NECESSARY AND SUFFICIENT CONDITIONS FOR PARAMETER INSENSITIVE DISTURBANCE-REJECTION PROBLEMS WITH STATIC FEEDBACK[1]

Naohisa Otsuka *

* *Department of Information Sciences, Tokyo Denki University, Saitama 350-0394, Japan*

Abstract: In this paper, necessary and sufficient conditions for the parameter insensitive disturbance-rejection problem with state feedback for uncertain linear systems to be solvable which was pointed out as an open problem by Bhattacharyya are proved by using relationship between generalized invariant subspaces and simultaneously invariant subspaces. Further, the problem with static output feedback is also investigated.

Keywords: Geometric approach, Invariant subspaces, Uncertain linear systems, Disturbance rejection, State feedback, Static output feedback

1. INTRODUCTION

Since the notions of (A, B)-invariant and (C, A)-invariant subspaces were studied, many control problems for linear multivariable systems have been studied in the framework of the so-called geometric approach (e.g., (Basile, 1991), (Wonham, 1984), (Schumacher, 1980)).

Further, the notion of generalized (A, B)-invariant subspace which is an extension of (A, B)-invariant subspace was studied for uncertain linear systems from the practical viewpoint (Bhattacharyya, 1983). Among this, parameter insensitive disturbance-rejection problem with state feedback for uncertain linear systems whose matrices depend linearly on uncertain parameters was formulated and sufficient conditions for the problem to be solvable were given but it's necessary conditions were pointed out as an open problem. After that, generalized (C, A)-invariant subspace which is the dual concept of generalized (A, B)-invariant subspace and the generalized (A, B, C)-invariant subspace were studied, and then parameter-insensitive disturbance rejection problems with static output feedback and with dynamic compensator were studied ((Otsuka, 1999),(Otsuka, 2000)).

On the other hand, simultaneously (A, B)-invariant and simultaneously (C, A)-invariant subspaces were studied for a family of linear systems, and then parameter insensitive disturbance-rejection problems for systems whose uncertain matrices are expressed as convex combination of two given matrices were studied (Ghosh, 1986).

In this paper, necessary and sufficient conditions for parameter insensitive disturbance-rejection problem with state feedback to be solvable which was pointed out as an open problem by Bhattacharyya are proved. And then, parameter insensitive disturbance-rejection problem with static output feedback is also investigated. In Section 2, simultaneously invariant subspaces for a family of linear systems and a key lemma to prove the main result are discussed. In Section 3, relationship between generalized invariant subspaces for uncertain linear systems and simultaneously invariant subspaces for a family of linear systems are investigated. In Section 4 the problem formu-

[1] Partially supported by the Grant-in-Aid for the 21st Century COE(Center of Excellence) Research by the Ministry of Education, Culture, Sports, Science and Technology.

lations are given and the main results are proved. Finally, conclusions are given in Section 5.

2. PRELIMINARIES

At first, some notations used throughout this investigation are given. For a linear map A from a vector space $\mathcal{X}$ into a vector space $\mathcal{Y}$ and a subspace φ of $\mathcal{Y}$ the image, the kernel, the dimension and the inverse image are denoted by $\mathrm{Im}(A)$, $\mathrm{Ker}(A)$, $\dim(\varphi)$ and $A^{-1}\varphi := \{x \in \mathcal{X} \mid Ax \in \varphi\}$, respectively.

Next, consider a family of linear systems defined in $\mathcal{X} := \boldsymbol{R}^n$:

$$S_i : \begin{cases} \dfrac{d}{dt}x(t) = A_i x(t) + B_i u(t), \quad (i = 1, \cdots, \tau) \\ y(t) = C_i x(t), \quad (i = 1, \cdots, \tau) \end{cases}$$

where $x(t) \in \mathcal{X}$ is the state, $u(t) \in \mathcal{U} := \boldsymbol{R}^m$ is the input and $y(t) \in \mathcal{Y} := \boldsymbol{R}^\ell$ is the output. And coefficient matrices $A_i \in \boldsymbol{R}^{n\times n}$, $B_i \in \boldsymbol{R}^{n\times m}$ and $C_i \in \boldsymbol{R}^{\ell\times n}$.

Definition 1. Let $\mathcal{V}$, Ω and ε be subspaces of $\mathcal{X}$.

(i) $\mathcal{V}$ is said to be *simultaneously* $\{(A_i, B_i); 1 \le i \le \tau\}$*-invariant* if

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_\tau \end{bmatrix} V \subset \begin{bmatrix} \mathcal{V} \\ \mathcal{V} \\ \vdots \\ \mathcal{V} \end{bmatrix} + \mathrm{Im} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_\tau \end{bmatrix}.$$

$\boldsymbol{\mathcal{V}_s}((A_i, B_i); 1 \le i \le \tau \mid \Omega)$
$:= \{\mathcal{V} \mid \mathcal{V}$ is simultaneously $\{(A_i, B_i); 1 \le i \le \tau\}$-invariant and $\mathcal{V} \subset \Omega\}$.

(ii) $\mathcal{V}$ is said to be *simultaneously* $\{(C_i, A_i); 1 \le i \le \tau\}$*-invariant* if

$$\begin{bmatrix} A_1 \; A_2 \cdots A_\tau \end{bmatrix} \left(\begin{bmatrix} \mathcal{V} \\ \mathcal{V} \\ \vdots \\ \mathcal{V} \end{bmatrix} \cap \mathrm{Ker} \begin{bmatrix} C_1 \; C_2 \cdots C_\tau \end{bmatrix} \right) \subset \mathcal{V}$$

$\boldsymbol{\mathcal{I}_s}(\varepsilon \mid (C_i, A_i); 1 \le i \le \tau)$
$:= \{\mathcal{V} \mid \mathcal{V}$ is simultaneously $\{(C_i, A_i); 1 \le i \le \tau\}$-invariant and $\varepsilon \subset \mathcal{V}.\}$

(iii) $\mathcal{V}$ is said to be *simultaneously* $\{(A_i, B_i, C_i); 1 \le i \le \tau\}$*-invariant* if there exists an $H \in \boldsymbol{R}^{m\times\ell}$ such that

$$(A_i + B_i H C_i)\mathcal{V} \subset \mathcal{V}$$

for all $i = 1, \cdots, \tau$. ■

The following two Lemmas follow from Wonham(1984) and Ghosh(1986).

Lemma 2.

(i) $\mathcal{V}$ is simultaneously $\{(A_i, B_i); 1 \le i \le \tau\}$-invariant if and only if there exists an $F \in \boldsymbol{R}^{m\times n}$ such that

$$(A_i + B_i F)\mathcal{V} \subset \mathcal{V}$$

for all $i = 1, \cdots, \tau$.

(ii) The class $\boldsymbol{\mathcal{V}_s}((A_i, B_i); 1 \le i \le \tau \mid \Omega)$ has the unique maximal element $\mathcal{V}_s^{max}(\Omega)$ and it can be computed as the following sequence.

$\mathcal{V}^{(0)} := \Omega$,

$$\mathcal{V}^{(k)} := \Omega \cap \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_\tau \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathcal{V}^{(k-1)} \\ \mathcal{V}^{(k-1)} \\ \vdots \\ \mathcal{V}^{(k-1)} \end{bmatrix} + \mathrm{Im} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_\tau \end{bmatrix} \right),$$

$$(k = 1, 2, \cdots)$$

$\mathcal{V}_s^{max}(\Omega) := \mathcal{V}^{(\dim(\Omega))}$. ■

Lemma 3.

(i) $\mathcal{V}$ is simultaneously $\{(C_i, A_i); 1 \le i \le \tau\}$-invariant if and only if there exists a $G \in \boldsymbol{R}^{n\times\ell}$ such that

$$(A_i + GC_i)\mathcal{V} \subset \mathcal{V}$$

for all $i = 1, \cdots, \tau$.

(ii) The class $\boldsymbol{\mathcal{I}_s}(\varepsilon \mid (C_i, A_i); 1 \le i \le \tau)$ has the unique minimal element $\mathcal{V}_s^{min}(\varepsilon)$ and it can be computed as the following sequence.

$\mathcal{V}^{(0)} := \varepsilon$,

$\mathcal{V}^{(k)} := \mathcal{V}^{(k-1)} +$

$$\begin{bmatrix} A_1 \; A_2 \cdots A_\tau \end{bmatrix} \left(\begin{bmatrix} \mathcal{V}^{(k-1)} \\ \mathcal{V}^{(k-1)} \\ \vdots \\ \mathcal{V}^{(k-1)} \end{bmatrix} \cap \mathrm{Ker} \begin{bmatrix} C_1 \; C_2 \cdots C_\tau \end{bmatrix} \right),$$

$$(k = 1, 2, \cdots)$$

$\mathcal{V}_s^{min}(\varepsilon) := \mathcal{V}^{(n-\dim(\varepsilon))}$. ■

Now, the following is a key lemma to prove the main results.

Lemma 4. Consider a family of systems S_i $(i = 1, \cdots, \tau)$. And for a given subspace ε of $\mathcal{X}$ and a state feedback $F \in \boldsymbol{R}^{m\times n}$, define the following sequences of subspace as

Step1. $\mathcal{V}^{(0)} := \varepsilon$.

Step2. $\mathcal{V}_1^{(k)} := \mathcal{V}^{(k-1)} + (A_1 + B_1 F)\mathcal{V}^{(k-1)}$,

$\mathcal{V}_2^{(k)} := \mathcal{V}^{(k-1)} + (A_2 + B_2 F)\mathcal{V}^{(k-1)}$,

$\vdots$

$\mathcal{V}_\tau^{(k)} := \mathcal{V}^{(k-1)} + (A_\tau + B_\tau F)\mathcal{V}^{(k-1)}$,

$$(k = 1, 2, \cdots)$$

Step3. $\mathcal{V}^{(k)} := \mathcal{V}_1^{(k)} + \mathcal{V}_2^{(k)} + \cdots + \mathcal{V}_\tau^{(k)}$
$$(k = 1, 2, \cdots).$$

Then, $\mathcal{V}^{(k-1)} \subset \mathcal{V}^{(k)}$ $(k = 1, 2, \cdots)$ and there exists a μ ($\leq$ $n - \dim(\varepsilon)$) such that

$$\mathcal{V}^{(\mu)} = \mathcal{V}^{(k)} \quad \text{for} \quad \forall k \geq \mu$$

which satisfies $\mathcal{V}^{(\mu)}$ is simultaneously $\{(A_i, B_i); 1 \leq i \leq \tau\}$-invariant and contains ε, that is, $\mathcal{V}^{(\mu)} \in \boldsymbol{\mathcal{V}_s}((A_i, B_i); 1 \leq i \leq \tau \mid \mathcal{X})$ and $\varepsilon \subset \mathcal{V}^{(\mu)}$.

Proof. It follows from Steps 1-3 that

$$\begin{aligned}
\mathcal{V}^{(k)} &= \mathcal{V}_1^{(k)} + \mathcal{V}_2^{(k)} + \cdots + \mathcal{V}_r^{(k)} \\
&= \mathcal{V}^{(k-1)} + (A_1 + B_1 F)\mathcal{V}^{(k-1)} + \cdots \\
&\qquad + (A_\tau + B_\tau F)\mathcal{V}^{(k-1)} \\
&\supset \mathcal{V}^{(k-1)} \\
&\supset \varepsilon \quad (k = 1, 2, \cdots). \qquad (1)
\end{aligned}$$

Since $\mathcal{X}$ is finite-dimensional, there exists a μ ($\leq$ $n - \dim(\varepsilon)$) such that

$$\mathcal{V}^{(\mu)} = \mathcal{V}^{(k)} \quad \text{for} \quad \forall k \geq \mu.$$

Then, for all $i = 1, \cdots, \tau$

$$\begin{aligned}
&(A_i + B_i F)\mathcal{V}^{(\mu)} \\
&\quad \subset (A_1 + B_1 F)\mathcal{V}^{(\mu)} + \cdots + (A_\tau + B_\tau F)\mathcal{V}^{(\mu)} \\
&\quad \subset \mathcal{V}_1^{(\mu+1)} + \cdots + \mathcal{V}_\tau^{(\mu+1)} \\
&\quad = \mathcal{V}^{(\mu+1)} \\
&\quad = \mathcal{V}^{(\mu)}. \qquad (2)
\end{aligned}$$

It follows from Lemma 2(i), (1) and (2) that $\mathcal{V}^{(\mu)}$ is simultaneously $\{(A_i, B_i); 1 \leq i \leq \tau\}$-invariant and contains ε. ■

Lemma 5. Consider sequence of subspaces $\mathcal{V}^{(k)}$ $(k = 0, 1, \cdots)$ in Lemma 4 for HC_i replaced F in $\mathcal{V}_i^{(k)}$ $(i = 1, \cdots, \tau)$. Then, there exists a μ ($\leq$ $n -$ $\dim(\varepsilon)$) such that

$$\mathcal{V}^{(\mu)} = \mathcal{V}^{(k)} \quad \text{for} \quad \forall k \geq \mu$$

which satisfies $\mathcal{V}^{(\mu)}$ is simultaneously $\{(A_i, B_i, C_i);$ $1 \leq i \leq \tau\}$-invariant and contains ε.

Proof. The proof follows from the same manner as the proof of Lemma 4. ■

3. GENERALIZED INVARIANT SUBSPACES

In this section some important properties of generalized invariant subspaces which are used to prove the main results are investigated.

Consider the following linear systems defined in $\mathcal{X} := \boldsymbol{R}^n$:

$$S(\alpha, \beta, \gamma) : \begin{cases} \dfrac{d}{dt}x(t) = A(\alpha)x(t) + B(\beta)u(t), \\ y(t) = C(\gamma)x(t), \end{cases}$$

where $x(t) \in \mathcal{X}$ is the state, $u(t) \in \mathcal{U} := \boldsymbol{R}^m$ is the input and $y(t) \in \mathcal{Y} := \boldsymbol{R}^\ell$ is the output. And coefficient matrices $A(\alpha), B(\beta)$ and $C(\gamma)$ have uncertain parameters in the sense that

$$A(\alpha) = A_0 + \alpha_1 A_1 + \cdots + \alpha_p A_p := A_0 + \Delta A(\alpha),$$

$$B(\beta) = B_0 + \beta_1 B_1 + \cdots + \beta_q B_q := B_0 + \Delta B(\beta),$$

$$C(\gamma) = C_0 + \gamma_1 C_1 + \cdots + \gamma_r C_r := C_0 + \Delta C(\gamma),$$

where $\alpha := (\alpha_1, \cdots, \alpha_p)$, $\beta := (\beta_1, \cdots, \beta_q)$, $\gamma :=$ $(\gamma_1, \cdots, \gamma_r)$, and $\alpha_i, \beta_i, \gamma_i \in \boldsymbol{R}$ are all arbitrary real numbers.

In system $S(\alpha, \beta, \gamma)$, (A_0, B_0, C_0) and $(\Delta A(\alpha),$ $\Delta B(\beta),$ $\Delta C(\gamma))$ represent the nominal system model and a specific uncertain perturbation, respectively. Now, the definitions of generalized invariant subspaces are introduced as follows.

Definition 6. Let $\mathcal{V}$, Ω and ε be subspaces of $\mathcal{X}$.

(i) $\mathcal{V}$ is said to be *generalized* (A, B)-*invariant* if there exists an $F \in \boldsymbol{R}^{m \times n}$ such that

$$(A(\alpha) + B(\beta)F)\mathcal{V} \subset \mathcal{V}$$

for all $(\alpha, \beta) \in \boldsymbol{R}^p \times \boldsymbol{R}^q$.

$\boldsymbol{\mathcal{V}_g}((A, B)|\Omega) := \{\mathcal{V} \mid \exists F \in \boldsymbol{R}^{m \times n} : (A(\alpha) + B(\beta)F)$
$\mathcal{V} \subset \mathcal{V}$ for all $(\alpha, \beta) \in \boldsymbol{R}^p \times \boldsymbol{R}^q$ and $V \subset \Omega\}$.

(ii) $\mathcal{V}$ is said to be *generalized* (C, A)-*invariant* if there exists a $G \in \boldsymbol{R}^{n \times \ell}$ such that

$$(A(\alpha) + GC(\gamma))\mathcal{V} \subset \mathcal{V}$$

for all $(\alpha, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r$.

$\boldsymbol{\mathcal{I}_g}(\varepsilon \mid (C, A)) := \{\mathcal{V} \mid \exists G \in \boldsymbol{R}^{n \times \ell} : (A(\alpha) + GC(\gamma))$
$\mathcal{V} \subset \mathcal{V}$ for all $(\alpha, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r$ and $\varepsilon \subset V\}$.

(iii) $\mathcal{V}$ is said to be *generalized* (A, B, C)-*invariant* if there exists an $H \in \boldsymbol{R}^{m \times \ell}$ such that

$$(A(\alpha) + B(\beta)HC(\gamma))\mathcal{V} \subset \mathcal{V}$$

for all $(\alpha, \beta, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$. ■

The following result gives the relationship between simultaneously $\{(A_i, B_i); 1 \leq i \leq \tau\}$- invariant and generalized (A, B)-invariant subspace.

Theorem 7. $\mathcal{V}$ is generalized $(A,$ $B)$-invariant if and only if

$$\begin{bmatrix} A_0 \\ A_0+A_1 \\ \vdots \\ A_0+A_p \\ A_0 \\ \vdots \\ A_0 \end{bmatrix} V \subset \begin{bmatrix} \mathcal{V} \\ \mathcal{V} \\ \vdots \\ \mathcal{V} \\ \mathcal{V} \\ \vdots \\ \mathcal{V} \end{bmatrix} + \mathrm{Im} \begin{bmatrix} B_0 \\ B_0 \\ \vdots \\ B_0 \\ B_0+B_1 \\ \vdots \\ B_0+B_q \end{bmatrix} \quad (3)$$

which implies $\mathcal{V}$ is simultaneously $\{(A_0, B_0), (A_0 + A_i, B_0), (A_0, B_0 + B_j);\ i = 1, \cdots, p, j = 1, \cdots, q\}$-invariant. Namely, $\boldsymbol{\mathcal{V}_g}((A, B) \mid \Omega) = \boldsymbol{\mathcal{V}_s}((A_0, B_0), (A_0 + A_i, B_0), (A_0, B_0 + B_j);\ i = 1, \cdots, p, j = 1, \cdots, q \mid \Omega)$.

Proof. Choose a new parameter γ satisfying

$$\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j + \gamma = 1.$$

Then, the following equality holds.

$$\begin{aligned} &A(\alpha) + B(\beta)F \\ &= \gamma(A_0 + B_0F) + \sum_{i=1}^{p} \alpha_i\{(A_0 + A_i) + B_0F\} \\ &\qquad + \sum_{j=1}^{q} \beta_j\{A_0 + (B_0 + B_j)F\}. \quad (4) \end{aligned}$$

Then, the following equivalences can be obtained from Lemma 2(i) and the above equality (4).

$(A(\alpha) + B(\beta)F)\mathcal{V} \subset \mathcal{V}$ for all $(\alpha, \beta) \in \boldsymbol{R}^p \times \boldsymbol{R}^q$.

$$\Leftrightarrow \begin{cases} (A_0 + B_0F)\mathcal{V} \subset \mathcal{V}, \\ \{(A_0 + A_i) + B_0F)\mathcal{V} \subset \mathcal{V} \quad (i = 1, \cdots, p), \\ (A_0 + (B_0 + B_j)F)\mathcal{V} \subset \mathcal{V} \quad (j = 1, \cdots, q). \end{cases}$$

$\Leftrightarrow$ $\mathcal{V}$ is simultaneously $\{(A_0, B_0), (A_0 + A_i, B_0), (A_0, B_0 + B_j);\ i = 1, \cdots, p, j = 1, \cdots, q\}$ -invariant.

$\Leftrightarrow$ Equation (3). ∎

Similarly, the following result can be obtained.

Theorem 8. $\mathcal{V}$ is generalized $(C,\ A)$-invariant if and only if

$$[A_0, A_0 + A_1, \cdots, A_0 + A_p, A_0, \cdots, A_0] \times \left(\begin{bmatrix} \mathcal{V} \\ \mathcal{V} \\ \vdots \\ \mathcal{V} \\ \mathcal{V} \\ \vdots \\ \mathcal{V} \end{bmatrix} \cap \mathrm{Ker}[\, C_0, C_0, \cdots, C_0, C_0 + C_1, \cdots, C_0 + C_r \,] \right) \subset \mathcal{V},$$

which implies $\mathcal{V}$ is simultaneously $\{(C_0, A_0), (C_0, A_0 + A_i), (C_0 + C_k, A_0);\ i = 1, \cdots, p, k = 1, \cdots, r\}$-invariant. Namely, $\boldsymbol{\mathcal{I}_g}(\varepsilon \mid (C, A)) = \boldsymbol{\mathcal{I}_s}(\varepsilon \mid (C_0, A_0), (C_0, A_0 + A_i), C_0 + C_k, A_0)\ ;\ i = 1, \cdots, p,\ k = 1, \cdots, r\}$. ∎

Now, the following two theorems give new computational algorithms to compute the maximal element $\mathcal{V}_g^{max}(\Omega)$ and the minimal element $\mathcal{V}_g^{min}(\varepsilon)$, respectively.

Theorem 9. The class $\boldsymbol{\mathcal{V}_g}((A, B); \Omega)$ has the unique maximal element $\mathcal{V}_g^{max}(\Omega)$ and it can be computed as the following steps.

Step1. $\mathcal{V}^{(0)} := \Omega$.

Step2. $\mathcal{V}^{(k)} :=$

$$\Omega \cap \begin{bmatrix} A_0 \\ A_0+A_1 \\ \vdots \\ A_0+A_p \\ A_0 \\ \vdots \\ A_0 \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathcal{V}^{(k-1)} \\ \mathcal{V}^{(k-1)} \\ \vdots \\ \mathcal{V}^{(k-1)} \\ \mathcal{V}^{(k-1)} \\ \vdots \\ \mathcal{V}^{(k-1)} \end{bmatrix} + \mathrm{Im} \begin{bmatrix} B_0 \\ B_0 \\ \vdots \\ B_0 \\ B_0+B_1 \\ \vdots \\ B_0+B_q \end{bmatrix} \right).$$

$(k = 1, 2, \cdots)$

Step3. $\mathcal{V}_g^{max}(\Omega) := \mathcal{V}^{(\dim(\Omega))}$.

Proof. The proof follows from Theorem 7 and Lemma 2(ii). ∎

Theorem 10. The class $\boldsymbol{\mathcal{I}_g}(\varepsilon \mid (C, A))$ has the unique minimal element $\mathcal{V}_g^{min}(\varepsilon)$ and it can be computed as the following steps.

Step1. $\mathcal{V}^{(0)} := \varepsilon$.

Step2. $\mathcal{V}^{(k)} := \mathcal{V}^{(k-1)} +$

$$[A_0, A_0 + A_1, \cdots, A_0 + A_p, A_0, \cdots, A_0] \times \left(\begin{bmatrix} \mathcal{V}^{(k-1)} \\ \mathcal{V}^{(k-1)} \\ \vdots \\ \mathcal{V}^{(k-1)} \\ \mathcal{V}^{(k-1)} \\ \vdots \\ \mathcal{V}^{(k-1)} \end{bmatrix} \cap \mathrm{Ker}[\, C_0, C_0 \cdots C_0, C_0 + C_1 \cdots C_0 + C_r \,] \right).$$

$(k = 1, 2, \cdots)$

Step3. $\mathcal{V}_g^{min}(\varepsilon) := \mathcal{V}^{(n-\dim(\varepsilon))}$.

Proof. The proof follows from Theorem 8 and Lemma 3(ii). ∎

The following lemma is used in the next section.

Lemma 11. (Otsuka, 1999) $\mathcal{V}$ is generalized $(A,\ B,\ C)$-invariant subspace if and only if $\mathcal{V}$ is generalized (A, B)-invariant and generalized (C, A)-invariant subspace. ∎

4. NECESSARY AND SUFFICIENT CONDITIONS

This section gives the results for an open problem which was pointed out by Bhattacharyya, that is, necessary and sufficient conditions for the parameter insensitive disturbance-rejection problem with state feedback to be solvable by using useful results in the previous sections are proved.

Consider the following uncertain linear system $S(\alpha,\beta,\gamma,\delta,\sigma)$ defined in $\mathcal{X} := \boldsymbol{R}^n$:

$$\begin{cases} \frac{d}{dt}x(t) = A(\alpha)x(t) + B(\beta)u(t) + E(\sigma)\xi(t), \\ y(t) = C(\gamma)x(t), \\ z(t) = D(\delta)x(t), \end{cases}$$

where $x(t) \in \mathcal{X}$ is the state, $u(t) \in \mathcal{U} := \boldsymbol{R}^m$ is the input, $y(t) \in \mathcal{Y} := \boldsymbol{R}^\ell$ is the output to be measured, $z(t) \in \mathcal{Z} := \boldsymbol{R}^\mu$ is the output to be controlled and $\xi(t) \in \boldsymbol{R}^\eta$ is the disturbance. And coefficient matrices have the following uncertain parameters:

$$A(\alpha) = A_0 + \alpha_1 A_1 + \cdots + \alpha_p A_p := A_0 + \Delta A(\alpha),$$

$$B(\beta) = B_0 + \beta_1 B_1 + \cdots + \beta_q B_q := B_0 + \Delta B(\beta),$$

$$C(\gamma) = C_0 + \gamma_1 C_1 + \cdots + \gamma_r C_r := C_0 + \Delta C(\gamma),$$

$$D(\delta) = D_0 + \delta_1 D_1 + \cdots + \delta_s D_s := D_0 + \Delta D(\delta),$$

$$E(\sigma) = E_0 + \sigma_1 E_1 + \cdots + \sigma_t E_t := E_0 + \Delta E(\sigma),$$

where A_i, B_i, C_i are the same as system $S(\alpha,\beta,\gamma)$ in Section 3, $D_i \in \boldsymbol{R}^{\mu\times n}$ and $E_i \in \boldsymbol{R}^{n\times\eta}$. Further, $\alpha := (\alpha_1,\cdots,\alpha_p) \in \boldsymbol{R}^p$, $\beta := (\beta_1,\cdots,\beta_q) \in \boldsymbol{R}^q$, $\gamma := (\gamma_1,\cdots,\gamma_r) \in \boldsymbol{R}^r$, $\delta := (\delta_1,\cdots,\delta_s) \in \boldsymbol{R}^s$, $\sigma := (\sigma_1,\cdots,\sigma_t) \in \boldsymbol{R}^t$.

In system $S(\alpha,\beta,\gamma,\delta,\sigma)$, $(A_0, B_0, C_0, D_0, E_0)$ and $(\Delta A(\alpha),\ \Delta B(\beta),\ \Delta C(\gamma),\ \Delta D(\delta),\ \Delta E(\sigma))$ represent the nominal system model and a specific uncertain perturbation, respectively.

Now, parameter insensitive disturbance-rejection problems with static feedback are formulated as follows.

[Parameter Insensitive Disturbance Rejection Problem with State Feedback] (PIDRP SF)

Given matrices A_i,B_i,D_i,E_i for system $S(\alpha,\ \beta,\ \delta,\ \sigma)$, find if possible a state feedback gain $F \in \boldsymbol{R}^{m\times n}$ such that

$$< A(\alpha) + B(\beta)F \mid \mathrm{Im}E(\sigma) > \;:= \sum_{i=1}^{n}(A(\alpha)+B(\beta)F)^{i-1}(\mathrm{Im}E(\sigma)) \subset \mathrm{Ker}D(\delta)$$

for all parameters $(\alpha,\beta,\delta,\sigma) \in \boldsymbol{R}^p\times\boldsymbol{R}^q\times\boldsymbol{R}^s\times\boldsymbol{R}^t$. ■

[Parameter Insensitive Disturbance Rejection Problem with Static Output Feedback] (PIDRPSOF)

Given matrices A_i, B_i, C_i, D_i, E_i for system $S(\alpha,\beta,\gamma,\delta,\sigma)$, find if possible an output feedback gain $H \in \boldsymbol{R}^{m\times\ell}$ such that

$$< A(\alpha) + B(\beta)HC(\gamma) \mid \mathrm{Im}E(\sigma) > \;:= \sum_{i=1}^{n}(A(\alpha) + B(\beta)HC(\gamma))^{i-1}(\mathrm{Im}E(\sigma)) \subset \mathrm{Ker}D(\delta)$$

for all parameters $(\alpha,\beta,\gamma,\delta,\sigma) \in \boldsymbol{R}^p\times\boldsymbol{R}^q\times\boldsymbol{R}^r\times\boldsymbol{R}^s\times\boldsymbol{R}^t$. ■

The following theorem is the main result. The sufficiency was given by Bhattacharyya(1983) but the necessity has not been solved as an open problem. Now, necessary and sufficient conditions for the problem to be solvable are proved for completeness.

Theorem 12. PIDRPSF is solvable if and only if

$$\sum_{i=0}^{t}\mathrm{Im}E_i \subset \mathcal{V}_g^{max}(\Omega),$$

where $\mathcal{V}_g^{max}(\Omega)$ is the maximal element of $\boldsymbol{\mathcal{V}_g}((A, B) \mid \Omega)$, $\Omega := \bigcap_{i=0}^{s}\mathrm{Ker}D_i$.

Sketch of Proof. (Necessity) Suppose that the PIDRPSF is solvable. Then, there exists a state feedback gain $F \in \boldsymbol{R}^{m\times n}$ such that

$$< A(\alpha)+B(\beta)F \mid \mathrm{Im}E(\sigma) >\subset \mathrm{Ker}D(\delta) \qquad (5)$$

for all parameters $(\alpha,\beta,\delta,\sigma) \in \boldsymbol{R}^p\times\boldsymbol{R}^q\times\boldsymbol{R}^s\times\boldsymbol{R}^t$.

It follows from (5) that the following relations hold.

$$\sum_{i=0}^{t}\mathrm{Im}E_i \;\subset\; < A(\alpha) + B(\beta)F \mid \sum_{i=0}^{t}\mathrm{Im}E_i > \;\subset \bigcap_{i=0}^{s}\mathrm{Ker}D_i = \Omega. \qquad (6)$$

for all parameters (α,β). Noticing (6), define the sequence of subspaces as

$$\mathcal{V}^{(0)} := \sum_{i=0}^{t}\mathrm{Im}E_i,$$

$$\mathcal{V}^{(k)} := \mathcal{V}_1^{(k)} + \mathcal{V}_2^{(k)} + \cdots + \mathcal{V}_{p+q+1}^{(k)} \quad (k = 1,2,\cdots),$$

where for $k = 1,2,\cdots$

$$\mathcal{V}_1^{(k)} := \mathcal{V}^{(k-1)} + (A_0 + B_0F)\mathcal{V}^{(k-1)}, \quad (\subset \Omega)$$

$$\mathcal{V}_2^{(k)} := \mathcal{V}^{(k-1)} + ((A_0 + A_1) + B_0F)\mathcal{V}^{(k-1)},$$

$$(\subset \Omega)$$

$$\cdots$$

$$\mathcal{V}_{p+1}^{(k)} := \mathcal{V}^{(k-1)} + ((A_0 + A_p) + B_0 F)\mathcal{V}^{(k-1)},$$

$$(\subset \Omega)$$

$$\mathcal{V}_{p+2}^{(k)} := \mathcal{V}^{(k-1)} + (A_0 + (B_0 + B_1)F)\mathcal{V}^{(k-1)},$$

$$(\subset \Omega)$$

$$\cdots$$

$$\mathcal{V}_{p+q+1}^{(k)} := \mathcal{V}^{(k-1)} + (A_0 + (B_0 + B_q)F)\mathcal{V}^{(k-1)}.$$

$$(\subset \Omega)$$

Then, it follows from Lemma 4 that there exists a μ $(\leq n - \dim(\sum_{i=0}^{t} \mathrm{Im} E_i))$ such that

$$\mathcal{V}^{(\mu)} = \mathcal{V}^{(k)} \quad \text{for all} \quad k \geq \mu$$

and $\mathcal{V}^{(\mu)}$ is simultaneously $\{(A_0, B_0), (A_0 + A_i, B_0), (A_0, B_0 + B_j);\ i = 1, \cdots, p, j = 1, \cdots, q\}$-invariant. Since $\mathcal{V}^{(\mu)}$ is contained in Ω, it follows from Theorem 7 that $\mathcal{V}^{(\mu)}$ is an element of $\boldsymbol{\mathcal{V}_g}((A, B) \mid \Omega)$. Further, from definition of $\mathcal{V}^{(\mu)}$ the following relations hold.

$$\sum_{i=0}^{t} \mathrm{Im} E_i \subset \mathcal{V}^{(\mu)} \subset \mathcal{V}_g^{max}(\Omega).$$

This completes the proof of necessity.

(Sufficiency) Suppose that $\sum_{i=0}^{t} \mathrm{Im} E_i \subset \mathcal{V}_g^{max}(\Omega)$.

Since $\mathcal{V}_g^{max}(\Omega)$ is a generalized invariant subspace, there exists an $F \in \boldsymbol{R}^{m \times n}$ such that

$$(A(\alpha) + B(\beta)F)\mathcal{V}_g^{max}(\Omega) \subset \mathcal{V}_g^{max}(\Omega)$$

for all $(\alpha, \beta) \in \boldsymbol{R}^p \times \boldsymbol{R}^q$.

Then, the following relations hold.

$$\begin{aligned}
&< A(\alpha) + B(\beta)F \mid \mathrm{Im} E(\sigma) > \\
&\subset < A(\alpha) + B(\beta)F \mid \sum_{i=0}^{t} \mathrm{Im} E_i > \\
&\subset < A(\alpha) + B(\beta)F \mid \mathcal{V}_g^{max}(\Omega) > \\
&= \mathcal{V}_g^{max}(\Omega) \\
&\subset \Omega \\
&\subset \mathrm{Ker} D(\delta)
\end{aligned}$$

for all parameters $(\alpha, \beta, \delta, \sigma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^s \times \boldsymbol{R}^t$, which imply PIDRPSF is solvable. This completes the proof of theorem. ■

The following theorem follows from Lemmas 5 and 11 and the same manner as the proof of Theorem12.

Theorem 13. PIDRPSOF is solvable if and only if there exists a generalized (A, B)-invariant and generalized (C, A)-invariant subspace $\mathcal{V}$ such that

$$\sum_{i=0}^{t} \mathrm{Im} E_i \subset \mathcal{V} \subset \bigcap_{i=0}^{s} \mathrm{Ker} D_i. \ ■$$

Corollary 14. Suppose that $\mathcal{V}_g^{min}(\varepsilon)$ is the minimal element of $\boldsymbol{\mathcal{I}_g}(\varepsilon \mid (C, A))$ and $\mathcal{V}_g^{max}(\Omega)$ is the maximal element of $\boldsymbol{\mathcal{V}_g}((A, B) \mid \Omega)$, where $\varepsilon := \sum_{i=0}^{t} \mathrm{Im} E_i$, $\Omega := \bigcap_{i=0}^{s} \mathrm{Ker} D_i$. If $\mathcal{V}_g^{min}(\varepsilon) \in \boldsymbol{\mathcal{V}_g}((A, B) | \Omega)$ or $\mathcal{V}_g^{max}(\Omega) \in \boldsymbol{\mathcal{I}_g}(\varepsilon \mid (C, A))$, then the PIDRPSOF is solvable. ■

5. CONCLUSIONS

In this paper, the relationships between generalized invariant subspaces and simultaneously invariant subspaces were firstly investigated. And then, necessary and sufficient conditions for the parameter insensitive disturbance-rejection problem with state feedback for uncertain linear systems to be solvable which was pointed out as an open problem by Bhattacharyya were proved. Further, the problem with static output feedback was also investigated.

6. REFERENCES

Basile, G. and Marro, G. (1991). *Controlled and Conditioned Invariants in Linear System Theory.* Prentice Hall. New Jersey.

Bhattacharyya, S. P (1983). Generalized Controllability, (A,B)-invariant Subspaces and Parameter Invariant Control. *SIAM Journal on Algebraic and Discrete Methods* **4**, 529–533.

Ghosh, B. K. (1986). *A geometric approach to simultaneous system design: parameter insensitive disturbance decoupling by state and output feedback.* Modeling, Identification and Robust Control, C.I.Byrnes and A.Lindquist(ed.). North-Holland.

Otsuka, N. (1999). Generalized (C, A, B)-pairs and Parameter Insensitive Disturbance-Rejection Problems with Dynamic Compensator. *IEEE Transactions on Automatic Control* **44**, 2195-2200.

Otsuka, N. (2000). Generalized Invariant Subspaces and Parameter Insensitive Disturbance Rejection Problem with Static Output Feedback. *IEEE Transactions on Automatic Control* **45**, 1691–1697.

Schumacher, J. M. (1980). Compensator synthesis using (C, A, B)-pair. *IEEE Transactions on Automatic Control* **25**, 1133–1138.

Wonham, W. M. (1984). *Linear multivariable control : A geometric approach.* Springer Verlag. New York.