# NONLINEAR DIAGNOSTIC FILTER DESIGN: ALGEBRAIC AND GEOMETRIC POINTS OF VIEW

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Abstract: The problem of diagnostic filter design is studied. Algebraic and geometric approaches to solve this problem are investigated. Certain relations between these approaches are established. It is shown that algebraic approach can be used for wider class of nonlinear systems than geometric one. 

Keywords: diagnostic filter, nonlinear systems, geometric approach, algebraic approach, fault detection and isolation, observers.

## 1. INTRODUCTION

This paper deals with the problem of diagnostic filter design. By definition, the diagnostic filter is an observer (or a bank of observers) whose output (residual) is structured according to the faults arising in the system under monitoring. Up to now, two basic approaches to diagnostic filter design were developed: geometric approach and algebraic one.

In the framework of geometric approach, the solution of diagnostic filter design was firstly proposed by Massoumnia (1986), Massoumnia *et al*. (1989) for linear systems. Later, this solution was developed for nonlinear systems by De Persis and Isidori (1999, 2000); Join *et*. *al* (2002a, b).

Algebraic approach (so-called the algebra of functions) was firstly proposed for fault detection in nonlinear systems by Zhirabok and Shumsky (1987). Then, the algebra of functions was developed for solving different diagnostic tasks (Shumsky, 1988, 1991; Zhirabok, 1997) and for nonlinear systems research (Zhirabok and Shumsky, 1993a, b).

The goal of this paper is to investigate the relations that exist between algebraic and geometric approaches.

## 2. PROBLEM DESCRIPTION

Consider the system

$$dx(t)/dt = f(x(t), u(t), \vartheta(t)) \qquad (1)$$

$$y(t) = h(x(t)) \qquad (2)$$

where $x(t)\in X\subseteq R^n$ is the state vector, $u(t)\in U\subseteq R^m$ is the vector of control, $y(t)\in Y\subseteq R^l$ is the measurable output vector, $\vartheta(t)\in R^s$ is the parameters vector, $f$ and $h$ are nonlinear vector functions assumed to be smooth for $x(t)$ and $\vartheta(t)$. It is also assumed that for healthy system it holds $\vartheta(t)=\vartheta^0$ $\forall t$, where $\vartheta^0$ is a given nominal value of parameters vector.

The set of faults considered for the design of diagnostic filter is specified by a list of faults $\{\rho_1, \rho_2, \ldots, \rho_d\}$, $d \geq s$. Single and multiple faults are distinguished. It is assumed that every single fault $\rho_i$, $i$=1, …, $s$, results in unknown time behavior of appropriate parameter $\vartheta_i(t)$ such that $\vartheta_i(t) \neq \vartheta_i^0$. The multiple fault is considered as a collection of single faults occurring simultaneously. Notice, that this representation of faults corresponds not only to actuator or plant faults but also to sensor faults, considered as pseudoactuator faults, see, e.g. (Massoumnia *et al*., 1989; Park *et al*., 1994).

To detect and isolate the faults, diagnostic filter in the form of a bank of reduced-order nonlinear observers is involved. Every observer generates appropriate subvector of the residuals $r^{(i)}$, $i$=1, …, $q$, and the residual vector $r$ is composed from these subvectors.

Usually, see, e.g. (Gertler and Kunwer, 1993), the structure properties of the residual vector are characterized by binary matrix $S$ of fault syndromes (FS) with element $S_{ij}$ =1 if subvector $r^{(i)}$ is sensitive to single fault $\rho_j$, otherwise (if $r^{(i)}$ is insensitive to $\rho_j$) $S_{ij}$ =0, $i$=1, …, $q$, $j$=1, …, $s$. Different ways for choosing FS matrix were discussed in literature (Gertler and Kunwer, 1993; Chen and Patton, 1994). It was shown that the diagonal structure of this matrix guarantees the isolation of multiple faults but puts strong demands on the system. Also, the matrix with zeros only on its diagonal allows to isolate only single faults but gives more possibilities for the design.

For nonlinear systems, the delay among the first distortion of system output due to some fault and the instant of time when subvector $r^{(i)}$ takes nonzero value depends on control and may be significant (or even infinite) to prevent making the decision timely. As result, in nonlinear case, the characteristics of the residual structure becomes more exhaustive if to use instead of the term "sensitivity" the term "detectability" of the fault via residual subvector, drawing a distinction between weak and strong detectability. Let $t_0$ be an instant of time when fault $\rho_j$ results in distortion of system output.

*Definition* 1. Fault $\rho_j$ is called weakly detectable via residual $r^{(i)}$ if there exist the state $x(t_0)$, finite time interval $T=[t_0, t]$ and control $u(\tau)\in U$, $\tau\in[t_0, t]$, such that $r^{(i)}(t)\neq 0$. Clearly, the notion of weak detectability is equal to the notion of sensitivity to the fault.

*Definition* 2. Fault $\rho_j$ is called strongly detectable via residual $r^{(i)}$ if it holds $r^{(i)}(t_0)\neq 0$.

As soon as the notions of weak and strong detectability are introduced, the elements of FS matrix may take three values : $S_{ij}$ =1 if fault $\rho_j$ is strongly detectable via residual $r^{(i)}$; $S_{ij}$ =0 if $r^{(i)}$ is insensitive to fault $\rho_j$; $S_{ij}$ = $z$ if fault $\rho_j$ is weakly detectable via residual $r^{(i)}$. It makes reasonable to introduce the definitions of weak and strong fault distinguishability and isolability.

*Definition* 3. Faults $\rho_i$ and $\rho_j$ are called weakly (strongly) distinguishable if corresponding to these faults columns of FS matrix do not coincide under $z$=1 ($z$=0).

*Definition* 4. Faults $\rho_1$, $\rho_2$, …, $\rho_d$ are called weakly (strongly) isolable if every two columns of FS matrix do not coincide under $z$=1 ($z$=0).

Weak distinguishability (isolability) of the faults means that these faults (all faults) are distinguishable (isolable) under some "favorable" control. In contrast to this, strong distinguishability (isolability) means that these faults are distinguishable (isolable) under every control.

The key problem of finding FS matrix for a given system and the set of faults deals with solving two tasks : i) full decoupling effects of the faults in output space of diagnostic filter and ii) analysis of fault detectability via subvectors of the residual.

An idea of full decoupling is based on the compensation of fault effects in output space of observer. If no assumption is made about time behavior of system parameters affected by the faults, such compensation is possible only if there exist at least two different ways (channels) of fault effect propagation (Petrov's two channels principle). To illustrate the way for realization of this principle in the framework of the problem under consideration, let us address to the structure interpretation given in Fig. 1 (Shumsky, 1991).

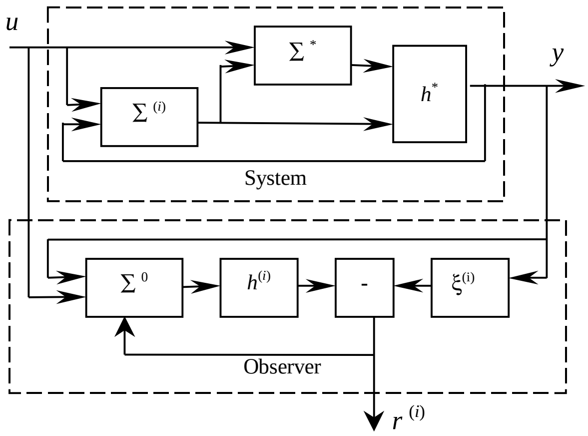


Fig.1. Structure interpretation of observer-based residual generation involving two channels principle

In Fig.1 system (1) is decomposed into subsystems $\Sigma^{(i)}$ (the first channel), $\Sigma^*$ and function $h^*$, specified as follows :

$$\Sigma^{(i)}: \quad dx^{(i)}(t)/dt = f^{(i)}(x^{(i)}(t), y(t), u(t), \vartheta^{(i)}(t)) \qquad (3)$$

$$\Sigma^{*}: \quad dx^{*}(t)/dt = f^{*}(x^{*}(t), x^{(i)}(t), u(t), \vartheta(t)) \qquad (4)$$

$$h^{*}: \quad h^{*}(x^{*}(t), x^{(i)}(t)) = h(x(t)) \qquad (5)$$

where $\vartheta^{(i)}$ is some subvector of $\vartheta$ unaffected by fault $\rho_i$. Assume that

$$\Sigma^{0}: \quad dx^{(0)}(t)/dt = f^{(i)}(x^{(0)}(t), y(t), u(t), \vartheta^{(i,0)}) + G(x^{(0)}(t), u(t), y(t))\, r^{(i)}(t) \qquad (6)$$

$$\xi^{(i)}, h^{(i)}: \quad \xi^{(i)}(h^{*}(x^{*}(t), x^{(i)}(t))) = h^{(i)}(x^{(i)}(t)) \qquad (7)$$

$$r^{(i)}: \quad r^{(i)}(t) = h^{(i)}(x^{(0)}(t)) - \xi^{(i)}(y(t)) \qquad (8)$$

where $G$ is a gain matrix function and $\vartheta^{(i,0)}$ is a nominal value of subvector $\vartheta^{(i)}$. In this description the subsystem $\Sigma^0$ plays a role of the second channel.

Consider faulty free case and let $x^{(0)}(0) = x^{(i)}(0)$. Notice firstly, that from (2), (5), (7), (8) it follows $r^{(i)}(0) = 0$. Assuming that $r^{(i)}(t) = 0\ \forall t$, one obtains from (3), (6) $x^{(0)}(t) = x^{(i)}(t)$. Using again (2), (5), (7), (8), it is easily to prove that it really holds $r^{(i)}(t) = 0\ \forall t$. Then, because $\vartheta^{(i)}$ is unaffected by fault $\rho_i$, $x^{(0)}(t) = x^{(i)}(t)$ and $r^{(i)}(t) = 0$ also hold under presence of this fault. Let now $x^{(0)}(0) \neq x^{(i)}(0)$. The design of asymptotically stable observer with property $t \to \infty \Rightarrow x^{(0)}(t) - x^{(i)}(t) \to 0$ ($r^{(i)}(t) \to 0$) involves an appropriate choice of the gain matrix function.

The problem of finding the gain matrix function has been extensively studied (see, e.g., survey by Misawa and Hedrick, 1989; papers by Birk and Zeitz, 1988; Ding and Frank, 1990; so on). It is a reason to concentrate below only on the problem of finding the functions $f^{(i)}$, $h^{(i)}$, $\xi^{(i)}$, $i$=1, …, $q$, assuming that $x^{(0)}(0) = x^{(i)}(0)$.

According to Shumsky (1991), solution of the last problem is based on the following assumption : there exists coordinate transformation given by smooth vector function $\alpha^{(i)}$ such that for healthy system and every $t$ it holds

$$x^{(i)}(t) = \alpha^{(i)}(x(t)). \qquad (9)$$

Using (1), (3), (9), one obtains defining equation for $f^{(i)}$ :

$$f^{(i)}(\alpha^{(i)}(x), h(x), u, \vartheta^{(i)}) = (\partial\alpha^{(i)}/\partial x)\, f(x, u, \vartheta). \qquad (10)$$

Because $\vartheta^{(i)}$ is unaffected by fault $\rho_i$, one has

$$(\partial\alpha^{(i)}/\partial x)\,(\partial f(x, u, \vartheta)/\partial\vartheta_j) = 0 \qquad (11)$$

for every $\vartheta_j$ subjected to distortion by this fault. Then, from (2), (5), (7), (9) one also obtains defining equation for $h^{(i)}$ :

$$h^{(i)}(\alpha^{(i)}(x)) = \xi^{(i)}(h(x)). \qquad (12)$$

Thus, the functions $f^{(i)}$ and $h^{(i)}$ are found from (10) and (12) respectively under known functions $\alpha^{(i)}$, $\xi^{(i)}$. This is why in the next section an attention is paid to finding the functions $\alpha^{(i)}$, $\xi^{(i)}$ and studying their properties, taking into account both solvability condition for (10) - (12) and the demands imposed on the structure of FS matrix by the set of faults.

## 3. ALGEBRAIC APPROACH

In this section, the algebra of functions is basically used for solving the general problem of finding $\alpha^{(i)}$, $\xi^{(i)}$ for every $i$ =1, … $d$ and determining FS matrix.

### *3.1. Algebraic tools*

Denote $\Im_S$ the set of smooth vector functions with domain $S$. For $\alpha, \beta \in \Im_S$ partial preordering relation $\leq$ is defined as follows : $\alpha \leq \beta$ if and only if there exists some differentiable function $\gamma$, determined on the set of values of $\alpha$, such that $\beta = \gamma \circ \alpha$, where $\circ$ is the symbol of functions composition. To verify if $\alpha \leq \beta$ one can check the equality of ranks for functional (Jacobian) matrices $J_\alpha(s) = \partial\alpha(s)/\partial s$ and $J_{\alpha\times\beta}(s) = \partial(\alpha(s)\times\beta(s))/\partial s$: $\alpha \leq \beta \Leftrightarrow rank\ J_\alpha(s) = rank\ J_{\alpha\times\beta}(s)\ \forall s \in S$, where the symbol $\times$ is given to simplify (but not only) the writing for composed vector function, namely, $\alpha\times\beta = (\alpha^T, \beta^T)^T$, and $^T$ is the symbol of transposition. If $\alpha \leq \beta$ and $\beta \leq \alpha$ then $\alpha$, $\beta$ are called equivalent : $\alpha \sim \beta$. Thus, relation $\sim$ splits the set $\Im_S$ on equivalent function classes.

Every function $\alpha \in \Im_S$ specifies equivalence relation $E_\alpha$ on $S$ : $(s^1, s^2) \in E_\alpha \Leftrightarrow \alpha(s^1) = \alpha(s^2)$. Relation $E_\alpha$ gives appropriate partition of $S$. One can easily see that equivalent functions give the same partitions of $S$. Moreover, if $E_\alpha$ and $E_\beta$ are equivalence relations corresponding to functions $\alpha$ and $\beta$, then

$$[\alpha \leq \beta] \Leftrightarrow [(s^1, s^2) \in E_\alpha \Rightarrow (s^1, s^2) \in E_\beta\ \forall s^1, s^2 \in S].$$

Therefore, there exists the ordering set of partitions of $S$, corresponding to functions from $\Im_S$. This set is a grid with zero, given by arbitrary one to one function (e.g., identity function $i(s)$ = s $\forall s \in S$), and unity, given by arbitrary constant function ($c(s) = const\ \forall s \in S$). The problem of finding the maximal bottom for every pare of partitions of this grid has constructive solution : if these partitions are specified by functions $\alpha$, $\beta$ then the function $\alpha\times\beta$ corresponds to maximal bottom partition. As soon as maximal bottom of two partitions is their product, operation $\times$ acquires the definite sense : the product of partitions given by functions $\alpha$ and $\beta$ is equal to the partition given by function $\alpha\times\beta$. To find minimal top for every pare of partitions, special operation $\oplus$ for vector functions is introduced. Function $\alpha\oplus\beta$ corresponds to the sum of partitions specified by functions $\alpha$, $\beta$. Finally, operations $\times$, $\oplus$ are defined as follows :

$$[\alpha\times\beta \in \Im_S]\ \&\ [\gamma \leq \alpha, \gamma \leq \beta \Rightarrow \gamma \leq \alpha\times\beta]$$

$$[\alpha\oplus\beta \in \Im_S]\ \&\ [\alpha \leq \gamma, \beta \leq \gamma \Rightarrow \alpha\oplus\beta \leq \gamma].$$

For healthy system (1), relation $\Delta \subset \Im_X \times \Im_X$ and operator $m : \Im_X \to \Im_X$ are introduced as follows:

$$[(\alpha, \beta) \in \Delta] \Leftrightarrow [\pi_u \times \alpha\circ\pi_x \leq J_\beta f]$$

$$[(\alpha, m(\alpha)) \in \Delta]\ \&\ [(\alpha, \beta) \in \Delta \Rightarrow m(\alpha) \leq \beta]$$

where $\pi_u$, $\pi_u(x, u) = u$, and $\pi_x$, $\pi_x(x, u) = x$, are projections.

### 3.2. *Fault decoupling*

Let $\alpha^{(i,0)}$ be vector function such that

$$(\partial\alpha^{(i,0)}/\partial x)\,(\partial f(x, u, \vartheta)/\partial\vartheta_j) = 0 \qquad (13)$$

for every $\vartheta_j$ subjected to distortion by fault $\rho_i$ and for every function $\alpha^{(i)}$, satisfying (10), it holds $\alpha^{(i,0)} \le \alpha^{(i)}$. Solvability condition for (9) – (11) is given by the following theorem (Shumsky, 1991).

*Theorem* 1. Equations (9) – (11) are solvable if and only if

$$(h \times \alpha^{(i)}, \alpha^{(i)}) \in \Delta, \qquad \alpha^{(i,0)} \le \alpha^{(i)} \tag{14}$$

$$\alpha^{(i)} \le \xi^{(i)} \circ h. \tag{15}$$

The next theorem (Shumsky, 1991) gives a regular rule for finding minimal function $\alpha^{(i)}$, satisfying (14). Notice, that minimal function $\alpha^{(i)}$ corresponds to subsystem $\Sigma^{(i)}$ of maximally possible dimension.

*Theorem* 2. Let $\alpha^{(i,\,j+1)} \sim \mathrm{m}\,(\alpha^{(i,\,j)} \times h) \oplus \alpha^{(i,\,j)}$, $j \ge 0$, and there exists natural $k$ such that $\alpha^{(i,\,k+1)} \sim \alpha^{(i,\,k)}$. Then :
(i) the function $\alpha^{(i,\,k)}$ satisfies (14);
(ii) for every function $\alpha^{(i)}$, satisfying (14), it holds

$$\alpha^{(i,\,k)} \le \alpha^{(i)}. \tag{16}$$

*Corollary* 1 (from theorem 2). It holds :

$$\alpha^{(i,\,k)} \oplus h \le \xi^{(i)} \circ h. \tag{17}$$

Relation (17) follows immediately from (15), (16) and the definition of operation $\oplus$.

Theorems 1, 2 result in the following algorithm for finding functions $\alpha^{(i)}$, $\xi^{(i)}$ such that the residual subvector $r^{(i)}$ is insensitive to fault $\rho_i$.

*Algorithm* 1.
1. Calculate the function $\alpha^{(i,0)}$ with maximum number of functional independent components from equation (13).
2. Calculate the function $\alpha^{(i,\,k)}$, using the rule of theorem 2.
3. Calculate the function $\xi^{(i)}$ :

$$\xi^{(i)} \circ h \sim \alpha^{(i,\,k)} \oplus h. \tag{18}$$

*Remark* 1. Relation (18) gives the minimal function $\xi^{(i)}$, satisfying (17).

The use of algorithm 1 needs in calculating operation $\oplus$ and operator $m$. If necessary, one can find the rules for their calculation in Shumsky (1989), Zhirabok and Shumsky (1993a, b). Notice, that in Section 4 these rules are given in geometric terms.

So, using algorithm 1, one obtains the functions $\alpha^{(i)}$, $\xi^{(i)}$ for every fault $\rho_i$, $i = 1, \dots d$.

### *3.3. Detectability analysis and FS matrix construction*

Let for the faults $\rho_i$ and $\rho_j$ it holds

$$\xi^{(i)} \ge \xi^{(j)}. \tag{19}$$

It means that the residual subvector $r^{(i)}$ is insensitive not only to fault $\rho_i$ but also to fault $\rho_j$. Indeed, because of (19) one can write $\xi^{(i)} = \gamma \circ \xi^{(j)}$, where $\gamma$ is some vector function. According to (4) and (6) – (9), equality $r^{(j)}=0$ holds if $h^{(j)}(\alpha^{(j)})=\xi^{(j)}(h)$. But from the last equality it follows $\gamma \circ h^{(j)}(\alpha^{(j)}) = \gamma \circ \xi^{(j)}(h)$. Then, according defining equations (11), $h^{(i)}(\alpha^{(i)})=\xi^{(i)}(h)$. Taking into account equalities written above, one obtains $r^{(i)}=h^{(i)}(\alpha^{(i)})-\xi^{(i)}(h)= \gamma \circ h^{(j)}(\alpha^{(j)}) - \gamma \circ \xi^{(j)}(h)=0$ as soon as $r^{(j)}=h^{(j)}(\alpha^{(j)})-\xi^{(j)}(h)=0$.

If inequality (19) does not hold, then $r^{(i)}$ is not insensitive to fault $\rho_j$ (that follows Remark 1), i.e. sensitive to this fault. Thus, violation of (19) is a condition of weak detectability of $\rho_j$ via residual subvector $r^{(i)}$.

*Remark* 2. To check if (19) is violated, it is sufficient to prove the following rank condition for some $y \in Y$ :

$$rank\, \partial(\xi^{(i)} \times \xi^{(j)})/\partial y > rank\, \partial\xi^{(j)}/\partial y. \tag{20}$$

*Theorem 3*. (Shumsky, 1988). Fault $\rho_i$ is strongly detectable via residual $r^{(j)}$, $j \ne i$, if

$$\xi^{(i)} \times \xi^{(j)} \sim i_Y \tag{21}$$

where $i_Y$ is identity function with domain $Y$.

*Remark* 3. To check if (21) holds, it is sufficient to prove rank condition

$$rank\, \partial(\xi^{(i)} \times \xi^{(j)})/\partial y = l \quad \forall y \in Y. \tag{22}$$

Primary FS matrix of dimension $d \times d$ is constructed as follows. The diagonal elements of this matrix are taken equal to zero, because residual subvector $r^{(i)}$ is insensitive to the fault $\rho_i$. Then, applying conditions (20), (22), one fills in nondiagonal elements of this matrix. Involving primary FS matrix, fault isolability is investigated. Final FS matrix is obtained by excluding redundant rows (i.e. rows whose excluding do not influence on fault isolability).

## 4. GEOMETRIC INTERPRETATION

In this section, the connection among algebraic and geometric tools is investigated for nonlinear systems, whose dynamics is affine in control and fault action:

$$dx(t)/dt = f(x(t)) + g(x(t))\,u(t) + w(x(t))\vartheta(t), \tag{23}$$

where $g(x)$ and $w(x)$ are smooth matrix functions of appropriate dimensions.

### *4.1. Preliminary remarks*

For vector function $\alpha \in \Im_S$ the codistribution $\Omega_\alpha$ is introduced as follows: $\Omega_\alpha(s) = span\{J_{\alpha i}(s), 1 \le i \le p\}$, where $J_{\alpha i}(s)$ is the $i$-th row of Jacobian matrix $J_\alpha(s)$ and $p$ is the dimension of vector function $\alpha$. Let $\alpha$, $\beta \in \Im_S$. It is easily to see that if $\alpha \le \beta$ then $\Omega_\alpha \supseteq \Omega_\beta$.

Codistribution $\Omega_{\alpha\times\beta}$ is the minimal one that contains both codistributions $\Omega_\alpha$ and $\Omega_\beta$, i.e. $\Omega_{\alpha\times\beta} = \Omega_\alpha+\Omega_\beta$. Codistribution $\Omega_{\alpha\oplus\beta}$ is the maximal one that is included into intersection of codistributions $\Omega_\alpha$ and $\Omega_\beta$, i.e. $\Omega_{\alpha\oplus\beta}\subseteq\Omega_\alpha\cap\Omega_\beta$. At a given point $s$, the intersection $\Omega_\alpha(s)\cap\Omega_\beta(s)$ can be found by solving the homogeneous equation

$$\sum_{i=1}^{p_\alpha} a_i(s)\, J_{\alpha i}(s)^T - \sum_{i=1}^{p_\beta} b_i(s)\, J_{\beta i}(s)^T = 0 \qquad (24)$$

for the unknown functions $a_i(s)$, $1 \le i \le p_\alpha$, and $b_i(s)$, $1 \le i \le p_\beta$, where $p_\alpha$ and $p_\beta$ denote the dimensions of appropriate vector functions. Because codistribution $\Omega_{\alpha\oplus\beta}$ must correspond to some vector function, the coefficient matrix ($a_1(s)$, $a_2(s)$, …, $a_{p_\alpha}(s)$, $b_1(s)$, $b_2(s)$, …, $b_{p_\beta}(s)$) must satisfy an additional demand to make possible integration of $\Omega_{\alpha\oplus\beta}$:

$$\partial(a_i(s)\, J_{\alpha i}(s)^T)/\partial s_j = \partial(a_j(s)\, J_{\alpha j}(s)^T)/\partial s_i \qquad (25)$$

(the similar equation can be written for coefficients $b_i(s)$, $1\le i\le p_\beta$, and function $\beta$). The set of independent solutions of (24), (25) gives the basis for codistribution $\Omega_{\alpha\oplus\beta}$.

Consider the construction $\beta \sim m(\alpha\times h)\oplus\alpha$ that is similar to one from theorem 2. Notice firstly, that from definitions of operator $m$ and relation $\Delta$ for system (23) and every $i$, $1 \le i \le p_\beta$, and $j$, $1 \le j \le m$, it follows $\alpha \times h \le J_{\beta i}\, f$, $\alpha \times h \le J_{\beta i}\, g_j$, or, that is the same, $L_\varphi\, J_{\beta i} \subseteq \Omega\alpha+\Omega_h$, $\varphi\in\{f, g_1,…, g_m\}$, where $L_\varphi\, J_{\beta i}$ denotes the Lie derivative of covector field $J_{\beta i}(x)$ along vector field $\varphi$. Let $\Lambda_\alpha$ be distribution such that $\Lambda_\alpha^\perp=\Omega_\alpha$ where $\Lambda_\alpha^\perp$ is an annihilator of $\Lambda_\alpha$. Let also $\omega\in\Lambda_\alpha \cap \ker J_h$. Clearly, that $\langle L_\varphi J_{\beta i}, \omega\rangle = 0$ where the brackets $\langle *,*\rangle$ denote an inner product. Because of $\alpha\le\beta$, one also has $\langle J_{\beta i},\omega\rangle=0$. Taking into account the identity (Isidori, 1989) $L_\varphi \langle J_{\beta i}, \omega\rangle = \langle L_\varphi J_{\beta i}, \omega\rangle + \langle J_{\beta i}, [\varphi, \omega]\rangle = 0$, where [*, *] denotes the Lie brackets, one obtains $\langle J_{\beta i}, [\varphi, \omega]\rangle = 0$. Let now $\Phi_\beta$ be minimal involutive distribution, containing $\Lambda_\alpha+ span\{[\varphi, \omega], \varphi\in\{f, g_1, …, g_m\}, \omega\in\Lambda_\alpha \cap \ker J_h\}$. According to Frobenius theorem (Isidori, 1989) distribution $\Phi_\beta$ is integrable that means solvability of partial differential equation

$$J_{\beta i}\, \Phi_\beta = 0 \qquad (27)$$

for unknown function $\beta_i$. Under this, all independent solutions of (27) specify all components of $\beta$.

### 4.2. *Realisation of geometric approach*

Denote $w^{(i)}(x)$ the matrix containing such (and only such) columns of the matrix $w(x)$ that correspond to components of $\vartheta$ affected by the fault $\rho_i$. Let $\Phi\alpha^{(i, 0)}$ be minimal involutive distribution, containing $span\{w^{(i)}\}$. According to written above, the rule for finding minimal function $\alpha^{(i)}$, satisfying (14), obtains the following geometric interpretation.

*Corollary* 2 (from theorem 2). Let $\Phi\alpha^{(i, j+1)}$ be minimal involutive distribution, containing $\Phi\alpha^{(i, j)}+ span\{[\varphi, \omega], \varphi\in\{f, g_1, …, g_m\}, \omega\in\Phi\alpha^{(i, j)} \cap \ker J_h\}$, and there exists natural $k$ such that $\Phi\alpha^{(i, k+1)}=\Phi\alpha^{(i, k)}$. Then the function $\alpha^{(i, k)}$ can be found by integration of $\Phi\alpha^{(i, k)}$.

*Remark* 4. Corollary 2 results in the construction for distribution $\Phi\alpha^{(i, j+1)}$ calculating that is similar to one proposed by De Persis and Isidori (1999).

The codistribution $\Omega_{\xi^{(i)}\circ h}$ can be found as $\Omega\alpha^{(i, k+1)}{}_{\oplus h}$ under $\Omega\alpha^{(i, k+1)} = \Phi\alpha^{(i, k+1)\perp}$ involving equations of the form (24), (25). From remark 2 (relation (20)) it follows that fault $\rho_j$ is weakly detectable via residual subvector $r^{(i)}$ if

$$rank\,(\Omega_{\xi^{(i)}\circ h} + \Omega_{\xi^{(j)}\circ h}) > rank\,\Omega_{\xi^{(j)}\circ h}\,. \qquad (24)$$

It also follows from remark 3 (relation (21)) that fault $\rho_j$ is strongly detectable via residual subvector $r^{(i)}$ if

$$rank\,(\Omega_{\xi^{(i)}\circ h} + \Omega_{\xi^{(j)}\circ h}) = l. \qquad (25)$$

### 4.3. *Example*

Consider the system described by (23) and (2) with matrix functions $f(x)=col(x_1x_4,\ x_3(1\text{-}x_3),\ 0,\ 0)$, $h(x)= col(x_1, x_2)$

$$g(x) = w(\mathrm{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & x_1 \\ 1 & 0 \end{bmatrix}.$$

It holds for single faults $\Phi\alpha^{(1, 0)} = span\{w^{(1)}\}$, $\Phi\alpha^{(2, 0)} = span\{w^{(2)}\}$ and for multiple fault $\Phi\alpha^{(3, 0)} = span\{w^{(1)}, w^{(2)}\}$. Making further calculations, one obtains

$$\Phi\alpha^{(1, 1)}(x) = \Phi\alpha^{(1, 2)}(\mathrm{x}) = span\left\{\begin{bmatrix} 0 & x_1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}\right\};$$

$$\Phi\alpha^{(2, 1)}(x) = \Phi\alpha^{(2, 2)}(\mathrm{x}) = span\left\{\begin{bmatrix} 0 & 0 \\ 0 & x_1(1\text{-}2\,x_3) \\ x_1 & -\,x_1\,x_4 \\ 0 & 0 \end{bmatrix}\right\};$$

$\Phi\alpha^{(3, 1)}(x) = \Phi\alpha^{(3, 2)}(\mathrm{x}) = \Phi\alpha^{(1, 1)}(x) + \Phi\alpha^{(2, 1)}(x)$; $\Omega_{\xi^{(1)}\circ h}(x) = \Phi\alpha^{(1, 2)}(\mathrm{x})^\perp \cap span\{Jh_i(\mathrm{x}), i=1, 2\}= span\{0,1,0,0\}$; $\Omega_{\xi^{(2)}\circ h}(x) = \Phi\alpha^{(2, 2)}(\mathrm{x})^\perp \cap span\{Jh_i(\mathrm{x}), i=1, 2\} = span\{1,0,0,0\}$; $\Omega_{\xi^{(3)}\circ h}(x) = \Phi\alpha^{(3, 2)}(\mathrm{x})^\perp \cap span\{Jh_i(\mathrm{x}), i=1,2\} = \varnothing$. Appropriate vector functions are found as follows : $\alpha^{(1)}(x) = col(x_2, x_3)$; $\alpha^{(2)}(x)=col(x_1, x_4)$; $\alpha^{(3)}(x)=const$; $\xi^{(1)}(y)=y_2$; $\xi^{(2)}(y)=y_1$; $\xi^{(3)}(y)=const$.

The results of calculating are applied to isolability analysis. Because of fulfilling (22) (or, that is the same, (25)) for $i$=1, $j$=2 and $i$=2, $j$=1, one concludes that fault $\rho_2$ is strongly detectable via residual sub-

Table 1. FS matrix

| *Residual* | *Faults* | | |
|---|---|---|---|
| | $\rho_1$ | $\rho_2$ | $\rho_3$ |
| $r^{(1)}$ | 0 | 1 | z |
| $r^{(2)}$ | 1 | 0 | z |
| $r^{(3)}$ | 0 | 0 | 0 |

vector $r^{(1)}$ and, respectively, $\rho_1$ is strongly detectable via residual subvector $r^{(2)}$. In contrast to this, for $i$=1, $i$=2 and $j$=3 only (20) (or (24)) holds, i.e. fault $\rho_3$ is only weakly detectable via residuals $r^{(1)}$, $r^{(2)}$. Then, because (19) holds for $i$=3, $j$=1 and $j$=2, residual subvector $r^{(3)}$ is insensitive both to faults $\rho_1$ and $\rho_2$. Thus, primary FS matrix is as given in Table 1.

Analysis of this matrix shows that single faults $\rho_1$ and $\rho_2$ are strongly distinguishable whereas every single fault and multiple fault $\rho_3$ are only weakly distinguishable. Indeed, let fault $\rho_1$ affects on system output such that $y_1(t_0)=0$. In this case $y_2(t)$ becomes insensitive to fault $\rho_2$ for every $t \geq t_0$ and every control $u(\tau)\in U$, $\tau\in[t_0, t]$. Thus, the system with single fault $\rho_1$ will have the same behavior as the system with multiple fault $\rho_3$ at $t \geq t_0$. Similarly, if fault $\rho_2$ distorts output $y_2$ at $t_0$ and $y_1(t_0)=0$ is true, then the system with single fault $\rho_2$ will have the same behavior as the system with multiple fault $\rho_3$ at $t \geq t_0$.

Clearly, final MS matrix is obtained by excluding the third row.

## 5. CONCLUSION

This paper considers the problem of diagnostic filter design. Algebraic and geometric approaches to solve this problem were investigated. It was shown that certain relations exist between these approaches. As evident from models (1) and (23), algebraic approach can be used for wider class of nonlinear systems than geometric one. Moreover, algebraic approach is used to solve different tasks for discrete-time systems (Zhirabok and Shumsky, 1993a).

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