STABILIZATION OF A FLEXIBLE MANIPULATOR MODEL WITH PASSIVE JOINTS

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Abstract: To describe the dynamic behavior of a fire rescue turntable ladder the flexibility of the ladder set has to be analyzed. It turned out, that it can be modelled as a flexible manipulator consisting of an arbitrary number of links governed by the Euler-Bernoulli beam equation. The first link is fixed at one end and is driven by a control torque. The last link carries a payload. A peculiarity of the model is that the density and flexural rigidity parameters are allowed to be discontinuous but piecewise constant functions of the spatial coordinate. For the Galerkin approximation of the control system considered, we derive a result on stabilization of the equilibrium. The controller design scheme is based on explicit construction of a Lyapunov function and application of the invariance principle. A simulation of the closed-loop dynamics is carried out in order to show the efficiency of the controller proposed.

Keywords: Stabilizing feedback, Lyapunov function, Modal control, Flexible arms, Galerkin approximation

1. INTRODUCTION

Fire turntable ladders are manipulators with a fairly large workspace. As these manipulators are mobile vehicles and the area for strutting the vehicle is limited, the aim is to achieve the maximum of outreach. Therefore, this can be only achieved by reducing the weight of the ladder set itself. The result is a light weight construction of the ladder set, which is then characterized by a significant flexibility. In order to develop an oscillation damping control for these systems, the modelling of the ladder set as a flexible beam plays a crucial role.

Figure 1 shows a typical fire rescue turntable ladder. The manipulator is equipped with four active axes: the turning axis φ_T; the raising of the ladder set φ_R; another rotating joint φ_J for the upper ladder part 6; and the telescoping of the ladder parts 1 to 5, described by the variable l.

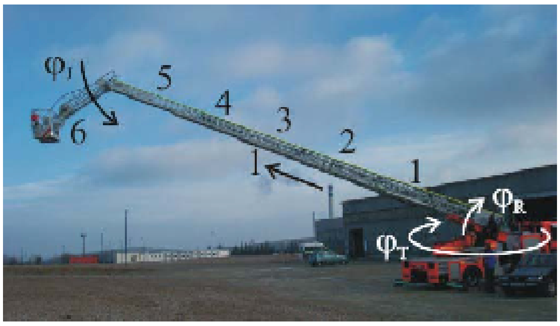

Figure 1: Fire turntable ladder.

The bearings of the ladder parts can be assumed to be passive joints of the system.

The dynamical behavior of the distributed parameter vibration systems, such as beams and flexible-

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link manipulators, is generally described by partial differential equations; see (Chen *et al.*, 1987; Bloch and Titi, 1991; Xu and Baillieul, 1993; Coron and d'Andrea Novel, 1998; Luo *et al.*, 1999) and references therein. However, finite dimensional approximate models obtained by the assumed modes and finite elements methods are used more frequently for solving the motion planning and stabilization problems (Talebi *et al.*, 2001; Sawodny *et al.*, 2002). The goal of this paper is to derive both complete and approximate dynamical models for the multi-link flexible manipulator representing a fire-rescue turntable ladder. In our case, mathematical modelling of the ladder dynamics as a homogeneous beam is not adequate since the real object consists of several segments having different mechanical parameters.

2. MOTION EQUATIONS

Consider a flexible manipulator performing planar motion under the action of a control torque M (see Figure 2). The manipulator is fixed at one end (hub at the point O), while the other end carries a payload of mass m.

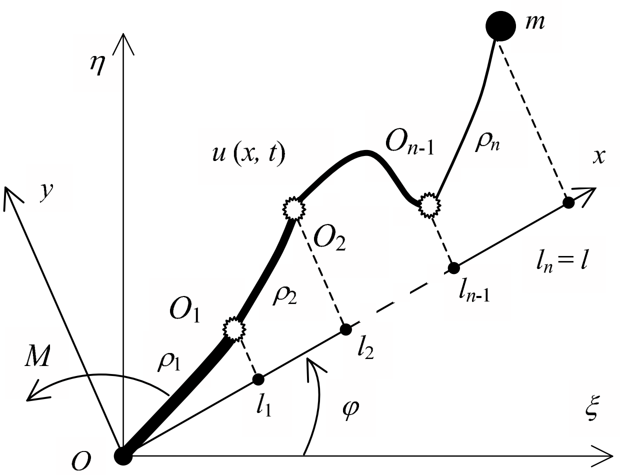

Figure 2: A multi-link flexible manipulator.

We assume that the manipulator consists of n flexible links connected via elastic joints at O_1, ..., O_{n-1}. It is assumed also that the manipulator length, l, is much greater than the width of each link, thus we shall use the Euler-Bernoulli beam equation for modelling the links oscillations. For each time $t \geq 0$, the links configuration is described by the graph of $u(x,t)$, $x \in [0,l]$ in the Cartesian frame Oxy. Given a partition $0 = l_0 < l_1 < ... < l_n = l$ of $[0,l]$, the deflection $u(\cdot,t)$ is assumed to be of class $C^0[0,l] \cap C^4((0,l) \setminus \{l_1,...,l_{n-1}\})$ for all $t \geq 0$. For a function $f(x,t)$, we denote by f_{a+0}, f_{a-0}, and f_a the values $\lim_{x\to a+0} f(x,t)$, $\lim_{x\to a-0} f(x,t)$, and $f(a,t)$, respectively. Thus, $u(x,t)$ satisfies the following geometric boundary condition:

$$(u)_0 = \left(\frac{\partial u}{\partial x}\right)_0 = 0.$$

Neglecting the effects of shear deformation and rotary inertia of the beams, the kinetic energy T is given by the following expression:

$$2T = \int_0^l \left((u\dot{\varphi})^2 + \left(x\dot{\varphi} + \frac{\partial u}{\partial t} \right)^2 \right) \rho(x)\, dx +$$

$$+ m\left((u_l \dot{\varphi})^2 + \left(l\dot{\varphi} + \frac{\partial u}{\partial t} \right)_l^2 \right) + J\left(\dot{\varphi} + \left(\frac{\partial^2 u}{\partial t \partial x} \right)_l \right)^2,$$

where $\varphi(t)$ is the angle between Ox and the fixed axis $O\xi$, $\rho(x)$ is the mass per unit length of the beam, J is the payload moment of inertia.

The potential energy U of the mechanical system considered takes the form:

$$2U = \int_0^l \left(\frac{\partial^2 u}{\partial x^2} \right)^2 c^2(x) \rho(x)\, dx +$$

$$+ \sum_{j=1}^{n-1} \varkappa_j^2 \left(\left(\frac{\partial u}{\partial x} \right)_{l_j+0} - \left(\frac{\partial u}{\partial x} \right)_{l_j-0} \right)^2,$$

where $c^2(x) = E(x)I(x)/\rho(x)$ ($E(x)I(x)$ is the flexural rigidity per unit length of the beam), $\varkappa_j^2$ is the stiffness coefficient of the torsion spring at O_j. In this paper, $\rho(x)$ and $c(x)$ are assumed to be piecewise-constant functions, i.e. $\rho(x) = \rho_j$ and $c(x) = c_j$ for $x \in [l_{j-1}, l_j)$, $j = 1, 2, ..., n$.

Suppose that $\varphi(t)$ and $u(x,t)$ define motion of the system for given $t \in [t_1, t_2]$ and control torque $M(t)$. Then Hamilton's principle (Goldstein, 1980) yields

$$\delta\left(\int_{t_1}^{t_2} L\, dt \right) + \int_{t_1}^{t_2} M(t) \delta\varphi(t) dt = 0, \qquad (1)$$

for each variation $\delta\varphi(t)$, $\delta w_i(x,t)$ such that

$$\delta\varphi \in C^2[t_1,t_2],\ \delta\varphi(t_1) = \delta\varphi(t_2) = 0,$$
$$\delta u \in C^2\left(((0,l) \setminus \{l_1,...,l_{n-1}\}) \times [t_1,t_2] \right),$$
$$\delta u(x,t_1) = \delta u(x,t_2) = 0, \quad \forall x \in [0,l],$$
$$(\delta u)_0 = \left(\frac{\partial \delta u}{\partial x} \right)_0 = 0,\ \delta u(\cdot,t) \in C^0[0,l],\ \forall t \in [t_1,t_2], \qquad (2)$$

where $L = T - U$ is the Lagrangian of the system.

The variation of $\int_{t_1}^{t_2} L\, dt$ in (1) is obtained by performing several integrations by parts [2]:

$$-\delta\left(\int_{t_1}^{t_2} L\, dt \right) - \int_{t_1}^{t_2} M \delta\varphi\, dt =$$

$$= \int_{t_1}^{t_2} \left[\frac{d}{dt} \left(m(u^2 + l^2)\dot{\varphi} + ml\frac{\partial u}{\partial t} + J\dot{\varphi} + J\frac{\partial^2 u}{\partial t \partial x} \right)_l + \right.$$

$$\left. + \frac{d}{dt} \int_0^l \left((u^2 + x^2)\dot{\varphi} + x\frac{\partial u}{\partial t} \right) \rho\, dx - M \right] \delta\varphi\, dt +$$

$$+ \int_{t_1}^{t_2} \mu\left(\frac{\partial^2 u}{\partial t^2}, u, \delta u, \ddot{\varphi}, \dot{\varphi} \right) dt = 0, \qquad (3)$$

[2] We omit cumbersome computations here.

where

$$\mu = \int_0^l \left(\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} + x\ddot{\varphi} - \dot{\varphi}^2 u \right) \delta u \rho \, dx +$$

$$+ m \left[\left(\frac{\partial^2 u}{\partial t^2} - \frac{c^2 \rho}{m} \frac{\partial^3 u}{\partial x^3} + l\ddot{\varphi} - \dot{\varphi}^2 u \right) \delta u \right]_l +$$

$$+ J \left[\left(\frac{\partial^3 u}{\partial t^2 \partial x} + \frac{c^2 \rho}{J} \frac{\partial^2 u}{\partial x^2} + \ddot{\varphi} \right) \frac{\partial \delta u}{\partial x} \right]_l +$$

$$+ \sum_{j=1}^{n-1} \left\{ \left[\left(c^2 \rho \frac{\partial^3 u}{\partial x^3} \right)_{l_j+0} - \left(c^2 \rho \frac{\partial^3 u}{\partial x^3} \right)_{l_j-0} \right] (\delta u)_{l_j} + \right.$$

$$+ \left[\left(\frac{c^2 \rho}{\varkappa_j^2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \right)_{l_j-0} - \frac{\partial u}{\partial x}_{l_j+0} \right] \varkappa_j^2 \left(\frac{\partial \delta u}{\partial x} \right)_{l_j-0} +$$

$$\left. + \left[\left(\frac{\partial u}{\partial x} - \frac{c^2 \rho}{\varkappa_j^2} \frac{\partial^2 u}{\partial x^2} \right)_{l_j+0} - \frac{\partial u}{\partial x}_{l_j-0} \right] \varkappa_j^2 \left(\frac{\partial \delta u}{\partial x} \right)_{l_j+0} \right\}.$$

To simplify (3), we introduce new control v by means of the following feedback transformation:

$$v = \left(m {u_l}^2 + \int_0^l (2x^2 + u^2) \rho \, dx \right)^{-1} \times$$

$$\times \left(M + \left(c^2 \rho \frac{\partial^2 u}{\partial x^2} \right)_0 - m\dot{\varphi} \left(2u \frac{\partial u}{\partial t} + l\dot{\varphi} u \right)_l + \right.$$

$$\left. + \dot{\varphi} \int_0^l \left(\dot{\varphi} x - 2 \frac{\partial u}{\partial t} \right) u \rho \, dx \right). \qquad (4)$$

As (3) should vanish for each variation $(\delta\varphi, \delta u)$ satisfying (2), we have

$$\ddot{\varphi} = v, \; v \in \mathbb{R},$$

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = \dot{\varphi}^2 u - xv, \; x \in (0,l) \setminus \{l_1, ..., l_{n-1}\}, \qquad (5)$$

$$\left(\frac{\partial^2 u}{\partial t^2} - \dot{\varphi}^2 u + lv - \frac{{c_n}^2 \rho_n}{m} \frac{\partial^3 u}{\partial x^3} \right) \Big|_{x=l} = 0,$$

$$\left(\frac{\partial^3 u}{\partial t^2 \partial x} + v + \frac{{c_n}^2 \rho_n}{J} \frac{\partial^2 u}{\partial x^2} \right) \Big|_{x=l} = 0,$$

$$u|_{x=0} = \frac{\partial u}{\partial x}\Big|_{x=0} = 0, \; u|_{x=l_j-0} = u|_{x=l_j+0},$$

$$c_j^2 \rho_j \frac{\partial^k u}{\partial x^k}\Big|_{x=l_j-0} = c_{j+1}^2 \rho_{j+1} \frac{\partial^k u}{\partial x^k}\Big|_{x=l_j+0}, \; k = 2,3,$$

$$\frac{\partial u}{\partial x}\Big|_{x=l_j+0} = \left(\frac{\partial u}{\partial x} + \frac{c_j^2 \rho_j}{\varkappa_j^2} \frac{\partial^2 u}{\partial x^2} \right) \Big|_{x=l_j-0}, \; j = \overline{1, n-1}.$$

For the case without a payload, a similar system describing the dynamics of serially connected beams was considered in (Chen *et al.*, 1987).

Our goal is to stabilize the equilibrium $\varphi = \dot{\varphi} = 0$, $u = \frac{\partial u}{\partial t} = 0$ by means of a state feedback law $v = \gamma\left(\varphi, \dot{\varphi}, u, \frac{\partial u}{\partial t}\right)$. This problem will be solved in Section 5 for a finite dimensional approximation of the boundary value problem (5).

3. EIGENFUNCTIONS OF THE HOMOGENEOUS PROBLEM

To derive an approximate dynamical model, we first apply separation of variables by substituting

$$u(x,t) = \psi(x)\theta(t), \; \varphi(t) = \text{const}, \; v(t) = 0 \qquad (6)$$

into (5). This yields

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t)$$

and

$$\psi(x) = C_{j1}\sin(\eta_j x) + C_{j2}\cos(\eta_j x) + C_{j3}\sinh(\eta_j x) +$$

$$+ C_{j4}\cosh(\eta_j x) \;\; \text{for } x \in [l_{j-1}, l_j), \; \eta_j^2 c_j = \omega, \; j = \overline{1,n}, \qquad (7)$$

provided that C_{ji} satisfy the following system of linear algebraic equations

$$M(\omega, P) \cdot (C_{11}, ..., C_{14}, ..., C_{n1}, ..., C_{n4})^T = 0. \qquad (8)$$

Here $M(\omega,P)$ is a $4n \times 4n$-matrix whose coefficients depend on parameters ω and $P = (m, J, n, l_j, \rho_j, c_j, \varkappa_j)$.

For fixed P, the system (8) admits a non-trivial invariant subspace if and only if ω is a solution of the transcendent equation $\det M(\omega,P) = 0$ [3]. If ω is such a solution, we will refer to $\psi(x)$ as *a form corresponding to* ω if its coefficients $(C_{ji}) \neq 0$ satisfy (8).

Proposition 1. *Let ψ_1 and ψ_2 be forms corresponding to frequencies $\omega_1 \neq \omega_2$. Then ψ_1 and ψ_2 are orthogonal with respect to the following bilinear form*

$$\langle \psi_1, \psi_2 \rangle_X = \int_0^l \psi_1 \psi_2 \rho \, dx + m\psi_1(l)\psi_2(l) + J\psi_1'(l)\psi_2'(l).$$

The **proof** exploits integration by parts with regard to the boundary conditions at $x = l_j$.

In order to illustrate particular solutions (6) of the boundary value problem (5), we compute some number of first frequencies ω_k for the following (dimensionless) values of parameters:

$$n = 2, \; l_1 = \frac{l_2}{2} = c_1 = c_2 = \rho_1 = \rho_2 = \frac{\varkappa_1^2}{2} = m = J = 1. \qquad (9)$$

We have: $\omega_1 = 0.3875771806$; $\omega_2 = 1.455554174$; $\omega_3 = 5.629214811$; $\omega_4 = 16.22789077$. The corresponding forms ψ are shown in Figure 3. The forms are normalized there so that $\langle \psi_i, \psi_k \rangle_X = \delta_{ik}$ for each $1 \le i \le j \le 4$.

Remark. We see that the derivatives $\psi_k'(x)$ are discontinuous at $x = l_1$. Thus, $\psi_k(\cdot)$ do not belong to the Sobolev space $H^2(0,l)$ in contrast to the case of a beam having continuous density function.

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[3] The expression for $\det M(\omega,P)$ has been obtained via the Maple software. Computational details are omitted in this paper.

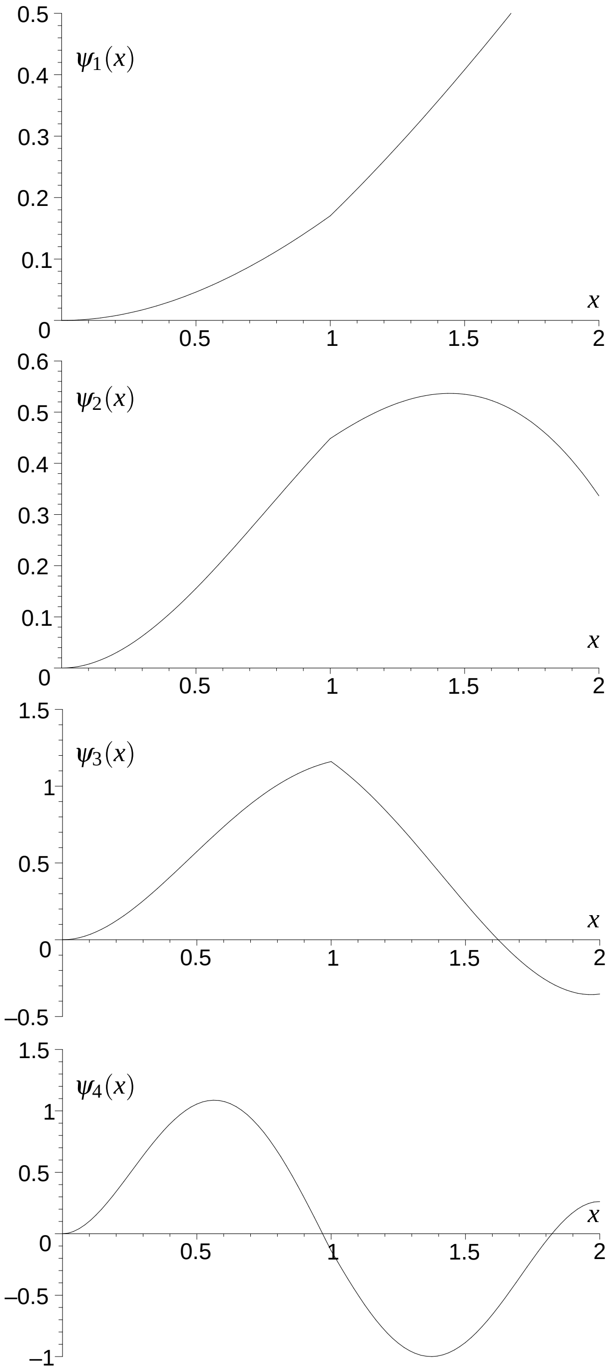

Figure 3: The graphs of $\psi_k(x)$ for $k \le 4$.

4. APPROXIMATE MODEL

Let $(\psi_1(\cdot), \ldots, \psi_N(\cdot))$ be forms corresponding to frequencies $\omega_1 < \omega_2 < \ldots < \omega_N$. To derive a Galerkin approximation of the boundary value problem (5), we restrict $u(\cdot,t)$ and $\delta u(\cdot,t)$ to the finite dimensional space

$$\mathscr{S}_N = \operatorname{span}\{\psi_1(\cdot), \ldots, \psi_N(\cdot)\}$$

in the variational form (3). Our goal is to find $\varphi(t)$ and $u_N(\cdot,t) \in \mathscr{S}_N$ satisfying (3) for each admissible variation $\delta\varphi(t)$ and $\delta u_N(\cdot,t) \in \mathscr{S}_N$; cf. (Donea and Huerta, 2003). Let $(q_1(t), \ldots, q_N(t))$ be the coordinates of $u_N(\cdot,t)$ with respect to $(\psi_1(\cdot), \ldots, \psi_N(\cdot))$ [4]:

$$u_N(x,t) = \sum_{k=1}^{N} \psi_k(x) q_k(t). \tag{10}$$

By substituting u_N into (3) and exploiting Proposition 1, we get the Galerkin approximation of (5) as follows:

$$\begin{gathered} \ddot\varphi = v,\ v \in \mathbb{R}, \\ \ddot q_k = -\omega_k{}^2 q_k + \left(q_k - \sum_{p=1}^{N} a_{kp} q_p\right)\dot\varphi^2 - b_k v, \\ k = 1, 2, \ldots, N, \end{gathered} \tag{11}$$

where v is the control, $(\varphi, \dot\varphi, q_1, \dot q_1, \ldots, q_N, \dot q_N)$ is the state,

$$a_{kp} = \frac{J\psi_k'(l)\psi_p'(l)}{\|\psi_k\|_X^2},\ b_k = \frac{\langle x, \psi_k\rangle_X}{\|\psi_k\|_X^2},\ \|\psi_k\|_X^2 = \langle \psi_k, \psi_k\rangle_X\,.$$

The control torque M is related to v in (11) by means of the feedback transformation (4) with $u = u_N(x,t)$.

5. STABILIZATION OF THE EQUILIBRIUM

The main result we shall prove is the following

Proposition 2. *Let $(\psi_1, \ldots, \psi_N)$ be forms corresponding to frequencies $\omega_1 < \omega_2 < \ldots < \omega_N$, and let, moreover, $\omega_1 > 0$. Then there exists a smooth feedback control $v = \gamma(\varphi, \dot\varphi, q_1, \dot q_1, \ldots, q_N, \dot q_N)$ ensuring global asymptotic stability of the origin for (11).*

Proof. Consider the following energy-based Lyapunov function candidate for (11):

$$\begin{gathered} 2V_N(\varphi, \dot\varphi, \ldots, q_N, \dot q_N) = k_1\varphi^2 + k_2\dot\varphi^2 + \\ + \sum_{j=1}^{N} \omega_j^2 \|\psi_j\|_X^2 q_j^2 + \sum_{j=1}^{N} \|\psi_j\|_X^2 \dot q_j^2 + 2\dot\varphi \sum_{j=1}^{N} \langle x, \psi_j\rangle_X \dot q_j + \\ + \left(\|x\|_X^2 + \sum_{j=1}^{N} \|\psi_j\|_X^2 q_j^2 - J\left(\sum_{j=1}^{N} \psi_j'(l) q_j\right)^2\right)\dot\varphi^2, \\ (k_1 > 0,\ k_2 > 0). \end{gathered}$$

To show that V_N is positive definite, let us first write its quadratic part Q_N as follows:

$$2Q_N = k_1\varphi^2 + k_2\dot\varphi^2 + \|\frac{\partial u_N}{\partial t} + \dot\varphi x\|_X^2 + \sum_{j=1}^{N} \omega_j^2 \|\psi_j\|_X^2 q_j^2,$$

where u_N is given by (10). We have

$$V_N = Q_N + \frac{\dot\varphi^2}{2}\left[\int_0^l \left(\sum_{j=1}^{N} \psi_j(x) q_j\right)^2 \rho\, dx + m\left(\sum_{j=1}^{N} \psi_j(l) q_j\right)^2\right]. \tag{12}$$

It is easy to see that $Q_N \ge 0$, Q_N vanishes only if $\varphi = \dot\varphi = q_1 = \ldots = q_N = 0$ and

$$\|\sum_{j=1}^{N} \psi_j \dot q_j\|_X = 0.$$

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[4] Functions $(\psi_1(\cdot), \ldots, \psi_N(\cdot))$ form a basis in $\mathscr{S}_N$ because of Proposition 1 and the assumption $\omega_i \neq \omega_k$ if $i \neq k$.

But the above expression implies $\dot{q}_1 = \ldots = \dot{q}_N = 0$ since the forms $(\psi_1, \ldots, \psi_N)$ are linearly independent on $[0,l]$. Therefore, the quadratic form Q_N is positive definite. This fact together with (12) implies positive definiteness of V_N and compactness of the level sets

$$L_c = \{(\varphi, \dot{\varphi}, q_1, \dot{q}_1, \ldots, q_N, \dot{q}_N) \in \mathbb{R}^{2N+2} \,|\, V_N \leq c\}$$

for each constant $c > 0$. The time-derivative of V_N with respect to the open-loop system (11) takes the form

$$\dot{V}_N = (\alpha_N + \beta_N v)\dot{\varphi},$$

where

$$\alpha_N = k_1\varphi + \sum_{j=1}^{N} \left(2\|\psi_j\|_X^2 \dot{\varphi}\dot{q}_j + \langle x, \psi_j \rangle_X (\dot{\varphi}^2 - \omega_j^2)\right) q_j -$$

$$-J\dot{\varphi}\left(\sum_{j=1}^{N} \psi_j'(l) q_j\right)\left(\sum_{j=1}^{N} \psi_j'(l)\left(2\dot{q}_j + \frac{\langle x, \psi_j \rangle_X}{\|\psi_j\|_X^2}\dot{\varphi}\right)\right)$$

and

$$\beta_N = k_2 + \|x\|_X^2 - J\left(\sum_{j=1}^{N} \psi_j'(l) q_j\right)^2 +$$

$$+\sum_{j=1}^{N}\left(\|\psi_j\|_X^2 q_j^2 - \frac{\langle x, \psi_j \rangle_X^2}{\|\psi_j\|_X^2}\right) > 0.$$

We define a feedback control $v = \gamma(\varphi, \dot{\varphi}, \ldots, q_N, \dot{q}_N)$ by the following expression

$$\gamma(\varphi, \dot{\varphi}, q_1, \dot{q}_1, \ldots, q_N, \dot{q}_N) = -\frac{\alpha_N + h\dot{\varphi}}{\beta_N}, \qquad (13)$$

where $h > 0$ is a constant. This yields $\dot{V}_N = -h\dot{\varphi}^2 \leq 0$ with respect to the closed-loop system (11), (13). It is easy to show that each positive semi-trajectory of the closed-loop system, restricted to the set

$$Z_0 = \{(\varphi, \dot{\varphi}, q_1, \dot{q}_1, \ldots, q_N, \dot{q}_N) \in \mathbb{R}^{2N+2} \,|\, \dot{V}_N = 0\},$$

satisfies the following relations:

$$\varphi(t) = \varphi_0 = \text{const},$$

$$q_k(t) = A_k \cos(\omega_k t) + B_k \sin(\omega_k t),\ k = 1, 2, \ldots, N,$$

$$k_1\varphi_0 = \sum_{j=1}^{N} \langle x, \psi_j \rangle\, \omega_j^2 q_j(t),\ \text{for all } t \geq 0.$$

The above relations imply $\varphi_0 = A_1 = B_1 = \ldots = B_N = 0$ since the functions $(\cos(\omega_1 t), \ldots, \sin(\omega_N t))$ are linearly independent on $\mathbb{R}^+$, and $\langle x, \psi_j \rangle_X \omega_j \neq 0$ under our assumptions. Therefore, the only solution of the closed-loop system, restricted to Z_0 for all $t \geq 0$, is the trivial one. Now global asymptotic stability of the closed-loop system (11), (13) follows from the Barbashin - Krasovskii and LaSalle invariance principle.

6. OBSERVABILITY PROBLEM

In order to implement the feedback law (13) in practice, one has to reconstruct the complete state vector of (11) from the outputs which can be measured.

An observability problem for a rigid body endowed with two elastic beams has been studied in (Kovalev *et al.*, 2002). It has been proved that the finite dimensional body-beams system (without a payload) is observable, provided that one measures the relative displacements of certain points at the beams. However, the values of displacements $u(x,t)$ cannot be directly measured in a real flexible ladder. Instead, there is a sensor [5] located at a certain point $x = \Delta$, $0 \leq \Delta \leq l_1$ that allows measurement of $\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=\Delta}$.

In the case of a single flexible beam without a payload, an output feedback controller was considered in (Luo and Guo, 1997) with $\Delta = 0$. That approach is not applicable in our case, as one should take into account the motion of a payload and assume $\Delta > 0$ for a turntable ladder.

By replacing $u(x,t)$ with $u_N(x,t)$, we assume that the following output signal is available for the finite dimensional approximation (11):

$$y_1(t) = \left.\frac{\partial^2 u_N(x,t)}{\partial x^2}\right|_{x=\Delta} = \sum_{k=1}^{N} \psi_k''(\Delta) q_k(t),\ y_2(t) = \varphi(t). \qquad (14)$$

Proposition 3. *Assume that $\psi_k''(\Delta) \neq 0$ for all $k = \overline{1,N}$, and $\omega_1 < \omega_2 < \ldots < \omega_N$. Then the linear approximation of (11) around zero is observable with respect to the output (14).*

The assertion of Proposition 3 follows from the rank observability condition (Wonham, 1985).

The above result justifies a possibility of applying the feedback control (13) at least for the linearized system. Indeed, if the assumptions of Propositions 2 and 3 are satisfied, there is a dynamic state observer for the linear approximation of (11). Then, by substituting the state estimate from the above observer into the linearized formulae (13) and (4), we can compute the control torque M and conclude about stability of the observer-based linear dynamics; see (Wonham, 1985). Certainly, this does not give a rigorous answer to the question of output stabilizability in the nonlinear case.

7. SIMULATION RESULTS

A simulation is carried out for the closed-loop approximation (11), (13) with $N = 2$ and the following initial conditions:

$$\varphi(0) = \frac{\pi}{2},\ \dot{\varphi}(0) = q_1(0) = q_2(0) = \dot{q}_1(0) = \dot{q}_2(0) = 0.$$

The mechanical parameters are chosen here as in (9), and the control parameters are: $h = 10$, $k_1 = k_2 = 1$. Figure 4 shows that the feedback proposed is able to steer the approximate system to the origin.

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[5] Such a sensor is implemented by a piezoelectric film patch attached to the beam around $x = \Delta$. Neglecting the small thickness of the piezoelectric film, the strain on the film can be considered the same as the train on the surface of the beam. Thus, the charge generated in the film is (approximately) proportional to $\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=\Delta}$.

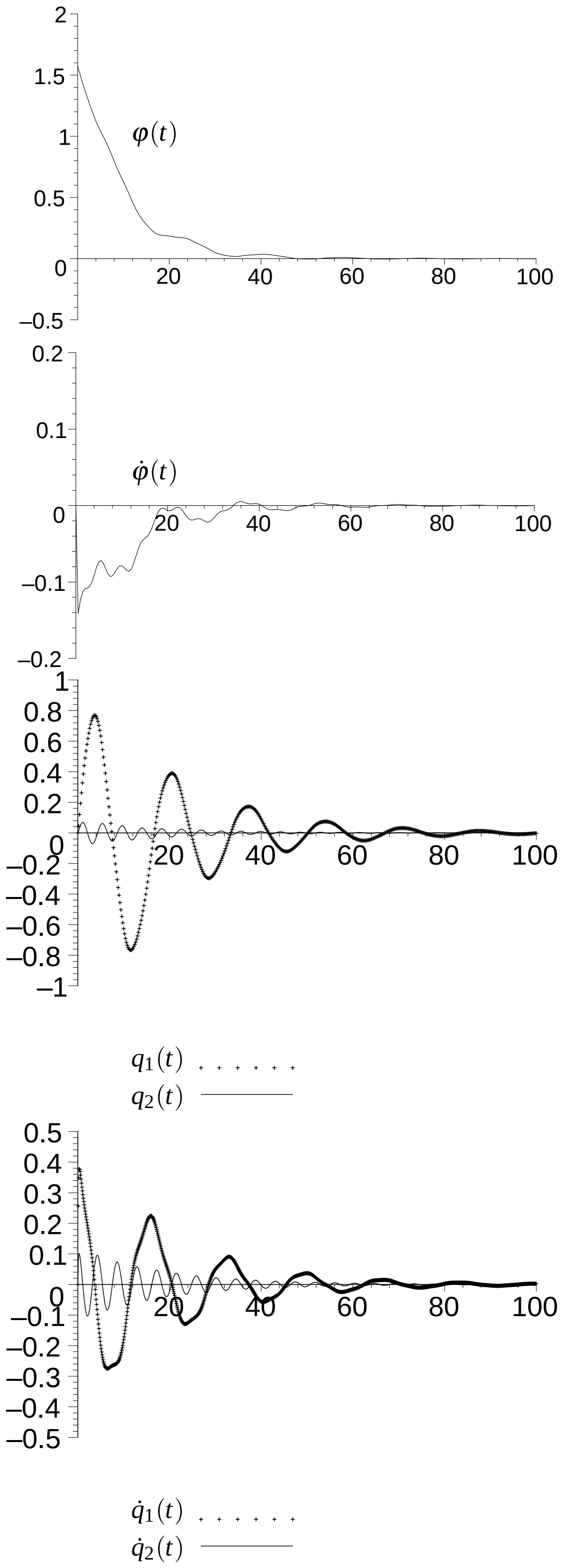

Figure 4: The closed-loop response of (11), (13).

8. CONCLUSION

In this paper, dynamical equations describing planar motion of a manipulator with an arbitrary number of flexible links, interconnected via elastic joints, have been derived. This model is motivated by the aim to improve dynamical properties of a controlled fire rescue turntable ladder. For an arbitrary order of elastic coordinates, we have proposed a feedback controller that stabilizes the origin of the Galerkin approximation (Proposition 2). We do not study the stabilization problem in infinite dimensions, partial asymptotic stability in the resonance case (Zuyev, 2005), and convergency issues in this paper as they require subtle estimates of the spectrum and nonlinear terms for the evolution equation. For the same reason we do not analyze the control spillover (Balas, 1978), leaving it for future work.

REFERENCES

Balas M.J. (1978). Modal control of certain flexible dynamical systems. *SIAM J. Control Optim.* **16**, 450–462.

Bloch A.M. and Titi E.S. (1991). On the dynamics of rotating elastic beams. In: *New Trends in Systems Theory, Proc. Jt. Conf., Genoa/Italy 1990*. Vol. 7. pp. 128–135.

Chen G., Delfour M.C., Krall F.M., and Payre G. (1987). Modeling, stabilization and control of serially connected beams. *SIAM J. Control Optim.* **25**, 526–546.

Coron J.-M. and d'Andrea Novel B. (1998). Stabilization of a rotating body beam without damping. *IEEE Trans. on Autom. Control* **44**, 608–618.

Donea J. and Huerta A. (2003). *Finite Element Methods for Flow Problems*. Wiley. Chichester.

Goldstein H. (1980). *Classical mechanics. 2nd ed.*. Addison-Wesley. Massachusets.

Kovalev A.M., Zuyev A.L., and Shcherbak V.F. (2002). Synthesis of a stabilizing feedback control for a rigid body with elastic attachments. *Journal of Automation and Information Sciences* **34**(11), 5–16.

Luo Z.-H. and Guo B.-Z. (1997). Shear force feedback control of a single-link flexible robot with a revolute joint. *IEEE Trans. on Autom. Control* **42**, 53–65.

Luo Z.-H., Guo B.-Z., and Morgul O. (1999). *Stability and Stabilization of Infinite Dimensional Systems*. Springer-Verlag. London.

Sawodny O., Aschemann H., and Bulach A. (2002). Mechatronical designed control of fire-rescue turntable ladders as flexible link robots. *Proc. 15th IFAC World Congress, Barcelona, Spain*.

Talebi H.A., Patel R.V., and Khorasani K. (2001). *Control of Flexible-link Manipulators Using Neural Networks*. Springer-Verlag. London.

Wonham W.M. (1985). *Linear multivariable control: a geometric approach (3rd ed.)*. Springer. New York.

Xu C.Z. and Baillieul J. (1993). Stabilizability and stabilization of a rotating body-beam system with torque control. *IEEE Trans. on Autom. Control* **38**, 1754–1765.

Zuyev A.L. (2005). Partial asymptotic stabilization of nonlinear distributed parameter systems. *Automatica* **41**, 1–10.