

**STRICT LYAPUNOV FUNCTION AND CHETAEV
FUNCTION FOR STABILITY/INSTABILITY
ANALYSIS OF THE PENDULUM¹**

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Abstract: Stability analysis of nonlinear systems can be carried out by the Lyapunov's first method (linearization technique). However, by utilizing this approach most insight about the physical system may be lost. Alternative approaches are available to exploit system characteristics such as those associated to energy concepts for stability and instability analysis. In this paper, with reference to a simple nonlinear system—the pendulum—, we proposed first, a simple strict Lyapunov function motivated by energy consideration to study asymptotic stability of some equilibria, and second a Chetaev function to analyze instability of the remaining ones. A remarkable point is that both functions have a common structure based on the system's energy, and differs only in a single scalar parameter. *Copyright ©2005 IFAC*

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1. INTRODUCTION

This paper considers autonomous systems described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, and the vector field $\mathbf{f}(\mathbf{x})$ is assumed sufficiently smooth.

The simplest and basic structure in dynamic systems is the equilibrium. Equilibria are related to important meaning in many physical processes.

They are those elements of the state space, say $\mathbf{x}^* \in \mathbb{R}^n$, where the vector field vanishes, i.e.

$$\mathbf{f}(\mathbf{x}^*) = \mathbf{0}.$$

The Lyapunov stability theory mainly focuses — at least as presented in important stability analysis textbooks— to study a number of attributes of equilibria such as: stability, asymptotic stability, instability, etc. The Lyapunov's theory offers a set of theorems to study these attributes of the equilibria (see e.g. Vidyasagar (1993), Khalil (1996)). Most of them resort to the concept of Lyapunov function below (Vidyasagar, 1993)

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Definition 1. Assume that the origin of the state space is an equilibrium of the system (1), i.e. $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. A continuous and differentiable function $V(\mathbf{x})$ is a *Lyapunov function* for the equilibrium $\mathbf{x}^* = \mathbf{0}$ if it is a positive definite function (at least locally) whose time derivative along the system (1):

$$\dot{V}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})^T}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})$$

is nonpositive (at least locally).

□

The basic stability theorem of the Lyapunov's direct method states that if a Lyapunov function exists then the equilibrium is stable (Khalil, 1996). A powerful class of Lyapunov function is defined in the following (Santibañez and Kelly, 1997)

Definition 2. A Lyapunov function for the equilibrium $\mathbf{x}^* = \mathbf{0}$ of the system (1) is a *strict Lyapunov function* if the time derivative along the system (1):

$$\dot{V}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})^T}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})$$

is negative definite (at least locally).

□

In other words, strict Lyapunov functions are continuously differentiable positive definite functions (at least locally) whose time derivative along the system trajectories are negative definite (at least locally).

For instability analysis, it is convenient to introduce the Chetaev functions as follows

Definition 3. A *Chetaev function* for the equilibrium $\mathbf{x}^* = \mathbf{0}$ of the system (1) is a continuously differentiable function $V(\mathbf{x})$ such that

- $V(\mathbf{0}) = 0$
- $V(\mathbf{x}) > 0$ at least for some arbitrarily small $\mathbf{x} \neq \mathbf{0}$

such that the time derivative along the system (1)

$$\dot{V}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})^T}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})$$

is positive definite (at least locally).

□

This definition obeys to conditions evoked by the Chetaev's instability theorem (Khalil, 1996). Indeed, Definition 3 imposes stronger requirements than those needed in the Chetaev's theorem.

Based in above definitions, we can summarize the following sufficient conditions for stability

and instability extracted from well-known theorems of the Lyapunov's stability theory (see e.g. Vidyasagar (1993), Khalil (1996)).

Theorem 1. Consider system (1) where the origin $\mathbf{x}^* = \mathbf{0}$ is an equilibrium. If there exists a continuously differentiable function $V(\mathbf{x})$ such that

- $V(\mathbf{x})$ is a Lyapunov function, then the equilibrium is stable
- $V(\mathbf{x})$ is a strict Lyapunov function, then the equilibrium is asymptotically stable
- $V(\mathbf{x})$ is a Chetaev function, then the equilibrium is unstable

□

The remaining of the paper is devoted to show the application of this theorem for the stability/instability analysis of a simple nonlinear system where its energy function is exploited to introduce a strict Lyapunov function and a Chetaev function with the novelty of a common structure.

2. PENDULUM WITH DAMPING

A simple and intuitive nonlinear physical system is the pendulum with damping. In normalized form, the unforced process dynamics is given by

$$\ddot{q} + \dot{q} + \sin(q) = 0$$

where q stands for the pendulum angular position. This system can be rewritten in state form as

$$\frac{d}{dt} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -\sin(q) - \dot{q} \end{bmatrix}.$$

This an autonomous system whose set of equilibria \mathcal{E} is given by

$$\mathcal{E} = \left\{ \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} n\pi \\ 0 \end{bmatrix} \right\}$$

where n is an integer number.

The stability/instability attributes of these equilibria can be studied straightforward by the Lyapunov's first method, that is, by the linearization approach. Notwithstanding, we pay attention to exploit the physical structure of the system to study its stability/instability without linearization. We believe that doing this, a systematic procedure may be derived for analysis of more complex physical systems.

The energy function of the system is composed by the sum of the kinetic energy $\frac{1}{2}\dot{q}^2$ plus the potential energy $1 - \cos(q)$.

The equilibria corresponding to n even (including $n = 0$) have common topological stability/instability attributes. This fact also applies to

those equilibria associated to n odd. In virtue of these reasons, it is enough to analyze the equilibria resulting for $n = 0$ and $n = 1$.

3. STRICT LYAPUNOV FUNCTION

The equilibrium associated to $n = 0$ is known to be asymptotically stable. One way to prove this fact is by finding a strict Lyapunov function and invoking the Theorem 1.

However, first we prove that the equilibrium is stable. The natural and classical way is by utilizing the pendulum total energy function

$$V_L(q, \dot{q}) = \frac{1}{2}\dot{q}^2 + [1 - \cos(q)]$$

which is locally positive definite. Its time derivative along the systems trajectories (1) yields

$$\dot{V}_L(q, \dot{q}) = -\dot{q}^2 \leq 0$$

In virtue of Definition 1, $V_L(q, \dot{q})$ qualifies as a Lyapunov function, but it is not a strict Lyapunov function because it fails to satisfy that $\dot{V}_L(q, \dot{q})$ is locally negative definite. Therefore, in view of Theorem 1, we have the conclusion that the equilibrium associated to $n = 0$ is stable. Typically, the asymptotic stability attribute is shown by application of the Krasovskii–LaSalle’s theorem (Vidyasagar, 1993). Instead, in the remaining of this Section we will utilize a strict Lyapunov function.

Now, in this paper we propose the following function inspired from the total energy function

$$\begin{aligned} V_{SL}(q, \dot{q}) &= V_L(q, \dot{q}) + \frac{1}{2} \sin(q)\dot{q}, \\ &= \frac{1}{2}\dot{q}^2 + [1 - \cos(q)] + \frac{1}{2} \sin(q)\dot{q} \end{aligned} \quad (2)$$

where the term $\frac{1}{2} \sin(q)\dot{q}$ has been added. We have to check that it is a locally positive definite function. To this end, it is sufficient to ensure: first, the function vanishes at the equilibrium. Second, its gradient is null at the equilibrium. Third, its Hessian evaluated at the equilibrium is positive definite.

Obviously $V_{SL}(0, 0) = 0$ by direct substitution. The gradient is given by

$$\begin{bmatrix} \sin(q) + \frac{1}{2} \cos(q)\dot{q} \\ \dot{q} + \frac{1}{2} \sin(q) \end{bmatrix}$$

which vanishes at the equilibrium. Finally the Hessian reads as

$$\begin{bmatrix} \cos(q) - \frac{1}{2} \sin(q)\dot{q} & \frac{1}{2} \cos(q) \\ \frac{1}{2} \cos(q) & 1 \end{bmatrix}.$$

Straightforward substitution indicates that the Hessian is positive definite at the equilibrium.

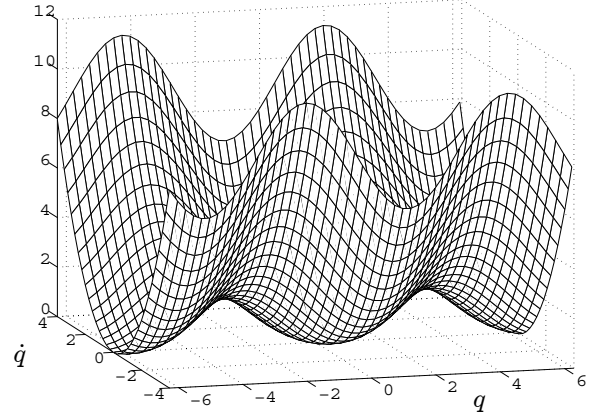


Fig. 1. Function $V_{SL}(q, \dot{q})$

These arguments imply that function $V_{SL}(q, \dot{q})$ is locally positive definite. Figure 1 depicts function $V_{SL}(q, \dot{q})$ which vanishes at

$$\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 2n\pi \\ 0 \end{bmatrix}.$$

After easy simplifications, the time derivative of $V_{SL}(q, \dot{q})$ along the system trajectories (1) leads to

$$\dot{V}_{SL}(q, \dot{q}) = -W(q, \dot{q})$$

where

$$W(q, \dot{q}) = \left[\left[1 - \frac{1}{2} \cos(q) \right] \dot{q}^2 + \frac{1}{2} \sin(q)^2 + \frac{1}{2} \sin(q)\dot{q} \right].$$

It will be proven that $W(q, \dot{q})$ is a locally positive definite function. We proceed by the same stages that those utilized to shown that $V_{SL}(q, \dot{q})$ is also locally positive definite.

First observe that $W(0, 0) = 0$. Second, the gradient can be written by

$$\begin{bmatrix} \frac{1}{2} \sin(q)\dot{q}^2 + \sin(q) \cos(q) + \frac{1}{2} \cos(q)\dot{q} \\ 2 \left[1 - \frac{1}{2} \cos(q) \right] \dot{q} + \frac{1}{2} \sin(q) \end{bmatrix}$$

which vanishes at $[q \ \dot{q}]^T = [0 \ 0]^T$. On the other hand, the Hessian matrix is

$$\begin{bmatrix} H_{11}(q, \dot{q}) & H_{12}(q, \dot{q}) \\ H_{21}(q, \dot{q}) & H_{22}(q, \dot{q}) \end{bmatrix}$$

where

$$H_{11}(q, \dot{q}) = \frac{1}{2} \cos(q)\dot{q}^2 + \cos(q)^2 - \sin(q)^2 - \frac{1}{2} \sin(q)\dot{q},$$

$$H_{12}(q, \dot{q}) = \sin(q)\dot{q} + \frac{1}{2} \cos(q),$$

$$H_{21}(q, \dot{q}) = \sin(q)\dot{q} + \frac{1}{2} \cos(q),$$

$$H_{22}(q, \dot{q}) = 2 \left[1 - \frac{1}{2} \cos(q) \right].$$

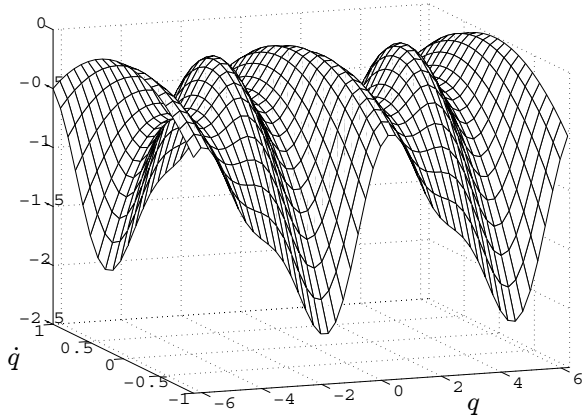


Fig. 2. Function $\dot{V}_{SL}(q, \dot{q})$

The Hessian evaluated at the equilibrium produces

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

which corresponds to a positive definite matrix. Therefore, these arguments ensure that $W(q, \dot{q})$ is a locally positive definite function, thus $\dot{V}_{SL}(q, \dot{q})$ is a locally negative definite function. The plot of $\dot{V}_{SL}(q, \dot{q})$ is shown in Figure 2; notice that it is negative and it vanishes only at

$$\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}.$$

This means that $V_{SL}(q, \dot{q})$ qualifies as a strict Lyapunov function, and therefore in agreement with Theorem 1, the equilibrium associated to $n = 0$ is asymptotically stable. As a matter of fact, this strict Lyapunov function applies naturally to prove asymptotic stability of all equilibria corresponding to n even.

4. CHETAEV FUNCTION

The equilibrium corresponding to $n = 1$ (and in general n odd) is known to be unstable. We shift the equilibrium to the origin by the change of variable $z = q - \pi$. The system (1) can be rewritten as

$$\frac{d}{dt} \begin{bmatrix} z \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -\sin(z + \pi) - \dot{q} \end{bmatrix}. \quad (3)$$

Thus, the origin $[z \ \dot{q}]^T = [0 \ 0]^T$ corresponds to equilibrium $[q \ \dot{q}]^T = [\pi \ 0]^T$. According to Theorem 1, it is sufficient to find a Chetaev function to prove that the equilibrium is unstable.

Motivated by the strict Lyapunov function (2), in this paper we propose

$$\begin{aligned} V_C(z, \dot{q}) &= V_L(z, \dot{q}) + 2 \sin(z) \dot{q}, \\ &= \frac{1}{2} \dot{q}^2 + [1 - \cos(z)] + 2 \sin(z) \dot{q}. \end{aligned} \quad (4)$$

It is worth noticing that above function and the strict Lyapunov function (2) have the same structure and differs only in a scalar of the third right hand term $\sin(z) \dot{q}$.

Certainly $V_C(z, \dot{q})$ is not anymore a locally positive definite function, but it satisfies

- $V_C(0, 0) = 0$
- $V_C(z, \dot{q}) > 0$ for $[z \ \dot{q}]^T = [0 \ \dot{q}]^T$ with \dot{q} as small as desired, and for $[z \ \dot{q}]^T = [z \ 0]^T$ with z arbitrarily small.

According to Definition 3, this means that $V_C(z, \dot{q})$ is a good candidate to be a Chetaev function. It remains to prove that its time derivative is locally positive definite.

After some tedious by straightforward manipulations, the time derivative of $V_C(z, \dot{q})$ along the trajectories of system (3) leads to

$$\dot{V}_C(z, \dot{q}) = [2 \cos(z) - 1] \dot{q}^2 + 2 \sin(z) \dot{q}. \quad (5)$$

Obviously $\dot{V}_C(0, 0) = 0$ and $\dot{V}_C(z, \dot{q}) > 0$ for z and \dot{q} small enough. This implies that $\dot{V}_C(z, \dot{q})$ in (4) is a locally positive definite function, hence in virtue of Definition 3 we have that $V_C(z, \dot{q})$ qualifies as a Chetaev function. This can be seen in Figure 3 where the domain in the state space where both $V_C(z, \dot{q})$ and $\dot{V}_C(z, \dot{q})$ are positive is shown. Observe that inside the circle the time derivative $\dot{V}_C(z, \dot{q})$ is positive. Finally, the instability attribute of the equilibrium follows from Theorem 1.

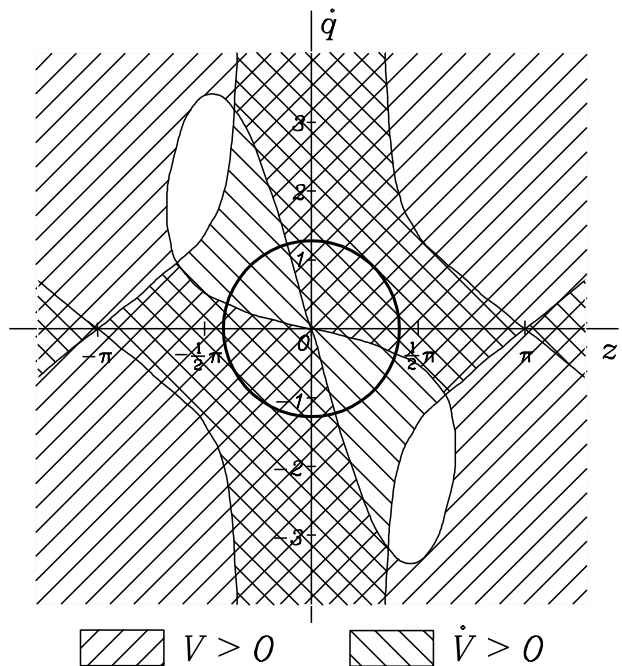


Fig. 3. Chetaev function

5. CONCLUSIONS

Although stability of physical systems can be studied by utilizing the Lyapunov's stability theory resorting to energy like functions, some times it is not sufficient to establish asymptotic stability. Then, typically the Krasovskii–LaSalle's theorem is invoked to support the Lyapunov's stability theorem. On the other hand, few studies concerning energy like function for study instability are available.

By means of a simple mechanical system, this paper have shown the analysis of asymptotic stability by proposing a strict Lyapunov function, thus obviating the Krasovskii–LaSalle's theorem. On the other hand, motivated by this strict Lyapunov function, we have proposed a Chetaev function having the same structure to show instability. Indeed, this has been shown on a simple process, notwithstanding, we believe that the ideas can be extended to more general systems, in particular for electromechanical systems.

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