IMPROVED PERTURBATION BOUNDS FOR THE CONTINUOUS-TIME H_∞-OPTIMIZATION PROBLEM

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Abstract: New local perturbation bounds are obtained for the continuous-time H_∞-optimization problem, which are nonlinear functions of the data perturbations and are tighter than the existing condition number based local bounds. The nonlinear local bounds are then incorporated into nonlocal perturbation bounds which are less conservative than the existing nonlocal perturbation estimates for the H_∞-optimization problem.

Keywords: H_∞ optimization, Robust control, Perturbation analysis, Riccati equations, Linear systems

1. INTRODUCTION

In this paper we present local and nonlocal perturbation analysis of the H_∞-optimization problem for continuous-time linear multivariable systems. First, nonlinear local perturbation bounds are derived for the matrix equations which determine the problem solution. The new local bounds are tighter than the existing condition number based linear perturbation estimates.

Then, using the nonlocal perturbation analysis techniques developed by the authors, the nonlinear local bounds are incorporated into nonlocal perturbation bounds which are less conservative than the existing nonlocal perturbation estimates for the H_∞-optimization problem. The nonlocal perturbation bounds are valid rigorously in contrast to the local bounds in which higher order terms are neglected.

We use the following notations: $\mathcal{R}^{m\times n}$ – the space of real $m \times n$ matrices; $\mathcal{R}^n = \mathcal{R}^{n\times 1}$; I_n – the unit $n \times n$ matrix; $A^\top$ – the transpose of A; $\|A\|_2 = \sigma_{\max}(A)$ – the spectral norm of A, where $\sigma_{\max}(A)$ is the maximum singular value of A; $\|A\|_{\mathrm{F}} = \sqrt{\mathrm{tr}(A^\top A)}$ – the Frobenius norm of A; $\|\,.\|$ is any of the above norms; $\mathrm{vec}(A) \in \mathcal{R}^{mn}$ – the column-wise vector representation of $A \in \mathcal{R}^{m\times n}$; $\Pi \in \mathcal{R}^{n^2\times n^2}$ – the vec-permutation matrix, so that $\mathrm{vec}(A^\top) = \Pi\mathrm{vec}(A)$ for $A \in \mathcal{R}^{n\times n}$; $A \otimes B$ – the Kronecker product of the matrices A and B. The notation ":=" stands for "equal by definition".

2. STATEMENT OF THE PROBLEM

Consider the linear continuous-time system

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + Ev(t) \\
y(t) &= Cx(t) + w(t) \\
z(t) &= \begin{bmatrix} Dx(t) \\ u(t) \end{bmatrix}
\end{aligned}
\tag{1}
$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^m$, $y(t) \in \mathcal{R}^r$ and $z(t) \in \mathcal{R}^p$ are the system state, input, output and performance vectors respectively, $v(t) \in \mathcal{R}^l$ and

$w(t) \in \mathcal{R}^r$ are disturbances and A, B, C, D, E are constant matrices of compatible dimensions.

The H_∞-optimization problem is stated as follows: Given the system (1) and a constant $\lambda > 0$, find a stabilizing controller

$$u(t) = -K\hat{x}(t)$$
$$\dot{\hat{x}}(t) = \hat{A}\hat{x} + L((y(t) - C\hat{x}(t))$$

which satisfies

$$\|H\|_\infty := \sup_{Re\ s \geq 0} \|H(s)\|_2 < \lambda$$

where $H(s)$ is the closed-loop transfer matrix from v, w to z.

If such a controller exists, then (Kwakernaak, 1993)

$$K = B^T X_0$$
$$\hat{A} = A - Y_0(C^T C - D^T D/\lambda^2)$$
$$L = Z_0 Y_0 C^T$$

where $X_0 \geq 0$ and $Y_0 \geq 0$ are the stabilizing solutions to the Riccati equations

$$A^T X + XA - X(BB^T - EE^T/\lambda^2)X + D^T D = 0 \quad (2)$$

$$AY + YA^T - Y(C^T C - D^T D/\lambda^2)Y + EE^T = 0 \quad (3)$$

and the matrix Z_0 is defined from

$$Z_0 = (I - Y_0 X_0/\lambda^2)^{-1} \quad (4)$$

under the assumption $\|Y_0 X_0\|_2 < \lambda^2$.

In the sequel we shall write equations (2), (3) as

$$A^T X + XA - XSX + Q = 0 \quad (5)$$

$$AY + YA^T - YRY + T = 0 \quad (6)$$

where $Q = D^T D,\ T = EE^T,\ S = BB^T - T/\lambda^2,\ R = C^T C - Q/\lambda^2$.

Suppose that the matrices $A, \ldots, E$ in (1) are subject to perturbations $\Delta A, \ldots, \Delta E$. Then we have the perturbed equations

$$(A + \Delta A)^T X + X(A + \Delta A) - X(S + \Delta S)X + Q + \Delta Q = 0 \quad (7)$$

$$(A + \Delta A)Y + Y(A + \Delta A)^T - Y(R + \Delta R)Y + T + \Delta T = 0 \quad (8)$$

$$Z = (I - YX/\lambda^2)^{-1} \quad (9)$$

where

$$\Delta Q = \Delta D^T D + D^T \Delta D + \Delta D^T \Delta D$$

$$\Delta T = \Delta E E^T + E \Delta E^T + \Delta E \Delta E^T$$

$$\Delta S = \Delta B B^T + B \Delta B^T + \Delta B \Delta B^T - \Delta T/\lambda^2$$

$$\Delta R = \Delta C^T C + C^T \Delta C + \Delta C^T \Delta C - \Delta Q/\lambda^2.$$

Denote by $\Delta_M = \|\Delta M\|$ the absolute perturbation of a matrix M. It is natural to use the Frobenius norm $\|\,.\|_F$ identifying the matrix perturbations with their vector-wise representations.

Since the Fréchet derivatives of the left-hand sides of (5), (6) in X and Y at $X = X_0$ and $Y = Y_0$ are invertible (see the next section) then, according to the implicit function theorem (Kantorovich *et al.*, 1977), the perturbed equations (7), (8) have unique solutions $X = X_0 + \Delta X$ and $Y = Y_0 + \Delta Y$ in the neighborhoods of X_0 and Y_0 respectively. Assume that $\|YX\| < \lambda^2$ and denote by $Z = Z_0 + \Delta Z$ the corresponding solution of the perturbed equation (9).

The sensitivity analysis of H_∞-optimization problem aims at determining perturbation bounds for the solutions X, Y and Z of equations (5), (6) and (4) as functions of the perturbations in the data A, S, Q, R, T.

Using the approach developed in (Konstantinov *et al.*, 1986; Konstantinov *et al.*, 1987), local perturbation bounds for the H_∞-optimization problem have been obtained in (Christov *et al.*, 1995; Konstantinov *et al.*, 1995), based on the condition numbers of equations (5), (6) and (4). However, the local estimates, based on condition numbers, may eventually produce pessimistic results. At the same time it is possible to derive local, first order homogeneous estimates, which are tighter in general (Konstantinov *et al.*, 1999a; Konstantinov *et al.*, 1999b). In this paper, using the local perturbation analysis technique developed in (Konstantinov *et al.*, 1999a; Konstantinov *et al.*, 1999b), we shall derive local first order perturbation bounds which are less conservative than the condition number based bounds in (Christov *et al.*, 1995; Konstantinov *et al.*, 1995).

Local perturbation bounds have a serious drawback: they are valid in a usually small neighborhood of the data $A, \ldots, T$, i.e. for $\Delta = [\Delta_A, \ldots, \Delta_T]^T$ asymptotically small. In practice, however, the perturbations in the data are always finite. Hence the use of local estimates remains (at least theoretically) unjustified unless an additional analysis of the neglected terms is made, which in most cases is a difficult task. In fact, to obtain bounds for the neglected nonlinear terms means to get a nonlocal perturbation bound.

Nonlocal perturbation bounds for the continuous-time H_∞-optimization problem have been obtained in (Christov *et al.*, 1995; Konstantinov *et al.*, 1995) using the Banach fixed point principle. In this paper, applying the method of nonlinear perturbation analysis proposed in (Konstantinov *et al.*, 1999a; Konstantinov *et al.*, 1999b) we shall derive new nonlocal perturbation bounds for the problem considered, which are less conservative than the nonlocal bounds in (Christov *et al.*, 1995; Konstantinov *et al.*, 1995).

3. LOCAL PERTURBATION ANALYSIS

Consider first the local sensitivity analysis of the Riccati equation (5). Denote by $F(X,\Sigma) = F(X,A,S,Q)$ the left-hand side of (5), where $\Sigma = (A,S,Q) \in \mathcal{R}^{n.n} \times \mathcal{R}^{n.n} \times \mathcal{R}^{n.n}$. Then $F(X_0,\Sigma) = 0$.

Setting $X = X_0 + \Delta X$, the perturbed equation (7) may be written as

$$\begin{aligned} &F(X_0+\Delta X, \Sigma+\Delta\Sigma) = \qquad (10)\\ &F(X_0,\Sigma) + F_X(\Delta X) + F_A(\Delta A) + F_S(\Delta S)\\ &+ F_Q(\Delta Q) + G(\Delta X, \Delta\Sigma) = 0 \end{aligned}$$

where $F_X(.)$, $F_A(.)$, $F_S(.)$ and $F_Q(.)$ are the Fréchet derivatives of $F(X,\Sigma)$ in the corresponding matrix arguments, evaluated for $X = X_0$, and $G(\Delta X, \Delta\Sigma)$ contains the second and higher order terms in ΔX, $\Delta\Sigma$.

A straightforward calculation leads to

$$\begin{aligned} F_X(M) &= A_c^T M + M A_c\\ F_A(M) &= X_0 M + M^T X_0\\ F_S(M) &= -X_0 M X_0\\ F_Q(M) &= M \end{aligned}$$

where $A_c = A - (BB^T - EE^T/\lambda^2)X_0$. Denote by $M_X \in \mathcal{R}^{n^2.n^2}$, $M_A \in \mathcal{R}^{n^2.n^2}$, $M_S \in \mathcal{R}^{n^2.n^2}$ the matrix representations of the operators $F_X(.)$, $F_A(.)$, $F_S(.)$:

$$\begin{aligned} M_X &= A_c^T \otimes I_n + I_n \otimes A_c^T\\ M_A &= I_n \otimes X_0 + (X_0 \otimes I_n)\Pi \qquad (11)\\ M_S &= -X_0 \otimes X_0 \end{aligned}$$

where $\Pi \in \mathcal{R}^{n^2.n^2}$ is the permutation matrix such that $\mathrm{vec}(M^T) = \Pi \mathrm{vec}(M)$ for each $M \in \mathcal{R}^{n.n}$ and $\mathrm{vec}(M) \in \mathcal{R}^{n^2}$ is the column-wise vector representation of M.

It follows from (10)

$$\begin{aligned} F_X(\Delta X) = &-F_A(\Delta A) - F_S(\Delta S) - \Delta Q\\ &- G(\Delta X, \Delta\Sigma). \end{aligned} \qquad (12)$$

Since A_c is stable, the operator $F_X(.)$ is invertible and (12) yields

$$\begin{aligned} \Delta X = &-F_X^{-1} \circ F_A(\Delta A) - F_X^{-1} \circ F_S(\Delta S)\\ &- F_X^{-1}(\Delta Q) - F_X^{-1}(G(\Delta X, \Delta\Sigma)). \end{aligned} \qquad (13)$$

The operator equation (13) may be written in a vector form as

$$\begin{aligned} \mathrm{vec}(\Delta X) = \; &N_1 \mathrm{vec}(\Delta A)\\ &+ N_2 \mathrm{vec}(\Delta S)\\ &+ N_3 \mathrm{vec}(\Delta Q)\\ &- M_X^{-1}\mathrm{vec}(G(\Delta X, \Delta\Sigma)) \end{aligned} \qquad (14)$$

where $N_1 = -M_X^{-1}M_A$, $N_2 = -M_X^{-1}M_S$, $N_3 = -M_X^{-1}$.

It is easy to show that the well-known condition number based perturbation bound (Christov *et al.*, 1995; Konstantinov *et al.*, 1995) is a corollary of (14). Indeed, it follows from (14)

$$\begin{aligned} \|\mathrm{vec}(\Delta X)\|_2 \le \; &\|N_1\|_2 \|\mathrm{vec}(\Delta A)\|_2\\ &+ \|N_2\|_2 \|\mathrm{vec}(\Delta S)\|_2\\ &+ \|N_3\|_2 \|\mathrm{vec}(\Delta Q)\|_2\\ &+ \mathrm{O}(\|\tilde\Delta\|^2). \end{aligned}$$

Having in mind that $\|\mathrm{vec}(\Delta M)\|_2 = \|\Delta M\|_F = \Delta_M$ and denoting $K_A^X = \|N_1\|_2$, $K_S^X = \|N_2\|_2$, $K_Q^X = \|N_3\|_2$, we obtain

$$\Delta_X \le K_A^X \Delta_A + K_S^X \Delta_S + K_Q^X \Delta_Q + \mathrm{O}(\|\tilde\Delta\|^2) \qquad (15)$$

where K_A^X, K_S^X, K_Q^X are the individual condition numbers of (5) and $\tilde\Delta = [\Delta_A, \Delta_S, \Delta_Q]^T$. Denoting $\Delta_{\max} = \max\{\Delta_A, \Delta_S, \Delta_Q\}$ and taking into account the inequalities

$$\begin{aligned} K_A^X &\le 2K_Q^X \|X_0\|\\ K_S^X &\le K_Q^X \|X_0\|^2 \end{aligned}$$

we get

$$\Delta_X \le K_Q^X (1 + \|X_0\|)^2 \Delta_{\max} \qquad (16)$$

where $K_Q^X(1 + \|X_0\|)^2$ is the overall condition number of (5).

Relation (14) also gives

$$\Delta_X \le \|\tilde N\|_2 \|\tilde\Delta\|_2 + \mathrm{O}(\|\tilde\Delta\|^2) \qquad (17)$$

where $\tilde N = [N_1, N_2, N_3]$.

Note that the bounds in (15) and (17) are alternative, i.e. which one is less depends on the particular value of $\tilde\Delta$.

There is also a third bound, which is always less than or equal to the bound in (15). We have

$$\Delta_X \le \sqrt{\tilde\Delta^T U(\tilde N)\tilde\Delta} + \mathrm{O}(\|\tilde\Delta\|^2)$$

where $U(\tilde N)$ is the 3×3 matrix with elements $u_{ij}(\tilde N) = \|N_i^T N_j\|_2$.

Since $\|N_i^T N_j\|_2 \leq \|N_i\|_2\|N_j\|_2$ we get

$$\sqrt{\tilde{\Delta}^T U(\tilde{N})\tilde{\Delta}} \leq \|N_1\|_2\Delta_A + \|N_2\|_2\Delta_S + \|N_3\|_2\Delta_Q.$$

Hence we have the overall estimate

$$\Delta_X \leq f(\tilde{\Delta}) + \mathrm{O}(\|\tilde{\Delta}\|^2),\ \tilde{\Delta} \to 0 \tag{18}$$

where

$$f(\tilde{\Delta}) = \min\{\|\tilde{N}\|_2\|\tilde{\Delta}\|_2, \sqrt{\tilde{\Delta}^T U(\tilde{N})\tilde{\Delta}}\,\} \tag{19}$$

is a first order homogeneous and piece-wise real analytic function in $\tilde{\Delta}$.

The local sensitivity of the Riccati equation (6) may be determined using the duality of (5) and (6). For the estimate of Δ_Y we have

$$\Delta_Y \leq g(\hat{\Delta}) + \mathrm{O}(\|\hat{\Delta}\|^2),\ \hat{\Delta} \to 0 \tag{20}$$

where

$$g(\hat{\Delta}) = \min\{\|\hat{N}\|_2\|\hat{\Delta}\|_2, \sqrt{\hat{\Delta}^T U(\hat{N})\hat{\Delta}}\,\} \tag{21}$$

$\hat{\Delta} = [\Delta_A, \Delta_R, \Delta_T]^T$ and $\hat{N}$ is determined replacing in (11) A_c and X_0 by $\hat{A}^T$ and Y_0, respectively.

Consider finally the local sensitivity analysis of equation (4). In view of (9) we have

$$\begin{aligned}\Delta Z &= [I_n - (Y_0 + \Delta Y)(X_0 + \Delta X)/\lambda^2]^{-1} - Z_0 \\ &= Z_0 W Z_0 + \mathrm{O}(\|W\|^2)\end{aligned} \tag{22}$$

where $W = (Y_0\Delta X + \Delta Y X_0 + \Delta Y \Delta X)/\lambda^2$.

It follows form (22)

$$\Delta_Z \leq \|Z_0^T \otimes Z_0\|_2\|W\|_F + \mathrm{O}(\|W\|^2)$$

and denoting $\zeta_0 = \|Z_0^T \otimes Z_0\|_2$ we get

$$\begin{aligned}\Delta_Z &\leq \zeta_0(\|Y_0\|_2\,\Delta_X + \|X_0\|_2\,\Delta_Y)/\lambda^2 \\ &\quad + \mathrm{O}(\|(\Delta X, \Delta Y)\|^2) \\ &\leq \zeta_0(\|Y_0\|_2\, f(\tilde{\Delta}) + \|X_0\|_2\, g(\hat{\Delta}))/\lambda^2 \\ &\quad + \mathrm{O}(\|\Delta\|^2).\end{aligned} \tag{23}$$

The relations (18), (20) and (23) give local first order perturbation bounds for the continuous-time H_∞-optimization problem.

4. NONLOCAL PERTURBATION ANALYSIS

The local perturbation bounds are obtained neglecting terms of order $\mathrm{O}(\|\Delta\|^2$, i.e. they are valid only asymptotically, for $\Delta \to 0$. That is why, their application for possibly small but nevertheless finite perturbations Δ requires additional justification. This disadvantage may be overcome using the methods of nonlinear perturbation analysis. As a result we obtain nonlocal (and in general nonlinear) perturbation bounds which guarantee that the perturbed problem still has a solution and are valid rigorously, unlike the local bounds. However, in some cases the nonlocal bounds may not exist or may be pessimistic.

Consider first the nonlocal perturbation analysis of the Riccati equation (5). The perturbed equation (13) can be rewritten in the form

$$\Delta X = \Psi(\Delta X) \tag{24}$$

where $\Psi : \mathcal{R}^{n.n} \to \mathcal{R}^{n.n}$ is determined by the right-hand side of (13). For $\rho > 0$ denote by $\mathcal{B}(\rho) \subset \mathcal{R}^{n.n}$ the set of all matrices $M \in \mathcal{R}^{n.n}$ satisfying $\|M\|_F \leq \rho$. For $U, V \in \mathcal{B}(\rho)$ we have

$$\|\Psi(U)\|_F \leq a_0(\tilde{\Delta}) + a_1(\tilde{\Delta})\rho + a_2(\tilde{\Delta})\rho^2$$

and

$$\|\Psi(U) - \Psi(V)\|_F \leq (a_1(\tilde{\Delta}) + 2a_2(\tilde{\Delta})\rho)\|U - V\|_F$$

where

$$\begin{aligned}a_0(\tilde{\Delta}) &:= f(\tilde{\Delta}) \\ a_1(\tilde{\Delta}) &:= 2\|M_X^{-1}\|_2\Delta_A + (\|M_X^{-1}(X_0 \otimes I_n)\|_2 \\ &\qquad + \|M_X^{-1}(I_n \otimes X_0)\|_2)\Delta_S \\ a_2(\tilde{\Delta}) &:= \|M_X^{-1}\|_2(\|S\|_2 + \Delta_S).\end{aligned} \tag{25}$$

Hence, the function $h(\rho, \tilde{\Delta}) = a_0(\tilde{\Delta}) + a_1(\tilde{\Delta})\rho + a_2(\tilde{\Delta})\rho^2$ is a Lyapunov majorant (Grebenikov *et al.*, 1979) for equation (24) and the majorant equation for determining a nonlocal bound $\rho = \rho(\tilde{\Delta})$ for Δ_X is

$$a_2(\tilde{\Delta})\rho^2 - (1 - a_1(\tilde{\Delta}))\rho + a_0(\tilde{\Delta}) = 0. \tag{26}$$

Suppose that $\tilde{\Delta} \in \tilde{\Omega}$, where

$$\tilde{\Omega} = \left\{\tilde{\Delta} \succeq 0 : a_1(\tilde{\Delta}) + 2\sqrt{a_0(\tilde{\Delta})a_2(\tilde{\Delta})} \leq 1\right\}. \tag{27}$$

Then equation (26) has nonnegative roots $\rho_1 \leq \rho_2$ with

$$\begin{aligned}\rho_1 &= \phi(\tilde{\Delta}) \\ &:= \frac{2a_0(\tilde{\Delta})}{1 - a_1(\tilde{\Delta}) + \sqrt{(1 - a_1(\tilde{\Delta}))^2 - 4a_0(\tilde{\Delta})a_2(\tilde{\Delta})}}.\end{aligned} \tag{28}$$

The operator Ψ maps the closed convex set $\mathcal{B}(\tilde{\Delta}) = \left\{M \in \mathcal{R}^{n.n} : \|M\|_F \leq \phi(\tilde{\Delta})\right\} \subset \mathcal{R}^{n.n}$ into itself and according to the Schauder fixed point principle there exists a solution $\Delta X \in \mathcal{B}(\tilde{\Delta})$ of equation (24), for which

$$\Delta_X \le \phi(\tilde{\Delta}), \quad \tilde{\Delta} \in \tilde{\Omega}. \tag{29}$$

The elements of ΔX are continuous functions of the elements of $\Delta\Sigma$.

If $\tilde{\Delta} \in \tilde{\Omega}_1$, where

$$\tilde{\Omega}_1 = \left\{ \tilde{\Delta} \succeq 0 : a_1(\tilde{\Delta}) + 2\sqrt{a_0(\tilde{\Delta}) a_2(\tilde{\Delta})} < 1 \right\} \subset \tilde{\Omega}$$

then $\rho_1 < \rho_2$ and the operator Ψ is a contraction on $\mathcal{B}(\tilde{\Delta})$. Hence according to the Banach fixed point principle the solution ΔX, for which the estimate (29) holds true, is unique. This means that the perturbed equation has an isolated solution $X = X_0 + \Delta X$. In this case the elements of ΔX are analytical functions of the elements of $\Delta\Sigma$.

In a similar way, replacing A_c with $\hat{A}^T$, S with R, Q with T and X_0 with Y_0, we obtain a nonlocal perturbation bound for ΔY. Suppose that $\hat{\Delta} \in \hat{\Omega}$, where

$$\hat{\Omega} = \left\{ \hat{\Delta} : b_1(\hat{\Delta}) + 2\sqrt{b_0(\hat{\Delta}) b_2(\hat{\Delta})} \le 1 \right\} \subset \mathcal{R}^3_+$$

and

$$\begin{aligned} b_0(\hat{\Delta}) &= g(\hat{\Delta}) \\ b_1(\hat{\Delta}) &= 2\|M_Y^{-1}\|_2 \Delta_{\hat{A}} + (\|M_Y^{-1}((Y_0 \otimes I_n))\|_2 \\ &\quad + \|M_Y^{-1}((I_n \otimes Y_0))\|_2)\Delta_R \\ b_2(\hat{\Delta}) &= \|M_Y^{-1}\|_2 (\|R\|_2 + \Delta_R). \end{aligned}$$

Then

$$\Delta_Y \le \psi(\hat{\Delta}), \quad \hat{\Delta} \in \hat{\Omega} \tag{30}$$

where

$$\psi(\hat{\Delta}) = \frac{2 b_0(\hat{\Delta})}{1 - b_1(\hat{\Delta}) + \sqrt{(1 - b_1(\hat{\Delta}))^2 - 4 b_0(\hat{\Delta}) b_2(\hat{\Delta})}}.$$

Finally, the nonlinear perturbation bound for ΔZ is obtained using (14) and (28), (29). If $1 \notin \mathrm{spect}(W Z_0)$ we have

$$\Delta Z = Z_0 W Z_0 (I_n - W Z_0)^{-1}.$$

Hence

$$\Delta_Z \le \zeta_0 \|W\|_F \|(I_n - W Z_0)^{-1}\|_2.$$

If $\|W\|_2 < 1/\|Z_0\|_2$ we have

$$\Delta_Z \le \frac{\zeta_0 \|W\|_F}{1 - \|Z_0\|_2 \|W\|_2}.$$

It is realistic to estimate $\|W\|$ when $\Delta X, \Delta Y$ vary independently. In this case one has to assume that

$$\begin{aligned} &\|Y_0\|_2 \phi(\tilde{\Delta}) + \|X_0\|_2 \psi(\hat{\Delta}) + \phi(\tilde{\Delta})\psi(\hat{\Delta}) \\ &< \lambda^2 / \|Z_0\|_2 \end{aligned}$$

and

$$\Delta_Z \le \frac{\zeta_0 \lambda^2 \xi_0}{\lambda^2 - \|Z_0\|_2 \xi_0} \tag{31}$$

where

$$\xi_0 = \|Y_0\|_2 \phi(\tilde{\Delta}) + \|X_0\|_2 \psi(\hat{\Delta}) + \phi(\tilde{\Delta})\psi(\hat{\Delta}).$$

Relations (29), (30) and (31) give nonlocal perturbation bounds for the continuous-time H_∞-optimization problem.

Note finally that one has to ensure the inequality

$$\|YX\|_2 < \lambda^2. \tag{32}$$

Since the unperturbed inequality $\|Y_0 X_0\|_2 < \lambda^2$ holds true, a sufficient condition for (32) to be valid is

$$\begin{aligned} &\|Y_0\|_2 \phi(\tilde{\Delta}) + \|X_0\|_2 \psi(\hat{\Delta}) + \phi(\tilde{\Delta})\psi(\hat{\Delta}) \\ &< \lambda^2 - \|Y_0 X_0\|_2. \end{aligned}$$

Note that $\tilde{\Delta}, \hat{\Delta}$ depend on λ^2 through Δ_S, Δ_R.

5. NUMERICAL EXAMPLE

Consider a third order Riccati equation of type (5) with matrices

$$A = V A^* V, \quad S = V S^* V \quad Q = V Q^* V$$

where

$$V = I_3 - 2 v v^T / 3, \quad v = [1, 1, 1]^T$$

and

$$\begin{aligned} A^* &= \mathrm{diag}(1, -0.1, -1) \\ S^* &= \mathrm{diag}(0.2, 1, 10) \\ Q^* &= \mathrm{diag}(0.1, 0.1, 0.1). \end{aligned}$$

The solution is given by

$$X = V X^* V, \quad X^* = \mathrm{diag}(x_1, x_2, x_3)$$

where

$$x_i = \frac{a_i + \sqrt{a_i^2 + s_i q_i}}{s_i}$$

and a_i, s_i and q_i are the corresponding diagonal elements of A^*, S^* and Q^*.

The perturbations considered in the data satisfy

$$\Delta A = V \Delta A^* V$$

$$\Delta S = V \Delta S^* V$$

$$\Delta Q = V \Delta Q^* V$$

where

$$\Delta F^* = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -9 \\ 0 & -9 & 5 \end{bmatrix} \times 10^{-i}$$

$$\Delta S^* = \begin{bmatrix} 10 & -5 & 7 \\ -5 & 1 & 3 \\ 7 & 3 & 10 \end{bmatrix} \times 10^{-i-1}$$

$$\Delta Q^* = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 10 \end{bmatrix} \times 10^{-i}$$

for $i = 12, 11, \ldots, 4$.

The perturbed solution $X + \Delta X$ of the equation is computed by the Schur method (Laub, 1979) in arithmetic with relative precision $\varepsilon = 2^{-52} \approx 2.22 \times 10^{-16}$.

The perturbations $\Delta_X = \|\Delta X\|_F$ in the solution are estimated by the well known linear bound (16), and the new nonlinear homogeneous bound (18) and nonlocal bound (29). The results obtained for different values of i are shown in Table 1. The actual changes in the solution are closed to the quantities predicted by the improved sensitivity analysis. The case when the conditions for existence of a nonlocal estimate are violated is denoted by asterisk.

Table 1

i	Δ_X	Est. (16)	Est. (18)	Est. (29)
12	$2.1\ 10^{-11}$	$2.6\ 10^{-9}$	$2.5\ 10^{-10}$	$2.5\ 10^{-10}$
11	$2.1\ 10^{-10}$	$2.6\ 10^{-8}$	$2.5\ 10^{-9}$	$2.5\ 10^{-9}$
10	$2.1\ 10^{-9}$	$2.6\ 10^{-7}$	$2.5\ 10^{-8}$	$2.5\ 10^{-8}$
9	$2.1\ 10^{-8}$	$2.6\ 10^{-6}$	$2.5\ 10^{-7}$	$2.5\ 10^{-7}$
8	$2.1\ 10^{-7}$	$2.6\ 10^{-5}$	$2.5\ 10^{-6}$	$2.5\ 10^{-6}$
7	$2.1\ 10^{-6}$	$2.6\ 10^{-4}$	$2.5\ 10^{-5}$	$2.5\ 10^{-5}$
6	$2.1\ 10^{-5}$	$2.6\ 10^{-3}$	$2.5\ 10^{-4}$	$2.5\ 10^{-4}$
5	$2.1\ 10^{-4}$	$2.6\ 10^{-2}$	$2.5\ 10^{-3}$	$2.6\ 10^{-3}$
4	$2.1\ 10^{-3}$	$2.6\ 10^{-1}$	$2.5\ 10^{-2}$	*

6. CONCLUSIONS

The local and nonlocal sensitivity of the continuous-time H_∞-optimization problem have been studied. New local perturbation bounds have been obtained for the matrix equations determining the problem solution. The new local bounds are nonlinear functions of the data perturbations and are tighter than the existing condition number based local bound. Using a nonlinear perturbation analysis technique, nonlocal perturbation bounds for the H_∞-optimization problem have then been derived. These bounds have two main advantages: they guarantee that the perturbed problem still has a solution, and are valid rigorously, unlike the local perturbation bounds.

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