DISCRETE-TIME TIME-VARYING ROBUST STABILIZATION FOR SYSTEMS IN POWER FORM

Dina Shona Laila and Alessandro Astolfi

Electrical and Electronic Engineering Department
Imperial College, Exhibition Road, London SW7 2AZ, UK.
E-mail: {d.laila, a.astolfi}@imperial.ac.uk

Abstract: A robust stabilization problem by means of continuous time-varying feedback for systems in power form is addressed. A Lyapunov based direct discrete-time design achieving input-to-state stability in a semiglobal practical sense for a discrete-time model of the system is presented. Two examples are presented to test the performance of the controller obtained using our design, in comparison with a controller obtained by emulation of a class of homogeneous controllers that are based on a similar construction.

Keywords: Discrete-time systems; Input-to-state stability; Lyapunov function; Nonholonomic systems; Power form; Time-varying systems.

1. INTRODUCTION

Consider a class of driftless control systems of the form

$$\dot{x} = \sum_{i=1}^{m} f_i(x)u_i + \sum_{j=1}^{l} e_j(x)d_j \ , \qquad (1)$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $d \in \mathbb{R}^l$ are the states, control inputs, and disturbances, respectively. f_i and e_j are smooth vector fields on $\mathbb{R}^n$. Robust stabilization using continuous feedback for systems (1) has been a difficult problem to solve. Let alone that the nominal system does not satisfy Brockett's necessary condition for smooth stabilizability using pure state feedback (Brockett, 1983), and hence it is necessary to use control that depends on time (time-varying control) or to use discontinuous feedback. The result of (Lizarraga *et al.*, 1999) states that there does not exist a continuous homogeneous stabilizer that robustly exponentially stabilizes system (1) against modeling uncertainties. The mentioned difficulties have motivated further research in this direction. Many researchers have been trying to solve this problem using discontinuous feedback (see (Lucibello and Oriolo, 2001; Morin and Samson, 1999; Prieur and Astolfi, 2003)), or to find special cases in which a continuous feedback can achieve robust stability (see (Maini *et al.*, 1999)).

Various results have been obtained for asymptotic stabilization of driftless systems via time-varying control (Pomet, 1992; Teel *et al.*, 1992; M'Closkey and Murray, 1997). Almost all available results concentrate on continuous-time design, and those that are based on Lyapunov approach rely on LaSalle Invariance Principle to complete the stability analysis, which unfortunately is not applicable for systems with uncertainty, and therefore this approach cannot be extended to solve a robust stabilization problem.

In this paper, we address a robust stabilitization problem for a class of systems with a special structure called *power form*. This class of systems is a particular case of (1) with $m = 2$ and is commonly used to model the kinematic equations of *nonholonomic* systems such as mobile robots. We focus on a type of robust stability called semiglobal practical input-to-state stability (SP-ISS) (Sontag, 2000). We exploit the results from

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(Laila and Astolfi, 2004) to design a SP-ISS controller for this class of systems, and provide an explicit construction of a SP-ISS control law and a strict Lyapunov function that can be used to analyze the SP-ISS of the closed-loop systems.

2. PRELIMINARIES

2.1 *Definitions and notation*

The set of real and natural numbers (including 0) are denoted respectively by $\mathbb{R}$ and $\mathbb{N}$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and zero at zero. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. Functions of class $\mathcal{K}_\infty$ are invertible. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class-$\mathcal{K}$ for each $\tau \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$. Given two functions $\alpha(\cdot)$ and $\gamma(\cdot)$, we denote their composition and multiplication as $\alpha \circ \gamma(\cdot)$ and $\alpha(\cdot) \times \gamma(\cdot)$, respectively. We denote $x_\circ := x(k_\circ)$, $k_\circ \geq 0$, and for any function or variable h we use a simplified notation $h(k, \cdot) := h(kT, \cdot)$. $|x|$ denotes the 1-norm of a vector $x \in \mathbb{R}^n$.

To begin with, we consider nonlinear time-varying systems described by

$$\dot{x} = f(t, x(t), d(t)) \ , \tag{2}$$

where $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^l$ are the states and exogenous disturbances, respectively. Assume that the system (2) is between a sampler and zero order hold. The discrete-time model of (2) is written as

$$x(k+1) = F_T(k, x(k), d(k)) \ . \tag{3}$$

We emphasize that for nonlinear systems the exact discrete-time model is usually not available, since it requires solving a nonlinear initial value problem which is unsolvable in general (see (Nešić and Laila, 2002) for more details). Therefore we assume that (3) is obtained by approximation, and it satisfies a type of consistency property to be a good approximation of the exact model (see (Laila and Astolfi, 2004; Stuart and Humphries, 1996)). We denote $x(k, k_\circ, x_\circ, d)$ the discrete-time trajectory of system (3) with initial state $x(k_\circ) = x_\circ$ and input d. We will use the following definitions to construct our main results.

Definition 2.1. The family of systems (3) is SP-ISS if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that for any strictly positive real numbers $\Delta_x, \Delta_d, \delta$ there exists $T^* > 0$ such that the solutions of the system satisfy

$$|x(k, k_\circ, x_\circ, d)| \leq \beta(|x_\circ|, (k - k_\circ)) + \gamma(\|d\|_\infty) + \delta \ ,$$

for all $k \geq k_\circ$, $T \in (0, T^*)$, $|x_\circ| \leq \Delta_x$, and $\|d\|_\infty \leq \Delta_d$. Moreover, if there is no disturbance, i.e. $d = 0$, the system is semiglobally practically asymptotically stable (SP-AS). ∎

Definition 2.2. A family of continuous functions $V_T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a family of SP-ISS Lyapunov functions for the family of systems (3) if there exist functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, a positive definite function α and a function $\chi \in \mathcal{K}$, and for any strictly positive real numbers $\Delta_x, \Delta_d, \nu_1, \nu_2$ there exists $T^* > 0$, such that the following inequalities

$$\underline{\alpha}(|x|) \leq V_T(k, x) \leq \overline{\alpha}(|x|) \ , \tag{4}$$

$$|x| \geq \chi(|d|) + \nu_1 \ \Rightarrow \ \Delta V_T \leq -T\alpha(|x|) \ , \tag{5}$$

$$V_T(k+1, F_T) \leq V_T(k, x) + \nu_2 \ , \tag{6}$$

with $\Delta V_T := V_T(k+1, F_T) - V_T(k, x)$, hold for all $k \geq k_\circ$, $T \in (0, T^*)$, $|x| \leq \Delta_x$, and $\|d\|_\infty \leq \Delta_d$. Moreover, if $d = 0$, the function V_T is called a SP-AS Lyapunov function. V_T is called a smooth Lyapunov function if it is smooth in $x \in \mathbb{R}^n$. ∎

Considering nonlinear time-varying systems with control input

$$\dot{x} = f(t, x(t), u(t), d(t)) \ , \tag{7}$$

where $u \in \mathbb{R}^m$ is a time-varying feedback control $u(t) := u(t, x(t))$, then the parameterized family of approximate discrete-time model of (7) is written as

$$x(k+1) = F_T(k, x(k), u(k), d(k)) \ . \tag{8}$$

If we use (8) for the design, we can obtain a discrete-time controller $u(k) = u_T(k, x(k))$ that is also parameterized by T. We have the following definition.

Definition 2.3. Let $\hat{T} > 0$ be given and for each $T \in (0, \hat{T})$ let the functions $V_T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $u_T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ be defined. The pair (u_T, V_T) is a semiglobally practically input-to-state stabilizing (SP-ISS) pair for the system (8) if there exist functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, a positive definite function α and a function $\chi \in \mathcal{K}$ such that for any strictly positive real numbers $\Delta_x, \Delta_d, \nu_1, \nu_2$ there exists a pair of strictly positive real numbers (T^*, M), with $T^* \leq \hat{T}$, such that (4), (5), (6) and

$$|u_T(k, x)| \leq M \ , \tag{9}$$

hold, for all $k \geq k_\circ$, $T \in (0, T^*)$, and all $|x| \leq \Delta_x$, $\|d\|_\infty \leq \Delta_d$. Moreover, if $d = 0$, the pair (V_T, u_T) is called a SP-AS pair. ∎

Systems in power form in the presence of disturbances are represented by

$$\dot{x} = \sum_{i=1}^{2} f_i(x) u_i + \sum_{j=1}^{l} e_j(x) d_j \ . \tag{10}$$

with the vector fields

$$f_1 = \frac{\partial}{\partial x_1}; \quad f_2 = \sum_{j=2}^{n} \frac{x_1^{j-2}}{(j-2)!} \frac{\partial}{\partial x_j} \ .$$

In this paper, we use the Euler approximate model of the system, i.e.

$$x(k+1) = x(k) + T[\sum_{i=1}^{2} f_i(x) u_i + \sum_{j=1}^{l} e_j(x) d_j]. \tag{11}$$

However, we note that the result can be generalized directly to the case of using other discrete-time models which are consistent with respect to the exact model of the continuous-time plant (1).

2.2 *ISS Lyapunov converse theorem for time-varying systems*

The following result is a converse Lyapunov theorem for SP-ISS of nonlinear discrete-time time-varying systems.

Theorem 2.1. (Laila and Astolfi, 2004) A parameterized family of discrete-time time-varying systems (3) is SP-ISS if and only if it admits a SP-ISS Lyapunov function V_T. ■

The following corollary is an application of Theorem 2.1 to time-varying periodic systems F_T which are periodic in k with period $\lambda > 0$, i.e.

$$F_T(k + m\lambda, x, d) := F_T(k, x, d) \ , \quad \forall m \in \mathbb{N} \ . \quad (12)$$

We will use the corollary to state our main result.

Corollary 1. (Laila and Astolfi, 2004) A time-varying periodic system (3) with period λ is SP-ISS if and only if it admits a SP-ISS periodic Lyapunov function with the same period λ. ■

3. LYAPUNOV STABILITY DESIGN FOR SYSTEMS IN POWER FORM

We propose a pair of SP-ISS Lyapunov function and discrete-time control law for systems in power form. We first focus on the SP-AS of the nominal system ($d = 0$), and since we have a strict Lyapunov function, we can extend the result to the SP-ISS of the system (11).

3.1 *Semiglobal practical asymptotic stabilization*

Theorem 3.1. Consider system (11) with $d = 0$, i.e.

$$x(k+1) = x(k) + T \sum_{i=1}^{2} f_i(x) u_i \ . \quad (13)$$

Suppose the functions $\rho : \mathbb{R} \to \mathbb{R}$ and $W : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfy the following properties.
P1. The function W is of class C^∞ on $\mathbb{R}^{n-1}$ and of class C^2 on $\mathbb{R}^{n-1} - \{0\}$, and is defined as

$$W(x) = \sum_{i=2}^{n} c_i \, |x_i|^{a_i} \ , \quad c_i > 0, \ a_i \in \{2, 3, \cdots\}.$$

P2. The function ρ is of class C^1 on $(0, \infty)$, and is defined by $\rho(s) = g_0 \, |s|^b \ , \quad b > 0, \ g_0 > 0$.
Then there exists $T^* > 0$ such that for all $T \in (0, T^*)$, the controller $u_T := (u_{1T}, u_{2T})^T$, where

$$\begin{aligned} u_{1T} &= -g_1 x_1 - \rho(W)\Big(\cos((k+1)T) \\ &\quad - \frac{\epsilon}{2} \sin((k+1)T) \Big) + \frac{\epsilon}{2} \Delta_\rho \sin((k+1)T) \\ u_{2T} &= -g_2 \, \mathrm{sign}(L_{f_2} W) \, |L_{f_2} W|^a \, \Big(2\rho(W) \\ &\quad + 2(g_1 x_1 + \rho(W) \cos((k+1)T)) \cos((k+1)T) \\ &\quad - \epsilon g_1 x_1 \sin((k+1)T) \Big), \end{aligned} \quad (14)$$

with $g_1 > 0$, $g_2 > 0$, $a > 0$ and a sufficiently small $\epsilon > 0$, is a SP-AS controller for the system (13) and the function

$$\begin{aligned} V_T(k, x) &= (g_1 x_1 + \rho(W) \cos(kT))^2 + \rho(W)^2 \\ &\quad - \epsilon g_1 x_1 \rho(W) \sin(kT) \end{aligned} \quad (15)$$

is a SP-AS Lyapunov function for the closed-loop system (13), (14). ■

Proof of Theorem 3.1: Pick the functions W and ρ satisfying **P1** and **P2**, respectively. We prove that (u_T, V_T) is a SP-AS pair for the system (13) by showing the existence of the positive numbers (T^*, M) such that the inequalities (4), (5), (6) and (9) of Definition 2.3 hold.

Fix strictly positive numbers Δ_x, ν_1 and ν_2. We consider arbitrary x with $|x| \le \Delta_x$. Let $T_1 > 0$ be such that for all $|x| \le \Delta_x$ and $T \in (0, T_1)$, we have $|x(k+1)| \le \Delta_x + 1$. Without loss of generality, assume that $T_1 < 1$. From **P1** and **P2** respectively, we see that the functions W and $\rho(W)$ are zero at zero, positive definite in $\mathbb{R}^{n-1}$ and radially unbounded. To show that the inequality (4) holds, we write the Lyapunov function (15) as

$$V_T(k, x) = \begin{bmatrix} x_1 & \rho(W) \end{bmatrix} P \begin{bmatrix} x_1 \\ \rho(W) \end{bmatrix} \ ,$$

with a symmetric matrix

$$P = \left[\begin{array}{c|c} g_1^2 & g_1(\cos(kT) - \frac{\epsilon}{2} \sin(kT)) \\ \hline * & \cos^2(kT) + 1 \end{array} \right] .$$

The determinant of the matrix P is

$$|P| = g_1^2 \Big[1 - \Big(\frac{\epsilon^2}{4} \sin^2(kT) - \epsilon \cos(kT) \sin(kT) \Big) \Big] \ .$$

Let $\epsilon > 0$ be sufficiently small, such that

$$\frac{\epsilon^2}{4} \sin^2(kT) - \epsilon \cos(kT) \sin(kT) \le \tilde{\epsilon} < 1 \ . \quad (16)$$

Hence, the matrix P is positive definite, and this implies that $V_T(k, x)$ is positive definite and radially unbounded. Therefore, inequality (4) holds.

We now prove (5) by showing that with the controller (14), the Lyapunov difference is negative definite in a semiglobal practical sense. We use the Mean Value Theorem to obtain

$$\begin{aligned} \Delta_\rho &:= \rho(W(x(k+1))) - \rho(W(x(k))) \\ &\le \frac{d\rho(W)}{dW} \bigg|_{W = W^*} (W(x(k+1)) - W(x(k))) \\ &\le b g_0 \, |W^*|^{b-1} \, \Delta_W \ , \end{aligned} \quad (17)$$

where $W^* = \theta_1 W(x(k+1)) + (1 - \theta_1) W(x(k))$ for $\theta_1 \in (0, 1)$, and

$$\begin{aligned} \Delta_W &:= W(x(k+1)) - W(x(k)) \\ &\le \frac{dW}{dx} \bigg|_{x = x^\star} (x(k+1) - x(k)) \\ &\le L_{f_2} W(x^\star) T u_{2T} \ , \end{aligned} \quad (18)$$

with $x^\star = \theta_2 x(k+1) + (1 - \theta_2) x(k)$ for $\theta_2 \in (0, 1)$. Let $T_2 > 0$ be sufficiently small, such that for $T \in$

$(0, T_2)$ we can assume that $L_{f_2}W(x^\star) \approx L_{f_2}W(x)$. Moreover, we use the following approximation

$$\cos((k+1)T) - \cos(kT) \approx O(T^2) \; , \qquad (19)$$
$$\sin((k+1)T) - \sin(kT) \approx O(T) \; . \qquad (20)$$

The Lyapunov difference can then be written as

$$\begin{aligned}
&\Delta V_T \\
&\quad = V_T(k+1, x(k+1)) - V_T(k, x(k)) \\
&\quad = \Big[g_1(x_1 + Tu_1) + (\rho(W) + \Delta_\rho)\cos((k+1)T)\Big]^2 \\
&\quad\quad - (g_1 x_1 + \rho(W)\cos(kT))^2 + 2\rho(W)\Delta_\rho \\
&\quad\quad - \epsilon g_1(x_1 + Tu_1)\,(\rho(W) + \Delta_\rho)\sin((k+1)T) \\
&\quad\quad + \epsilon g_1 x_1 \rho(W)\sin(kT) \; .
\end{aligned}$$

We use (17), (18), (19), (20) and ϵ sufficiently small ($\epsilon = O(T)$), and substitute (14) to obtain

$$\begin{aligned}
\Delta V_T \le & -2Tg_1(g_1x_1 + \rho(W)\cos((k+1)T))^2 \\
& - 2Tg_1(\frac{\epsilon}{2}\rho(W)\sin((k+1)T))^2 \\
& - TA(\epsilon g_1 x_1 \sin((k+1)T))^2 \\
& - 2Tg_1(\frac{\epsilon}{2}\Delta_\rho \sin((k+1)T))^2 \\
& - TA(2\rho(W) + 2(g_1x_1 + \rho(W) \\
& \quad \times \cos((k+1)T))\cos((k+1)T))^2 \\
& + O(T^2) \; ,
\end{aligned}$$

where $A := g_2 b g_0 \left|W^*\right|^{b-1} \left|L_{f_2}W\right|^{a+1} \ge 0$. We now focus on the state x_1 in the first term, and the states $x_i, \; i = 2, 3, \cdots, n$ in the second term. The first term is negative definite for $x_1 \neq -\rho(W)\cos((k+1)T)/g_1$. However, at these points, the third term is negative, and hence the sum of both terms is still negative. Moreover, the second term is negative definite for $(k+1)T \neq i\pi, \; i \in \mathbb{N}$. However, at these points the total quantity is still negative since $\cos((k+1)T)$ reaches its maximum and the nontrigonometric term is nonzero. Therefore, we can write

$$\Delta V_T \le -T\tilde{\alpha}(|x|) + O(T^2) \; , \qquad (21)$$

with $\tilde{\alpha}$ positive definite. Define $\tilde{\nu}_1 := \kappa\tilde{\alpha}(\nu_1), \; 0 < \kappa < 1$, and let $T_3 > 0$ be such that for all $T \in (0, T_3)$, the term $O(T^2) < T\tilde{\nu}_1$. Defining $T^* := \min\{\tilde{T}, T_1, T_2, T_3\}$, then for all $|x| \le \Delta_x$, and all $T \in (0, T^*)$, we have that

$$\Delta V_T \le -T\tilde{\alpha}(|x|) + T\tilde{\nu}_1 \; , \qquad (22)$$

and hence, (5) holds. Inequality (6) follows directly from (22). Finally, from **P1**, **P2**, (17) and (18), and since $|x(k+1)| \le \Delta_x + 1$, it is direct to show that (9) holds, and this completes the proof. ∎

Remark 2. Comparing the structure of the controller (14) with the homogeneous controller proposed in (Pomet and Samson, 1994), we can see that the former is a perturbed form of the latter. ∎

3.2 *An extension to SP-ISS stabilization*

In the presence of modeling uncertainties or disturbances it has been proven in (Lizarraga *et al.*, 1999) that smooth control that exponentially stabilizing affine systems, of which systems in power form are a special case, is not robust. Although the robust exponential stability definition of (Lizarraga *et al.*, 1999) is not general (ρ-exponential stability), it shows that robust stability design for this class of system is nontrivial. In Theorem 3.1, we have obtained V_T, a strict SP-AS Lyapunov function for the system. It is known that negative definiteness of ΔV_T makes possible to extend the result directly to the stabilization in the presence of disturbances. The following is an extension of Theorem 3.1 to SP-ISS using smooth feedback.

Theorem 3.2. Consider the Euler approximate model (11). Suppose that the functions ρ and W satisfy properties **P1** and **P2** respectively. Then there exist $T^* > 0$ such that for all $T \in (0, T^*)$, the controller (14) is a SP-ISS controller for the system (11) and the function (15) is a SP-ISS Lyapunov function for the closed-loop system (11), (14). ∎

Sketch of the proof of Theorem 3.2: The proof follows closely the proof of Theorem 3.1, by taking into account the disturbance $d \in \mathbb{R}^l$. Given a positive number $\Delta_d > 0$ such that the disturbance d satisfies $|d| \le \Delta_d$. Note that, while in the SP-AS case it is sufficient to show that (5) holds with a positive definite $\tilde{\alpha}$, for SP-ISS $\tilde{\alpha}$ is required to be a $\mathcal{K}_\infty$-function. Therefore we modify the last step in the following way. Using Young's inequality we split all terms containing the states and the disturbance. Through suitable majorization and since $\sin((k+1)T))^2 \le 1$ we get

$$\begin{aligned}
\Delta V_T \le & -T\bar{A}(x_1^2 + \rho(W)^2\sin((k+1)T)^2) \\
& + T\bar{\chi}(|d|) + T\bar{\nu}_1 \\
\le & -T\bar{A}(x_1^2 + \rho(W)^2)\sin((k+1)T)^2 \\
& + T\bar{\chi}(|d|) + T\bar{\nu}_1 \; ,
\end{aligned}$$

with $\bar{A} > 0$ and $\bar{\chi} \in \mathcal{K}$. We add and subtract the term $T\mu\bar{A}(x_1^2 + \rho(W)^2)$ with $0 < \mu \lll T$, so that $\mu\bar{A}(x_1^2 + \rho(W)^2) \le 0.1\bar{\nu}_1$ for all $|x| \le \Delta_x$. Hence,

$$\begin{aligned}
\Delta V_T \le & -T\bar{A}(x_1^2 + \rho(W)^2)(\sin((k+1)T)^2 + \mu) \\
& + T\bar{\chi}(|d|) + T(\bar{\nu}_1 + 0.1\bar{\nu}_1) \\
\le & -T\tilde{\alpha}(|x|) + T\bar{\chi}(|d|) + T\tilde{\nu}_1 \; ,
\end{aligned}$$

with $\tilde{\alpha} \in \mathcal{K}_\infty$ and $\tilde{\nu}_1 = 1.1\bar{\nu}_1$, which obviously implies that (5) holds. The rest follows exactly the proof of Theorem 3.1. ∎

4. DESIGN EXAMPLES

We present two examples to illustrate the proposed design. We compare the performance of the proposed controllers against the performance of the homogeneous controllers proposed in (Pomet and Samson, 1994).

4.1 *SP-AS design for a car-like mobile robot*

Consider a simple kinematic model of a car-like mobile robot moving on a plane (Teel *et al.*, 1992):

$$\begin{aligned}\dot{x} &= v\cos\theta\\ \dot{y} &= v\sin\theta\\ \dot{\phi} &= \omega\\ \dot{\theta} &= \frac{1}{L}\tan(\phi)v\ ,\end{aligned} \tag{23}$$

with $v-$the forward velocity, $\omega-$the steering velocity, $(x,y)-$the Cartesian position of the center of mass of the robot, $\phi-$the angle of the front wheels with respect to the car (the steering angle) and $\theta-$the orientation of the car with respect to some reference frame. Using a suitable coordinate transformation we obtain the dynamic model of system (23) in power form.

It has been shown in (Pomet and Samson, 1994) that the control

$$\begin{aligned}u_1 &= -3x_1 + 0.4\sqrt[6]{W(x)}\cos t\\ u_2 &= -0.03\kappa\,\mathrm{sign}(L_{f_2}W(x))\sqrt[5]{|L_{f_2}W(x)|}\ ,\end{aligned} \tag{24}$$

with $\kappa > 0$,

$$\begin{aligned}W(x) &= 0.5x_2^6 + 10^4\,|x_3|^3 + 1.5\times10^6 x_4^2\\ L_{f_2}W(x) &= 3x_2^5 + 3\times10^4\,\mathrm{sign}(x_3)x_3^2x_1\\ &\quad + 1.5\times10^6x_4x_1^2\ ,\end{aligned}$$

asymptotically stabilizes the mobile robot. Applying Theorem 3.1, we construct the controller

$$\begin{aligned}u_{1T} &= -3x_1 + \rho(W)(\cos((k+1)T)\\ &\quad - \frac{\epsilon}{2}\sin((k+1)T))\\ u_{2T} &= u_2\Big(2\rho(W) - 3\epsilon x_1\sin((k+1)T)\\ &\quad + 2\big(3x_1 + \rho(W)\cos((k+1)T)\big)\\ &\quad \times\cos((k+1)T)\Big),\end{aligned} \tag{25}$$

with $\rho(W) = 0.4\sqrt[6]{W(x)}$ and u_2 given by (24), which is a SP-AS controller for the Euler model of the system in power form.

Figure 1 shows the response when controller (25) with $\kappa = 1$ is applied to control the plant (23) compared to the response with the sample and hold version (emulation) of controller (24) with $\kappa = 25/6$ (chosen based on the average value of $2\rho(W)$). In the simulations, we have used $x_\circ = (0,0,0,1)'$, $T = 0.2$ and $\epsilon = 0.35$. We display the (x,y) position of the car, that is given by $x = x_1$ and $y = x_4 - x_1x_3 + \frac{1}{2}x_1^2x_2$. It is shown that in the absence of disturbance, the perturbed controller (25) that we propose performs as well as the homogeneous controller (24). Note that the controller (25) is in fact also a SP-ISS stabilizing controller for the same system with disturbance. The robustness of the proposed controller is demonstrated in the next example.

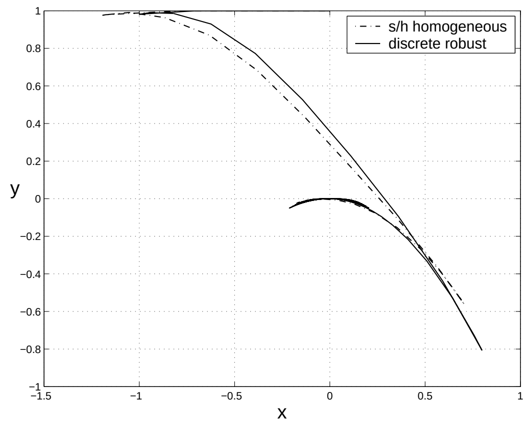

Figure 1. Position (x,y) of the car model with robust controller (25) and homogeneous controller (24).

4.2 *SP-ISS design for a unicycle mobile robot*

Consider the model of a unicycle mobile robot moving on a plane, with two independent rear motorized wheels (Pomet, 1992):

$$\begin{aligned}\dot{x} &= v\cos\theta + d\sin\theta\\ \dot{y} &= v\sin\theta - d\cos\theta\\ \dot{\theta} &= \omega\ ,\end{aligned} \tag{26}$$

with $v-$the forward velocity, $\omega-$the steering velocity, $(x,y)-$the Cartesian position of the center of mass of the robot, $\theta-$the heading angle from the horizontal axis, $d-$a disturbance (exogenous force) perpendicular to the forward direction. By the coordinate transformation

$$\begin{aligned}&x_1 = x;\quad x_2 = \tan\theta;\\ &x_3 = -y + x\tan\theta\ ,\end{aligned} \tag{27}$$

we obtain the dynamic model of system (26) in power form with disturbance:

$$\begin{aligned}\dot{x}_1 &= u_1 + \frac{d}{\sqrt{1+x_2^2}}\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_1u_2 + d\sqrt{1+x_2^2}\ ,\end{aligned} \tag{28}$$

where $u_1 := v\cos\theta$, and $u_2 := \omega\sec^2\theta$. Applying Theorem 3.2, and choosing

$$\begin{aligned}W &= 0.5x_2^4 + 10^4x_3^2\\ \rho(W) &= 0.1\sqrt[4]{|W|}\ ,\end{aligned} \tag{29}$$

it can be shown that the controller

$$\begin{aligned}u_{1T} &= -2x_1 + \rho(W)(\cos((k+1)T)\\ &\quad - \frac{\epsilon}{2}\sin((k+1)T))\\ u_{2T} &= -0.05\,\mathrm{sign}(L_{f_2}W)\,|L_{f_2}W|^{\alpha}\Big(2\rho(W)\\ &\quad + 2(2x_1 + \rho(W)\cos((k+1)T))\cos((k+1)T)\\ &\quad - 2\epsilon x_1\sin((k+1)T)\Big)\ ,\end{aligned} \tag{30}$$

and the Lyapunov function

$$\begin{aligned}V_T &= (2x_1 + \rho(W)\cos((k+1)T)^2 + \rho(W)^2\\ &\quad - 2\epsilon x_1\rho(W)\sin((k+1)T)\end{aligned} \tag{31}$$

is a SP-ISS pair for the closed-loop system which consist of the Euler model of (28) with the controller (30).

Figure 2 shows the simulation result for the system controlled using the proposed robust controller (30), in comparison with the sampled and hold version of the homogeneous controller, in the presence of a constant disturbance $d = 0.2$. We display the (x, y) position of the mobile robot, which are given by $x = x_1$ and $y = x_1 x_2 - x_3$. In the simulation we use the initial condition $x_0 = (0,\ 0,\ -1)^T$, $T = 0.5$ and $\epsilon = 0.415$. The controller parameters are chosen to give the best response for both controllers. It is shown by the simulation that for the chosen parameters, when applying the controller (30), which is designed using our proposed construction, the response exhibits lower overshoot and the steady state position of the vehicle is closer to the origin. This indicates that compared to the homogeneous controller, our proposed controller is somewhat more robust to the presence of disturbance. This behaviour is consistent for other simulation settings, with a careful choice of the parameters of the controller.

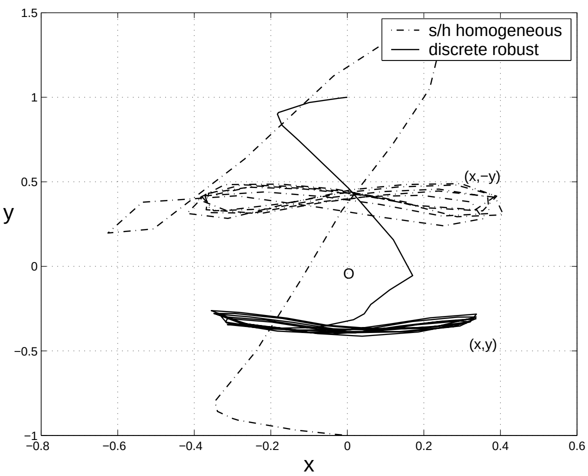

Figure 2. Position (x, y) with robust controller (30) vs (upside down) position $(x, -y)$ with homogeneous controller, for $d = 0.2$.

5. SUMMARY

We have presented a solution to a discrete-time robust stabilization problem for nonholonomic systems in power form. A construction of a discrete-time control law and a strict Lyapunov function for a SP-ISS problem has been presented. Examples are given to test the proposed design. It is shown that controllers designed using our construction are more robust, compared to homogeneous controllers that relies on a similar construction. The results have shown that robust stabilization using continuous control is possible, and this gives an alternative to emulation design for sampled-data stabilization of systems in power form.

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