GAIN REDUCTION IN SWITCHED SLIDING MODE CONTROL

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Abstract: A switched sliding mode control strategy for a class of nonlinear uncertain systems is presented in this paper. It is characterized by an event–driven gain reduction mechanism which relies on a decomposition of the system state into regions. By enforcing sliding mode behaviors on a suitable set of sliding manifolds, while avoiding the generation of limit cycles, the proposed strategy proves to globally asymptotically stabilize the origin of the system state space.

Keywords: Nonlinear systems, uncertain systems, sliding–mode control, switching algorithms, variable–structure systems.

1. INTRODUCTION

A huge number of scientific publications have been devoted to hybrid systems throughout the last decade: see, for instance, (A. S. Morse, 1997; Morse *et al.*, 1999; van der Schaft and Schumacher, 1999; Savkin and Evans, 2002), and the references therein cited. A remarkable class of hybrid systems, from the point of view of control engineering applications, is that of switched systems, i.e., dynamical systems capable of assuming different continuous–time mathematical models depending on a pre–specified switching rule. Also Sliding Mode Control (SMC) systems (Utkin, 1992; Hung *et al.*, 1993; DeCarlo *et al.*, 1998; Edwards and Spurgeon, 1998) belong to this class: they are, intrinsically, “switched systems” in the sense that the control design relies on a state space decomposition through a border, the so–called sliding manifold, which is a linear or nonlinear function of the full system state, so that the control law is switched on crossing it. Yet, in contrast to what happens in conventional switched systems, in SMC systems the state trajectory is not forced to instantaneously cross the commutation manifold, but to slide on it. Indeed, in this way, the desired dynamical features are assigned to the controlled system.

The aim of the present paper is to design and analyze a truly switched SMC strategy for a class of nonlinear uncertain systems which relies on a peculiar system state decomposition into countable partitions by means of a grid of conventional sliding manifolds, and a set of nested switching boundaries. With each partition of the state space a control law is univocally associated. The choice of the control laws is aimed at the attainment of one of the following two objectives: to reach a particular sliding manifold, or to cross the switching boundary closer to the origin.

The switched SMC system presented in this paper is an example of controller for which the switched version has a clear advantage over the conventional (non–switched) realization: it provides a solution to the problem of gain reduction in SMC systems. Indeed, the choice of the control amplitude in SMC is usually made taking into account the known bounds on the system uncertain dynamics, as well as the limits of the workspace to determine an upper bound on the variation of the state norm, which makes this choice quite conservative. A way to overcome this drawback is to include in the SMC design an adaptive tuning of the control gain (Bartolini *et al.*, 1998; Gessing, 2001). In the present work, an alternative approach is presented: as long as the state trajectory crosses a switching boundary, a gain variation is generated. More precisely, a state evolution approaching the state space origin tends to determine, on crossing switching boundaries closer to the origin, a reduction of the gain of the component of the control law which is discontinuous (as required to enforce sliding modes). In contrast to (Bartolini *et al.*, 1998) and (Gessing, 2001), where a continuous variation of the gain is generated, in the present proposal the gain reduction mechanism is event–driven and asynchronous in time. Most importantly, it does not require that the state is evolving along a sliding manifold, being active even during the reaching phases. As a result, the proposed switched SMC strategy with gain reduction proves to globally asymptotically stabilize the origin of the system state space, in spite of the presence of a bounded uncertain term in the system model.

The fact of dealing with dynamical systems with uncertainty is the motivation for using SMC to design a switched strategy. In fact, among the appreciable features of the SMC methodology, there is its robustness versus matched uncertainties and disturbances, which is naturally inherited by the proposed control approach.

Note that, the combination of SMC with hybrid control has already been investigated in (Bartolini *et al.*, 1999*a*) and (Bartolini *et al.*, 1999*b*). Yet, in such papers, the switching mechanism is driven by a logic based on the decomposition of the sliding variable phase plane, rather than of the original system state space, which can appear less intuitive as far as the control laws design is concerned. Rather, the present paper can be viewed as a development of (Ferrara *et al.*, 2002) in which a switched SMC strategy is designed for second order nonlinear systems through the decomposition of the system state space into a couple of regions, and no gain reduction mechanism is implemented.

The present paper is organized as follows. The next section is devoted to problem formulation and to the formal description of the proposed control strategy. The analysis of its stability and convergence properties is carried out in Section 3. Finally, in Section 4, a couple of examples are briefly discussed.

2. PROBLEM FORMULATION AND DESIGN OF THE SWITCHED SLIDING MODE CONTROLLER

Consider the nonlinear continuous–time dynamical system in controllable canonical form

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ &\vdots \\ \dot{x}_{n-1}(t) &= x_n(t) \\ \dot{x}_n(t) &= f(x(t)) + gu(t) \end{cases} \tag{1}$$

with $x(0) = \overline{x} \in \Re^n$, where $x = [x_1 \ \dots \ x_n]' \in \Re^n$ is the state vector, $f(x) = f_n(x(t)) + \bar{f}(x(t)) \in \Re$, with $f_n(x(t))$ known nominal part, and $\bar{f}(x(t))$ uncertain but such that

$$\left|\bar{f}(x(t))\right| < k, \quad \forall\, t \geq 0 \tag{2}$$

k being a positive constant, $u \in \Re^1$ is a scalar control variable which influences the state vector linearly through the known positive constant g.

Assume that the state space of system (1) is partitioned into different regions $\Omega^i(x)$, $i = 1, \dots, \nu$, bounded by nested switching boundaries $\bar{\varphi}_i$, $i = 1, \dots, \nu$, defined by $\varphi_i(x) = 0$, where

$$\varphi_i(x) = x' P_i x - c_i \tag{3}$$

with $P_i = P_i' = diag\{p_{i_1}, \ \dots, \ p_{i_n}\} > 0$. Note that also the origin of the state space can be interpreted as a switching boundary, i.e., $\bar{\varphi}_\nu$. More specifically,

$$\begin{aligned} \Omega^1(x) &= \{x : \varphi_1(x) > 0\} \\ \Omega^{i+1}(x) &= \{x : \varphi_{i+1}(x) > 0 \ \cap \ \varphi_i(x) < 0\} \end{aligned} \tag{4}$$

$i = 1, \dots, \nu - 1$.

With reference to the regions Ω^i, $i = 1, \dots, \nu$, introduce the linear functions

$$\sigma_i(x) = x_n + \sum_{j=1}^{n-1} \alpha_{ij} x_j \tag{5}$$

with $\alpha_{ij} = \frac{1}{\mu_i} \alpha_{(i-1)j}$, $i = 2, \dots, \nu$, $j = 1, \dots, n-1$, and the design parameter $\mu_i > 1$, as well as the corresponding sliding manifolds $\sigma_i(x) = 0$, $i = 1, \dots, \nu$. As usual in SMC control (Utkin, 1992), the switching functions $\sigma_i(x)$, $i = 1, \dots, \nu$, are selected so that when the state of system (1) is restricted to lay on the sliding manifolds, the system dynamics exhibits the desired behavior. Then, according to the SMC theory, define the switched control law

$$u(t) = u_n(t) + u_d(t) = -\frac{f_n(x(t))}{g} - K_i sign(\sigma_i(x(t))) \quad (6)$$

when $x \in \Omega^i(x)$, $i = 1, \ldots, \nu$, where K_i is a positive design parameter to be selected.

Then, the control problem is to design a switched control strategy to make the origin of the state space be a globally asymptotically stable equilibrium point of the controlled system. Moreover, the following constraint on the amplitude of the discontinuous component of the control law is considered

$$|u_d|_i > |u_d|_{i+1}, \quad i = 1, \ldots, \nu - 1 \quad (7)$$

$|u_d|_j$ being the amplitude of u_d in the region $\Omega^j(x)$. Clearly, constraint (7) prescribes a reduction of the gain of the discontinuous control action as the state trajectory moves through regions closer to the origin of the state space, driven by the event of crossing a switching boundary.

As it is well known, see e.g. (Branicky, 1988), a hybrid strategy where the controller switches between different control laws can result in an overall unstable closed–loop system even if each control law is designed so as to guarantee stability. So, the proposed switched SMC strategy does not guarantee by itself the global asymptotic stability of the origin of the controlled system state space, but some further conditions on the gains K_i must be imposed. To this end, define the functions

$$a = \begin{cases} 1 & if\ x_n > 0 \\ -1 & if\ x_n < 0 \end{cases}$$
$$b = \begin{cases} 1 & if\ sign(\sigma_i(x)) > 0 \\ -1 & if\ sign(\sigma_i(x)) < 0 \end{cases}$$
$$c = \begin{cases} 1 & if\ sign(\sigma_{i+1}(x)) > 0 \\ -1 & if\ sign(\sigma_{i+1}(x)) < 0 \end{cases}$$

Then, for any switching boundary $\bar{\varphi}_i$, $i = 1, ..., \nu - 1$, define

$$\Delta_{abc}^{i,i+1} = \left\{ x \in \Omega^i \,|\, |x_n| > \delta_1 \cap \|x - \bar{\varphi}_i\| < \delta_2 \right\} \quad (8)$$
$$\Delta_{abc}^{i+1,i} = \left\{ x \in \Omega^{i+1} \,|\, |x_n| > \delta_1 \cap \|x - \bar{\varphi}_i\| < \delta_2 \right\} \quad (9)$$

and let

$$\Delta^{i,i+1} = \cup \Delta_{abc}^{i,i+1} \quad (10)$$
$$\Delta^{i+1,i} = \cup \Delta_{abc}^{i+1,i} \quad (11)$$

With the ν regions Ω^i, the following positive values can be associated

$$\tilde{K}_i > \max \left\{ \frac{|x' P_{i-1} \mathcal{F}(x)|}{|x' P_{i-1} \mathcal{G}|} \ \forall x \in \Delta^{i,i-1}, \frac{|x' P_i \mathcal{F}(x)|}{|x' P_i \mathcal{G}|} \ \forall x \in \Delta^{i,i+1} \right\} \quad (12)$$

for $i = 1, \ldots, \nu$, with

$$\mathcal{F}(x) = [x_2 \ldots x_{n-1}\, f(x)]' \in \Re^n \quad (13)$$
$$\mathcal{G} = [0 \ldots 0\, g]' \in \Re^n \quad (14)$$

where the terms inside the maximum have to be regarded as zero for $x \in \Delta^{1,0}$ and $x \in \Delta^{\nu,\nu+1}$.

Thus, since a key design requirement, in the present case, is determined by constraint (7), assume that the control gains K_i, $i = 1, \ldots, \nu$, of the switched part of the control law (6) are chosen as follows

$$K_i \geq \max \left\{ \bar{K}_i, \tilde{K}_i \right\}, \quad i = 1, \ldots, \nu \quad (15)$$

with

$$K_{i-1} > K_i, \quad i = 2, \ldots, \nu \quad (16)$$

where $\bar{K}_i$ is a lower bound of the interval of values of the gain K_i for which the reaching condition $\sigma_i(x)\dot{\sigma}_i(x) < -\gamma|\sigma_i(x)|$, with γ positive constant, (Utkin, 1992), is fulfilled in $\Omega^i(x)$, $i = 1, \ldots, \nu$.

3. STABILITY AND CONVERGENCE

The stability of the origin of the closed-loop system (1), (6), (15) is now investigated by analyzing the behavior of the state trajectories in the vicinities $\Delta^{i,j}$ of the switching boundaries. For the sake of clarity, the analysis is carried out in the particular case of $\nu = 2$. Note, however, that the following results can be easily extended to the case $\nu > 2$.

A reaching condition is analyzed with reference to $\bar{\varphi} = \bar{\varphi}_1$ (setting $\varphi(x) = \varphi_1(x)$, for the sake of simplicity) in order to establish which parts of it exerts an attractive or repulsive action on the controlled state trajectories. Since $\nu = 2$, partitions $\Delta_{abc}^{1,0}$ and $\Delta_{abc}^{3,2}$ must not be considered, and the only partitions of interest are $\Delta_{abc}^{1,2}$ and $\Delta_{abc}^{2,1}$. For simplicity, let $\Delta_{abc}^{1} = \Delta_{abc}^{1,2}$ and $\Delta_{abc}^{2} = \Delta_{abc}^{2,1}$. In view of definition (3), system (1), and the switched control law (6), in Δ_{abc}^{i}, $i = 1, 2$, it results that

$$\dot{\varphi}(x) = 2x' P \mathcal{F}(x) - 2x' P \mathcal{G} K_i sign(\sigma_i(x)) \quad (17)$$

$P := P_1$. Moreover, in Δ_{abc}^1, one has $\varphi(x) > 0$ and, in view of (12), (15),

$$sign(x' P \mathcal{G} K_1 sign(\sigma_1(x))) = sign(ab) \quad (18)$$

and

$$sign(\varphi(x)\dot{\varphi}(x)) = -sign(ab) \qquad (19)$$

In contrast, in Δ^2_{abc}, one has $\varphi(x) < 0$,

$$sign(x'P\mathcal{G}K_2 sign(\sigma_2(x))) = sign(ac) \qquad (20)$$

and

$$sign(\varphi(x)\dot{\varphi}(x)) = sign(ac) \qquad (21)$$

Three different cases can occur:

case 1 : when $ab = -1$ and $ac = -1$, $\varphi(x)\dot{\varphi}(x) > 0$ in Δ^1_{abc}, while $\varphi(x)\dot{\varphi}(x) < 0$ in Δ^2_{abc}, so that the state trajectories move from Δ^2_{abc} to Δ^1_{abc};

case 2 : when $ab = 1$ and $ac = 1$, $\varphi(x)\dot{\varphi}(x) < 0$ in Δ^1_{abc}, while $\varphi(x)\dot{\varphi}(x) > 0$ in Δ^2_{abc}, so that the state trajectories move from Δ^1_{abc} to Δ^2_{abc};

case 3 : when $ab = -1$ and $ac = 1$, $\varphi(x)\dot{\varphi}(x) > 0$ both in Δ^1_{abc} and in Δ^2_{abc}. Hence, the state trajectories cannot go through the switching boundary $\bar{\varphi}$, which is "repulsive" both in Δ^1_{abc} and in Δ^2_{abc}.

The following results can be proved in a row.

Proposition 1. The trajectories of the switched closed–loop system do not present any limit cycle.

Proof: For the sake of simplicity, the first part of the proof refers to the case $n = 2$, $\nu = 2$, shown in Fig. 1. Limit cycles entirely included inside Ω^1 or Ω^2 cannot exist in view of the globally stabilizing property of the SMC law (6) guaranteed by the choice of the gains (12), (15). Hence, a limit cycle, if any, should belong to both Ω^1 and Ω^2. Specifically, and with reference to Fig. 1, only limit cycles of type 1 or type 2 are allowed by the attractive or repulsive properties of Δ^1_{abc} and Δ^2_{abc}. Any other possibility is forbidden, since it would imply that the closed trajectory intersects the sliding manifold $\sigma_1(x) = 0$ in Ω^1. However, in this case, the trajectory would follow such a sliding manifold until it reaches the origin. As for limit cycles of type 1, they should cross in Ω^1 a level line of the Lyapunov function $\frac{1}{2}\sigma_1^2$ in a forbidden direction, which is not allowed. The same kind of arguments can be used to show the infeasibility of cycles of type 2. When $n > 2$ and/or $\nu > 2$, the previous arguments can be used to draw the the same conclusions. △

Proposition 2. Any trajectory moving from Ω^2 to Ω^1, reaches in Ω^1 the sliding manifold $\sigma_1(x) = 0$.

Proof: This is a direct consequence of the fact that every trajectory starting in Ω^2 and passing in Ω^1 through $\Delta^2_{-1,1,1}$ should cross in Ω^1 a level line of the Lyapunov function $\frac{1}{2}\sigma_1^2$ in the forbidden direction. △

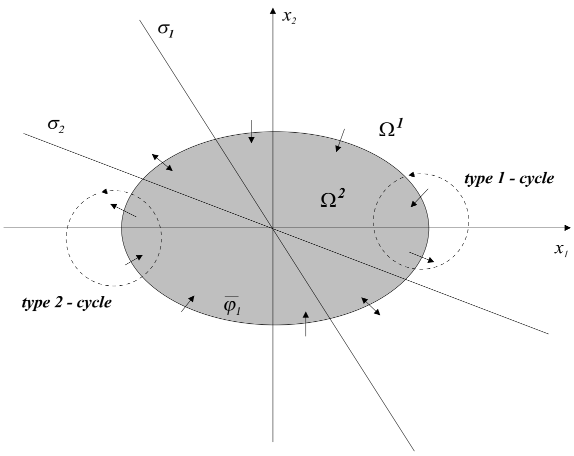

Fig. 1. The state space partition for $x \in \Re^2$ with the admissible crossing directions.

Proposition 3. The origin of the state space is a globally asymptotically stable equilibrium point for system (1) controlled by the switched SMC strategy (6), (15).

Proof: First assume that $x(0) \in \Omega^2$. Then, two cases are possible.

A1 The trajectory starting from $x(0)$ reaches the sliding manifold $\sigma_2(x) = 0$ and goes to the origin with the dynamics imposed by the choice of α_{2j}, $j = 1, \ldots, n-1$.

A2 The trajectory leaves Ω^2 and enters Ω^1. In view of Proposition 2, it reaches the sliding manifold $\sigma_1(x) = 0$; then, the case B1 below holds.

When $x(0) \in \Omega^1$, one of the following two cases holds.

B1 The trajectory starting from $x(0)$ reaches the sliding manifold $\sigma_1(x) = 0$; then, the trajectory enters in Ω^2 and reaches the sliding manifold $\sigma_2(x) = 0$ (case A1). Note that in this last case, it cannot exit Ω^2 without passing through $\sigma_2(x) = 0$.

B2 The trajectory enters Ω^2 and one of the cases A1–A2 applies.

We finally have to prove that the overall state trajectory goes to the origin asymptotically. To this end, note that by virtue of the choice of the control law (6) with the control gains in (15), and of the previous considerations, as well as of the results in Propositions 1 and 2, the sliding manifold $\sigma_1(x) = 0$ is reached in finite time. Then, classical results of the theory of SMC guarantee that the controlled system state is steered to the origin asymptotically (Utkin, 1992). △

4. SIMULATION EXAMPLES

As a first example, the proposed switched SMC strategy with event–driven reduction of the control gain is applied to the following system

$$\begin{cases} \dot{x}_1(t) & = x_2(t) \\ \dot{x}_2(t) & = x_3(t) \\ \dot{x}_3(t) & = \bar{f}(x(t)) + u(t) \\ \bar{f}(x(t)) & = -10 sin(x_2(t)) \end{cases} \quad (22)$$

i.e., the nominal part of $f(x(t))$ is equal to zero, and the uncertain part is bounded. Two regions Ω^i, $i = 1, 2$, are delimited by the switching boundary

$$\bar{\varphi}_1 \ : \ x_1^2 + x_2^2 + x_3^2 - 4 = 0 \quad (23)$$

Associated with the Ω^i's, the following sliding manifolds

$$\begin{aligned} \sigma_1(x) &= x_3 + 4x_2 + 6x_1 = 0 \\ \sigma_2(x) &= x_3 + 0.4x_2 + 0.6x_1 = 0 \end{aligned} \quad (24)$$

are selected. As a first choice, the corresponding control gains are set to the following values: $K_1 = 200$, $K_2 = 15$. The state trajectory of the controlled system, starting from $x(0) = [2\ 2\ 2]'$, and moving to the origin of the state space through a sequence of "reaching" and "sliding" phases, is shown in Fig. 2, while the switched control signal component u_d with gain reduction is illustrated in Fig. 3. Note that, with reference to system (22), if one sets the gain value equal to 15 $\forall\ t \geq 0$, then the state trajectory of the controlled system, starting from the same initial condition, does not reach any equilibrium point, as shown in Fig. 4. This is a sign of the fact that the gain selected in the inner region is very low, taking into account the system dynamics and the initial condition, and that the origin can be made an asymptotically stable equilibrium point of the controlled system only by virtue of the switched nature of the controller, which enables the control gain to have an initial higher value.

The class of systems considered in this paper is rather general, in the sense that it obviously includes all the nonlinear systems which can be transformed into the canonical form (1) through a suitable diffeomorphism. An example is given by the system

$$\begin{cases} \dot{x}_1(t) & = -x_2(t) - x_3(t) \\ \dot{x}_2(t) & = x_1(t) \\ \dot{x}_3(t) & = \bar{f}(x(t)) - u(t) \\ \bar{f}(x(t)) & = -cos(x_2(t)) \end{cases} \quad (25)$$

Relying on the global diffeomorphism $z = [z_1\ z_2\ z_3]' := [x_2\ \dot{x}_2\ \ddot{x}_2]' = [x_2\ x_1\ (-x_2 - x_3)]'$, system (25) can be transformed into the canonical form

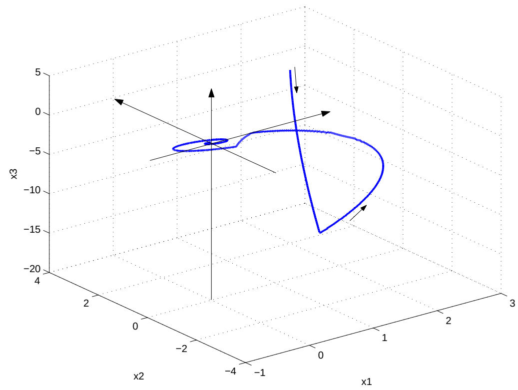

Fig. 2. The state trajectory of the controlled system when the control gains are $K_1 = 200$, and $K_2 = 15$.

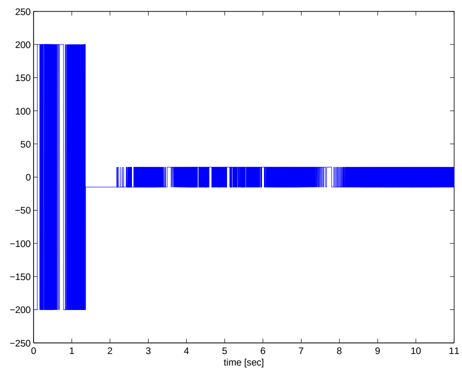

Fig. 3. The switched control with gain reduction from $K_1 = 200$ to $K_2 = 15$.

$$\begin{cases} \dot{z}_1(t) &= z_2(t) \\ \dot{z}_2(t) &= z_3(t) \\ \dot{z}_3(t) &= -z_2(t) + cos(z_1(t)) + u(t) \end{cases} \quad (26)$$

so that one has a nominal part $f_{n_z}(z(t)) = -z_2(t)$ (and, consequently, $u_n(t) = z_2(t)$), and a bounded uncertain part $\bar{f}_z(z(t)) = cos(z_1(t))$. Also in this example, two regions Ω^i, $i = 1, 2$, are considered. They are delimited by the switching boundary

$$\bar{\varphi}_1 \ : \ x_1^2 + 2x_2^2 + x_3^2 - 4 = 0 \quad (27)$$

while the corresponding sliding manifolds are

$$\begin{aligned} \sigma_1(x) &= x_3 + 5x_2 + 2x_1 = 0 \\ \sigma_2(x) &= x_3 + 0.5x_2 + 0.2x_1 = 0 \end{aligned} \quad (28)$$

The control gains in the two regions are set equal to: $K_1 = 200$, $K_2 = 20$. The state trajectory of the controlled system, starting from $x(0) = [3\ 3\ 3]'$, and moving to the origin is shown in Fig. 5, while the switched control signal component u_d with gain reduction is illustrated in Fig. 6.

5. CONCLUSIONS

A switched SMC strategy has been presented in the paper. Sliding mode behaviors are suitably

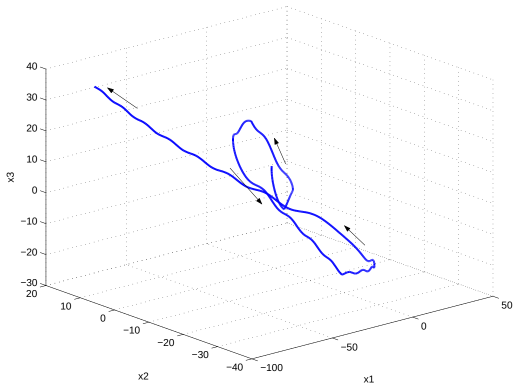

Fig. 4. The state trajectory of the controlled system when the control gains are both equal to 15.

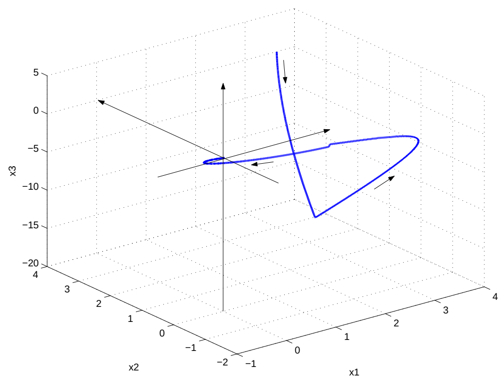

Fig. 5. The state trajectory of the controlled system (25).

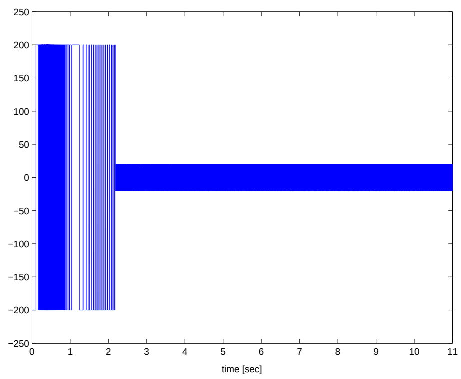

Fig. 6. The switched control with gain reduction in the second example.

generated so that they have finite duration when they occur on sliding manifolds which are separated from the origin by switching boundaries. Alternatively, they asymptotically steer the controlled system state trajectory to the origin of the state space. As a result, the proposed switched SMC strategy proves to globally asymptotically stabilize the origin of the system state space. The major advantage of the proposed controller is that its positive stability features are attained in spite of a reduction of the gain of the discontinuous component of the control law. The reduction mechanism is not of adaptive type, but is driven by the event of crossing a switching boundary, as the state trajectory moves through regions closer to the origin of the state space.

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