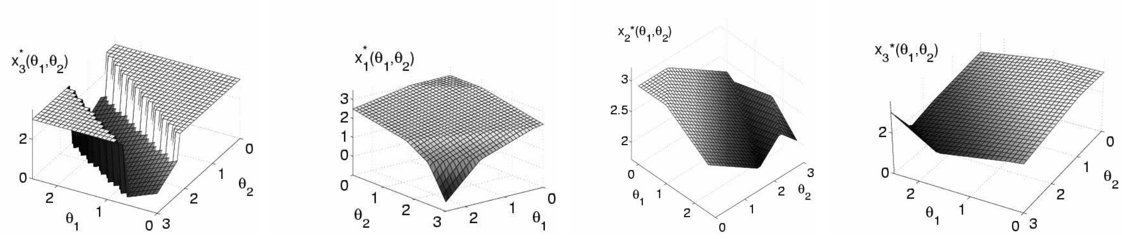
*Remark 9.* Algorithm 3 can always discard critical regions if the associated active set has already been found. This can not be done in Algorithm 1 if the LP has multiple primal solutions, since this may lead to sub-regions with different optimizer functions having intersecting interiors.

*Remark 10.* As an alternative to the last part of step 3 of Algorithm 2 a projection can be performed if regions with identical optimizer functions are not allowed to have intersecting interiors. The active constraints for problem (9) are projected onto the parameter space when LICQ is violated.

*Theorem 11.* (Continuity of solutions). The function $x^* : \Theta^* \mapsto \mathbb{R}^n$ obtained by Algorithm 3 is continuous.




(a) Partition, Geometric algorithm. (b) Partition, Modified geometric algorithm (c) $x_1^*(\theta_1, \theta_2)$, Geometric algorithm. (d) $x_2^*(\theta_1, \theta_2)$, Geometric algorithm.



(e) $x_3^*(\theta_1, \theta_2)$, Geometric algorithm. (f) $x_1^*(\theta_1, \theta_2)$, Modified geometric algorithm (g) $x_2^*(\theta_1, \theta_2)$, Modified geometric algorithm (h) $x_3^*(\theta_1, \theta_2)$, Modified geometric algorithm

Fig. 1. Comparison.

**PROOF.** We view problem (7) as a strictly convex mp-QP with the parameter vector $\hat{\theta} = [\theta, z^*(\theta)]^T$. Since $z^*(\theta)$ is a continuous function (Gal 1995, Theorem IV-4), $x_{qp}^*(\hat{\theta})$ is also continuous (Bemporad *et al.* 2002*b*).

## 4. EXAMPLE WITH NON-UNIQUE PRIMAL SOLUTION

Consider the following mp-LP

$$z^*(\theta) \triangleq \min_{x \in \mathbb{R}^3} \left\{ c^T x \,|\, Ax \leq b + S\theta, \theta \in \Theta \right\},$$

$$c^T \triangleq -\begin{bmatrix}1 & 1 & 1\end{bmatrix}, b^T \triangleq \begin{bmatrix}10 & 4 & 3 & 3 & 3 & 3 & 3 & 3 & 3\end{bmatrix},$$

$$A^T \triangleq \begin{bmatrix} 1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

$$S^T \triangleq \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Theta \triangleq \{\theta \in \mathbb{R}^p \,|\, 0 \leq \theta_1 \leq 2.5, 0 \leq \theta_2 \leq 3\}.$$

Note that in this section subscripts on variables refers to the elements of the vectors, that is, $x_i$ denotes the $i^{\text{th}}$ element of $x$.

### 4.1 Modified geometric algorithm

Let $\theta = \theta_0 = [1, 1]^T$, which is in the interior of $Y = \Theta$. The resulting LP has multiple primal solutions and a unique dual solution $\pi^*(\theta_0) = [-1 \ 0 \ \ldots \ 0]^T$. Algorithm 2 is then executed.

**1**: Noting that the dual solution is unique, $\mathcal{A}^* = \{1\}$ is uniquely identified.

**2**: Solving the QP obtained by fixing $\theta = \theta_0$ in (9a)-(9c) yields $\mathcal{A}_{qp} = \{1\}$.

**3**: The optimizer function is then given by

$$x_{qp}^*(\theta) = A_{\{1\}}^T (A_{\{1\}} A_{\{1\}}^T)^{-1} (b_{\{1\}} + S_{\{1\}}\theta),$$

and the sub-region becomes

$$R1 = \left\{ \theta \in \Theta \,\middle|\, A_{\{2:9\}} x_{qp}^*(\theta) \leq b_{\{2:9\}} + S_{\{2:9\}}\theta \right\}.$$

The parameter space is then partitioned into convex polyhedra that have not yet been explored.

Assume that the parameter vector $\theta = \theta_0 = [2, 2.5]^T$ is in the interior of the next unexplored polyhedron. Again this results in a non-unique primal solution and $\mathcal{A}^* = \{1\}$, however now $\mathcal{A}_{qp} = \{1, 2\}$, and the optimizer function and sub-region $R2$ is calculated as explained earlier.

The partition obtained by following Algorithm 3 is depicted in Figure 1(b).

### 4.2 Geometric algorithm

Let $\theta = \theta_0 = [1, 1]^T$, which is in the interior of $Y = \Theta$. The resulting LP has multiple primal solutions, and hence, a feasible vertex of the inequalities for which $x^*(\theta_0) \in X^*(\theta_0)$ is then chosen. A valid choice is $x^*(\theta_0) = [3, 2, 3]^T$, which results in $\mathcal{A}(x^*(\theta_0), \theta_0) = \{1, 4, 8\}$. The optimizer functions become

$$x_1^*(\theta) = 3, x_2^*(\theta) = -\theta_1 - \theta_2 + 4, x_3^*(\theta) = 3.$$

and the sub-region $R1$ is depicted in Figure 1(a).

Let $\theta = \theta_0 = [1, 2]^T$ be in the interior of the next unexplored polyhedron. This vector also yields a non-unique primal solution. A valid choice of $x^*(\theta_0)$ gives $\mathcal{A}(x^*(\theta_0), \theta_0) = \{1, 4, 6\}$. The optimizer functions become

$$x_1^*(\theta) = 3, x_2^*(\theta) = 3, x_3^*(\theta) = -\theta_1 - \theta_2 + 4.$$

and the sub-region $R2$ is shown in Figure 1(a). The hyperplane which separates $R1$ and $R2$ is given by $3\theta_1 + 4\theta_2 = 9$. Clearly, both $x_2^*(\theta_1, \theta_2)$ and $x_3^*(\theta_1, \theta_2)$ are discontinuous along this line. One of the possible solutions, following Algorithm 1, is depicted in Figure 1(a).

### 4.3 Comparison

Figures 1(c)-1(h) illustrate the optimizer functions. Clearly, all three functions are discontinuous for the geometric algorithm and for the proposed method all are continuous. By coincidence the solution consists of the same number of polytopes (Figure 1(a)-1(b)). Since the number of regions found by the geometric algorithm depends on the order in which they are found, it is not possible to determine which algorithm that generally yields the smallest number of polytopes.

## 5. CONCLUSION

A method for obtaining continuous solutions to multiparametric linear programming problems has been presented. The geometric algorithm presented by (Borrelli *et al.* 2003) has been modified such that no vertex is arbitrarily chosen in the case of non-unique primal solutions. When the mp-LP has multiple optimizers, the optimizer function is found from an mp-QP that has been constructed to maintain mp-LP optimality. The algorithm proposed by (Borrelli *et al.* 2003) is simpler to implement than the algorithm of (Gal and Nedoma 1972) and the proposed method is conceptually as simple as the geometric algorithm.

If regions with equal optimizer functions are allowed to have intersecting interiors, polytopes for which the active set has already been found can be discarded as the algorithm explores the parameter space.

The results of the present paper has also been extended to convex multiparametric quadratic programs (Spjøtvold *et al.* 2005).

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