A NEW ALGORITHM FOR SEARCHING RECIPROCAL MATRICES IN LMI BASED CONTROL DESIGN [1]

Dmitry V. Balandin * **and Mark M. Kogan** **

* *Department of Computational Mathematics, Nizhny Novgorod State University, Gagarin Ave., 23, Nizhny Novgorod 603950, Russia, e-mail: balandin@pmk.unn.runnet.ru*
** *Department of Mathematics, Architecture and Civil Engineering State University, Il'yinskaya str., 65, Nizhny Novgorod 603950, Russia, fax: +7 8312 305348, e-mail: mkogan@nngasu.ru*

Abstract: This paper deals with a new heuristic algorithm for reduced-order control problems using linear matrix inequalities. The algorithm is implemented in MATLAB as an iterative procedure on each iteration of which the problem of minimizing a linear function under LMI constraints is solved. The convergence of this procedure is proved. Numerical results for an inverted pendulum are given.

Keywords: reduced-order controllers, linear matrix inequalities, reciprocal matrices

1. INTRODUCTION

Linear matrix inequality (LMI) approach has been developed by many researchers (see, for example, Gahinet and Apkarian (1994), and Iwasaki and Skelton (1994)) for synthesizing controllers satisfying various performance specifications. It turned out that in framework of this approach the fixed-order control design is a nonconvex problem. For computational solution of this problem, some heuristic algorithms have been proposed by El Ghaoui et al.(1997), Iwasaki (1999), Apkarian and Tuan (2000) and others. In particular, this problem is reduced to finding two reciprocal matrices satisfying LMI's.

In this paper, we suggest a new algorithm for solving such a problem obtained by modification of the algorithm proposed in Balandin and Kogan (2004). The algorithm is implemented in MATLAB as an iterative procedure on each iteration of which the problem of minimizing a linear function under LMI constraints is solved. The convergence of this procedure is proved. Numerical results for an inverted pendulum are given.

2. PROBLEM STATEMENT

Consider a linear time-invariant controlled system described by the equations

$$\begin{aligned} \dot{x} &= Ax + Bu \ , \\ y &= Cx \ , \end{aligned} \tag{1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input, $y \in R^l$ is the measurement output. It is

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required to construct a linear dynamic controller of the k-th order in the form

$$\begin{aligned} \dot{x}_r &= A_r x_r + B_r y \ , \\ u &= C_r x_r + D_r y \ , \end{aligned} \tag{2}$$

where $x_r \in R^k$ is the controller state, to provide asymptotic stability for the closed-loop system

$$\dot{x}_c = A_c x_c \ , \quad A_c = \begin{pmatrix} A + BD_rC & BC_r \\ B_rC & A_r \end{pmatrix} , \tag{3}$$

where $x_c = \operatorname{col}(x, x_r)$, with a given degree of stability $\beta/2$. In particular, for $k = 0$ we get a static output controller $u = D_r y$.

3. PRELIMINARY

In accordance with the LMI approach the control objective is reformulated as a Lyapunov inequality

$$\dot{V} + \beta V < 0$$

for $V(x_c) = x_c^T X x_c$ with $X^T = X > 0$, which is equivalent to the matrix inequality

$$A_c^T X + X A_c + \beta X < 0 \ . \tag{4}$$

The controller parameters

$$\Theta = \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} \tag{5}$$

are introduced so as

$$\begin{aligned} A_c = A_0 + B_0 \Theta C_0 \ , \quad A_0 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \ , \\ B_0 = \begin{pmatrix} 0 & B \\ I & 0 \end{pmatrix} \ , \quad C_0 = \begin{pmatrix} 0 & I \\ C & 0 \end{pmatrix} \ . \end{aligned} \tag{6}$$

Then (4) is represented in the form of LMI

$$\Psi + P^T \Theta^T Q + Q^T \Theta P < 0 \tag{7}$$

with respect to the unknown parameters Θ, where

$$\begin{aligned} \Psi &= A_0^T X + X A_0 + \beta X \ , \\ P &= C_0 \ , \quad Q = B_0^T X \ . \end{aligned} \tag{8}$$

According to Lemma (see Gahinet and Apkarian (1994), and Iwasaki and Skelton (1994)) the inequality (7) holds if and only if

$$W_P^T \Psi W_P < 0 \ , \quad W_Q^T \Psi W_Q < 0 \ , \tag{9}$$

where W_S denotes any matrix whose columns form bases of the null bases of S. In the present case,

$$W_P = W_{C_0} \ , \quad W_Q = W_{B_0^T X} = X^{-1} W_{B_0^T} \ .$$

Substituting these expressions into (9), one gets the following statement.

Proposition 1. The problem of stabilization of the plant (1) with the degree of stability $\beta/2$ by means of the k-th order controller (2) is feasible if and only if there exists $X > 0$ satisfying the inequalities

$$\begin{aligned} W_{C_0}^T (A_0^T X + X A_0 + \beta X) W_{C_0} < 0 \ , \\ W_{B_0^T}^T (X^{-1} A_0^T + A_0 X^{-1} + \beta X^{-1}) W_{B_0^T} < 0. \end{aligned} \tag{10}$$

If the conditions (10) hold and such a matrix X has been found, the controller parameters are solutions to LMI (7), where Ψ, P, and Q are given in (8).

Introduce the matrix $Y = X^{-1}$ and rewrite (10) in the form of LMIs with respect to X and Y:

$$\Phi_1(X) < 0 \ , \quad \Phi_2(Y) < 0 \ , \tag{11}$$

where

$$\begin{aligned} \Phi_1(X) &= W_{C_0}^T (A_0^T X + X A_0 + \beta X) W_{C_0} \ , \\ \Phi_2(Y) &= W_{B_0^T}^T (Y A_0^T + A_0 Y + \beta Y) W_{B_0^T} \ . \end{aligned}$$

Then the problem under consideration is reduced to search for reciprocal matrices X and Y ($XY = I$) satisfying LMIs (11).

Remark 1. Taking into account the block structure of A_0, B_0, C_0 and representing X and Y in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix} \ , \quad Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{pmatrix} , \tag{12}$$

the inequalities (11) can be represented as

$$\begin{aligned} W_C^T (A^T X_{11} + X_{11} A + \beta X_{11}) W_C < 0 \ , \\ W_{B^T}^T (Y_{11} A^T + A Y_{11} + \beta Y_{11}) W_{B^T} < 0 \ . \end{aligned} \tag{13}$$

4. A NEW ALGORITHM

Problem 1: find two reciprocal matrices X and Y ($XY = I$) satisfying the LMIs $\Phi_1(X) < 0$, $\Phi_2(Y) < 0$.

To solve it consider another problem.

Problem 2: find

$$\begin{aligned} \lambda_{min} = \min\{\lambda : X - Y^{-1} < \lambda I, \, X > 0, \, Y > 0, \\ \Phi_1(X) < 0 \ , \Phi_2(Y) < 0 \ , \Phi_3(X, Y) < 0\} \ , \end{aligned}$$

where

$$\Phi_3(X, Y) = \begin{pmatrix} -X & I \\ I & -Y \end{pmatrix} \ .$$

In this problem, $\lambda_{min} \geq 0$ due to the inequality $\Phi_3(X, Y) < 0$. Equality holds if and only if X and Y are reciprocal matrices, and these are then solutions to the Problem 1 as well.

To solve the Problem 2 it is required to minimize a linear function under constraints one of which

$$X - Y^{-1} < \lambda I \tag{14}$$

is non-convex and, consequently, cannot be presented as LMI. For this problem we suggest an optimization algorithm that involves an iterative procedure at each step of which a minimum of a linear function subject to LMI constraints is found.

To this end, consider one more problem.

Problem 3: find

$$\lambda_{min} = \min\{\lambda : \Gamma(X,Y,G_1,G_2) < \lambda I,\ X > 0,$$
$$Y > 0\ , \Phi_1(X) < 0\ , \Phi_2(Y) < 0\ , \Phi_3(X,Y) < 0\},$$

where

$$\Gamma(X,Y,G_1,G_2) = (I\,G_1)\begin{pmatrix} X & I \\ I & Y \end{pmatrix}\begin{pmatrix} I \\ G_1 \end{pmatrix} +$$
$$(G_2\,I)\begin{pmatrix} X & I \\ I & Y \end{pmatrix}\begin{pmatrix} G_2 \\ I \end{pmatrix}\ ,$$

and $G_i = G_i^T$, $i = 1,2$ are some specified matrices.

Note that in the Problem 3, instead of the inequality (14) the LMI with respect to X and Y is taken. Furthermore, because of

$$\begin{aligned} &\Gamma(X,Y,G_1,G_2) = \\ &(G_1 + Y^{-1})Y(G_1 + Y^{-1}) + \\ &+(G_2 + X^{-1})X(G_2 + X^{-1}) + \\ &(X - Y^{-1}) + (Y - X^{-1}) \geq 0 \end{aligned} \tag{15}$$

under $\Phi_3(X,Y) < 0$, we have that, for $G_1 = -Y^{-1}$ and $G_2 = -X^{-1}$, when $\lambda_{min} = 0$, the corresponding X and Y are solutions to the Problem 1.

The algorithm consists of the following steps:

1. Set $j = 0$.
2. Fix $G_1 = G_1^{(j)}$ and $G_2 = G_2^{(j)}$.
3. Solve the Problem 3 by means of the command mincx in MATLAB and find its solutions λ_{j+1}, X_j, Y_j.
4. Set $G_1^{(j+1)} = -Y_j^{-1}$, $G_2^{(j+1)} = -X_j^{-1}$ and go to step 2 for $j = j+1$.

The process is assumed to be terminated when, at least, one of the following two inequalities $\lambda_j < \varepsilon$ or $|\lambda_{j+1} - \lambda_j| < \varepsilon$ holds, where $\varepsilon > 0$ is a given accuracy.

Theorem 1. For any initial matrices $G_1^{(0)}$ and $G_2^{(0)}$, the sequence λ_j generated by the algorithm is nondecreasing and

$$\lim_{j\to\infty} \lambda_j = \lambda_* \geq 0,\ \lim_{j\to\infty} X_j = X_*,\ \lim_{j\to\infty} Y_j = Y_*.$$

Proof. Evaluate variations of spectral radius ρ of the matrix function $\Gamma(X,Y,G_1,G_2)$ along algorithm trajectories. Let us present

$$\Delta\rho = \rho(\Gamma(X_{j+1}, Y_{j+1}, G_1^{(j+1)}, G_2^{(j+1)})) -$$
$$\rho(\Gamma(X_j, Y_j, G_1^{(j)}, G_2^{(j)}))$$

in the form

$$\Delta\rho = \Delta\rho_1 + \Delta\rho_2 =$$
$$[\rho(\Gamma(X_{j+1}, Y_{j+1}, G_1^{(j+1)}, G_2^{(j+1)})) -$$
$$\rho(\Gamma(X_j, Y_j, G_1^{(j+1)}, G_2^{(j+1)}))] +$$
$$[\rho(\Gamma(X_j, Y_j, G_1^{(j+1)}, G_2^{(j+1)})) -$$
$$\rho(\Gamma(X_j, Y_j, G_1^{(j)}, G_2^{(j)}))]\ .$$

Due to the algorithm the expression in the first square brackets is nonpositive since λ takes its minimal value for $X = X_{j+1}$, $Y = Y_{j+1}$. From (15) it follows that

$$\Gamma(X_j, Y_j, G_1^{(j+1)}, G_2^{(j+1)}) -$$
$$\Gamma(X_j, Y_j, G_1^{(j)}, G_2^{(j)}) =$$
$$(G_1^{(j+1)} + Y_j^{-1})Y_j(G_1^{(j+1)} + Y_j^{-1}) +$$
$$(G_2^{(j+1)} + X_j^{-1})X_j(G_2^{(j+1)} + X_j^{-1}) -$$
$$(G_1^{(j)} + Y_j^{-1})Y_j(G_1^{(j)} + Y_j^{-1}) -$$
$$(G_2^{(j)} + X_j^{-1})X_j(G_2^{(j)} + X_j^{-1})\ .$$

Now, in view of $G_1^{(j+1)} = -Y_j^{-1}$, $G_2^{(j+1)} = -X_j^{-1}$, we get

$$\Gamma(X_j, Y_j, G_1^{(j+1)}, G_2^{(j+1)}) -$$
$$\Gamma(X_j, Y_j, G_1^{(j)}, G_2^{(j)}) =$$
$$-(Y_j^{-1} - Y_{j-1}^{-1})Y_j(Y_j^{-1} - Y_{j-1}^{-1})$$
$$-(X_j^{-1} - X_{j-1}^{-1})X_j(X_j^{-1} - X_{j-1}^{-1}) \leq 0\ .$$

Since $A - B \leq 0$ implies $\rho(A) \leq \rho(B)$, we obtain $\Delta\rho \leq 0$. Therefore the sequence ρ_j is nonincreasing and bounded from below and, consequently, the limits mentioned in this theorem exist.

From the theorem it follows that the algorithm may result in one of two possible situations. If $\lambda_* = 0$, then $X_* Y_* = I$ and X_*, Y_* are solutions to the Problem 1. In this case, the control problem is feasible. If $\lambda_* > 0$, one cannot make a decision regarding feasibility of the above problem since the convergence of this algorithm to the global minimum in the Problem 3 is not guaranteed. In the latter situation, it is recommended, for example, to repeat the above process for other initial values of $G_1^{(0)}$ and $G_2^{(0)}$ as it is usually done in global optimization. The efficiency of the proposed algorithm will be demonstrated in the next section.

5. COMPUTATIONAL RESULTS

As an example we consider synthesizing a first order output controller for stabilizing (with the degree of stability $\beta/2 = 0.005$) inverted pendulum described by the equation

$$\ddot{\varphi} - \varphi = u$$

with the measured variable $y = \varphi$. In this case, we have

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad C = (1 \quad 0) \, .$$

We randomly generated the symmetric matrix $G_1^{(0)}$ with elements within the interval $[-1, 1]$ and chosen $G_2^{(0)} = [G_1^{(0)}]^{-1}$. The algorithm started 1000 times. The accuracy ε was equal to 10^{-6}. The algorithm was always successful and, in 990 cases, less than or equal to 7 iterations were needed. For example, for

$$G_1^{(0)} = \begin{pmatrix} 0.9 & -0.538 & 0.214 \\ -0.538 & -0.028 & 0.783 \\ 0.214 & 0.783 & 0.524 \end{pmatrix} ,$$

after 3 iterations the algorithm terminated and the differences between elements of the matrix XY and those of the unit matrix were 10^{-8} order.

For comparison, we examined this problem using the popular algorithm of El Ghaoui et al.(1997) that minimizes trace of the matrix XY under LMI constraints. After 100 iterations of this algorithm we obtained

$$XY = \begin{pmatrix} 1.01 & -0005 & 0 \\ -0.002 & 1.01 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

which is considerably worse than our result after three iterations.

6. CONCLUSION

In this paper, we suggested a new algorithm of searching reciprocal matrices satisfying LMI's and justified its convergence. The computational experiments confirmed its effectiveness. This algorithm can be utilized for various control design problems which are formulated in terms of LMI's plus nonconvex constraint.

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