DISCRETE SYSTEM ORDER-REDUCTION VIA BALANCING TRANSFORMATION USING SINGULAR PERTURBATIONS

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Abstract: Order-reduction techniques based on system balancing are developed for linear discrete systems using the method of singular perturbations. The generalized residualization technique is introduced with three variants as improved tools for order-reduction at low and medium frequencies. The newly introduced techniques have the same theoretical upper error bound with respect to the H_∞ norm of the reduced-order system as the existing techniques, but simulation results show the superiority of the new techniques at low and medium frequencies. Next, new techniques are proposed to perform order-reduction of lightly damped linear discrete-time systems by employing the method of singular perturbations and balancing transformation.

Keywords: Discrete system, system balancing, robust order reduction, singular perturbations, residualization.

1. INTRODUCTION

Internally balanced realizations of linear time-invariant systems were introduced by Mullis and Roberts (1976). The balancing transformation was used by Moore (1981) to develop the truncated balanced realization order-reduction method. Pernebo and Silverman (1982) provided a sufficient condition for asymptotic stability of the truncated system, which was generalized by Lam and Hung (1997). Pernebo and Silverman (1982) also showed that truncated continuous systems are balanced and minimal. The error bound for continuous systems was derived by Glover (1984) and Enns (1984). For discrete systems, an error bound was established by Al-Saggaf and Franklin (1987), which was improved by Hinrichsen and Pritchard (1990) to counterpart Glover's continuous error bound. Sreeram and Agathoklis (1989) pointed out that the balanced truncation provides a good approximation of the high frequency behaviour, but gives a poor approximation of the low frequency behaviour in the form of a large steady state error for the step response. More details related to continuous system order-reduction by balanced truncation can be found in the recent survey presented by Gugercin and Antoulas (2004).

Liu and Anderson (1989) proposed the balanced residualization order-reduction technique. They applied the technique to continuous and discrete systems. Order-reduction by residualizing continuous balanced realizations was also investigated by Samar, *et al* (1995). The results of Liu and Anderson (1989) for continuous systems were generalized by Gajic and Lelic (2001) and Lelic (2002) using the method of singular perturbations. They developed the generalized balanced residualization method along with some of its variants, and proposed an approximate order-reduction technique based on the fast subsystem.

This paper investigates several techniques for *discrete* linear systems order-reduction based on system balancing by employing singular perturbation method. These techniques counterpart the techniques presented by Gajic and Lelic (2001) and Lelic (2002), derived for order reduction of *continuous* linear systems.

1.1 Balancing Transformation for Discrete Systems

Consider the following *n*th order discrete linear time-invariant system

$$\begin{aligned}\mathbf{x}[k+1] &= \mathbf{\Phi}\mathbf{x}[k]+\mathbf{\Gamma}\mathbf{u}[k], \qquad \mathbf{x}[k_0]=\mathbf{x}_0 \\ \mathbf{y}[k] &= \mathbf{H}\mathbf{x}[k]+\mathbf{E}\mathbf{u}[k]\end{aligned} \tag{1}$$

with $\mathbf{x}\in R^n, \mathbf{u}\in R^m$, and $\mathbf{y}\in R^p$ representing the system state, input and output respectively. The transfer function of this system is

$$\mathbf{G}[z]=\mathbf{H}(z\mathbf{I}-\mathbf{\Phi})^{-1}\mathbf{\Gamma}+\mathbf{E} \tag{2}$$

The system is assumed to be asymptotically stable with a minimal realization transfer function; that is the pair ($\mathbf{\Phi}$, $\mathbf{\Gamma}$) is controllable and the pair ($\mathbf{\Phi}$, $\mathbf{H}$) is observable. The controllability Grammian $\mathbf{P}$ and the observability Grammian $\mathbf{Q}$ of the system are defined as the solutions to the algebraic Lyapunov equations

$$\begin{aligned}\mathbf{\Phi}\mathbf{P}\mathbf{\Phi}^T-\mathbf{P}+\mathbf{\Gamma}\mathbf{\Gamma}^T=0 \\ \mathbf{\Phi}^T\mathbf{Q}\mathbf{\Phi}-\mathbf{Q}+\mathbf{H}^T\mathbf{H}=0\end{aligned} \tag{3}$$

The solutions of these equations may be represented by (Gajic and Qureshi, 1995)

$$\begin{aligned}\mathbf{P} &= \sum_{k=0}^{\infty}\mathbf{\Phi}^k\mathbf{\Gamma}\mathbf{\Gamma}^T(\mathbf{\Phi}^T)^k \\ \mathbf{Q} &= \sum_{k=0}^{\infty}(\mathbf{\Phi}^T)^k\mathbf{H}^T\mathbf{H}\mathbf{\Phi}^k\end{aligned} \tag{4}$$

The discrete system is *internally balanced* if

$$\mathbf{P}=\mathbf{Q}=\mathbf{\Sigma}=diag\{\sigma_1,\sigma_2,\cdots,\sigma_n\} \tag{5}$$

with $\sigma_1\ge\sigma_2\ge\cdots\ge\sigma_n>0$.

The elements of the diagonal of the Grammian matrix $\mathbf{\Sigma}$ are known as the Hankel singular values. For the balanced system the algebraic Lyapunov equations in (3) become

$$\begin{aligned}\mathbf{\Phi}\mathbf{\Sigma}\mathbf{\Phi}^T-\mathbf{\Sigma}+\mathbf{\Gamma}\mathbf{\Gamma}^T=0 \\ \mathbf{\Phi}^T\mathbf{\Sigma}\mathbf{\Phi}-\mathbf{\Sigma}+\mathbf{H}^T\mathbf{H}=0\end{aligned} \tag{6}$$

If the system is not internally balanced, a nonsingular state transformation is applied to the system to make its controllability and observability Grammians identical and diagonal. This technique is called the *balancing transform* (Mullis and Roberts, 1976; Moore, 1981). The new balanced system is represented by

$$\begin{aligned}\mathbf{x}_b[k+1] &= \mathbf{\Phi}_b\mathbf{x}_b[k]+\mathbf{\Gamma}_b\mathbf{u}[k] \\ \mathbf{y}_b[k] &= \mathbf{H}_b\mathbf{x}_b[k]+\mathbf{E}_b\mathbf{u}[k]\end{aligned} \tag{7}$$

where

$$\begin{aligned}\mathbf{x}_b[k]=\mathbf{T}\mathbf{x}[k], \qquad \det(\mathbf{T})\neq 0 \\ \mathbf{\Phi}_b=\mathbf{T}\mathbf{\Phi}\mathbf{T}^{-1},\ \mathbf{\Gamma}_b=\mathbf{T}\mathbf{\Gamma},\ \mathbf{H}_b=\mathbf{H}\mathbf{T}^{-1},\ \mathbf{E}_b=\mathbf{E}\end{aligned} \tag{8}$$

For a controllable and observable system, the balanced system is also controllable and observable. The transfer function of the balanced system is the same as the transfer function of the original system since the state transformation is nonsingular.

$$\mathbf{G}_b[z]=\mathbf{H}_b(z\mathbf{I}-\mathbf{\Phi}_b)^{-1}\mathbf{\Gamma}_b+\mathbf{E}_b=\mathbf{G}[z] \tag{9}$$

1.2 System Order-Reduction via Direct Truncation

It was observed that the states that are associated with small Hankel singular values are weakly controllable and weakly observable, thus such states can be removed to produce a reduced-order system.

Consider the balanced discrete system defined in (7). The system can be partitioned as

$$\begin{aligned}&\mathbf{\Phi}_b=\begin{bmatrix}\mathbf{\Phi}_{11} & \mathbf{\Phi}_{12}\\ \mathbf{\Phi}_{21} & \mathbf{\Phi}_{22}\end{bmatrix},\ \mathbf{\Gamma}_b=\begin{bmatrix}\mathbf{\Gamma}_1\\ \mathbf{\Gamma}_2\end{bmatrix}, \\ &\mathbf{H}_b=[\mathbf{H}_1 \quad \mathbf{H}_2],\ \mathbf{E}_b=\mathbf{E},\ \mathbf{\Sigma}=\begin{bmatrix}\mathbf{\Sigma}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{\Sigma}_2\end{bmatrix}, \\ &\mathbf{\Sigma}_1=diag\{\sigma_1,\cdots,\sigma_r\},\ \mathbf{\Sigma}_2=diag\{\sigma_{r+1},\cdots,\sigma_n\}\end{aligned} \tag{10}$$

with $\mathbf{\Phi}_{11}$ and $\mathbf{\Sigma}_1$ are both $r\times r$ ($r < n$) matrices, $\mathbf{\Gamma}_1$ is $r\times m$ matrix, and $\mathbf{H}_1$ is $p\times r$ matrix.

Having $\sigma_r >> \sigma_{r+1}$, the reduced-order system of order r, obtained via the balancing truncation, is defined by

$$\begin{aligned}\mathbf{x}_1[k+1] &= \mathbf{\Phi}_{11}\mathbf{x}_1[k]+\mathbf{\Gamma}_1\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{H}_1\mathbf{x}_1[k]+\mathbf{E}\mathbf{u}[k]\end{aligned} \tag{11}$$

The transfer function of the reduced-order system is

$$\mathbf{G}_{tr}[z]=\mathbf{H}_1(z\mathbf{I}-\mathbf{\Phi}_{11})^{-1}\mathbf{\Gamma}_1+\mathbf{E} \tag{12}$$

The reduced-order system is asymptotically stable, controllable and observable (Liu and Anderson, 1989). Unlike the continuous case, the system defined in (11) is not balanced (Fernando and Nicholson, 1983). The H_∞ norm of the reduced-order system satisfies (Liu and Anderson, 1989)

$$\begin{aligned}\|\mathbf{G}[z]-\mathbf{G}_{tr}[z]\|_\infty &\le 2trace(\mathbf{\Sigma}_2) \\ &\le 2(\sigma_{r+1}+\sigma_{r+2}+\cdots+\sigma_n)\end{aligned} \tag{13}$$

The DC gain of the reduced-order system is different from the DC gain of the full-order system. This can be fixed by the discrete *corrected truncation method* similar to the continuous-time result of (Gajic and Lelic 2001)

$$\begin{aligned}\mathbf{G}_{tr}^{corr}[z]=\mathbf{H}_1(z\mathbf{I}-\mathbf{\Phi}_{11})^{-1}\mathbf{\Gamma}_1-\mathbf{H}_1(\mathbf{I}-\mathbf{\Phi}_{11})^{-1}\mathbf{\Gamma}_1 \\ +\mathbf{H}_b(\mathbf{I}-\mathbf{\Phi}_b)^{-1}\mathbf{\Gamma}_b+\mathbf{E}\end{aligned} \tag{14}$$

1.3 System Order-Reduction via Balanced Residualization

Consider the internally balanced linear time-invariant asymptotically stable discrete system in (7) that is partitioned as is (10), with the Hankel singular values satisfying $\sigma_r >> \sigma_{r+1}$. This system can be written as

$$\begin{aligned}\mathbf{x}_s[k+1] &= \mathbf{\Phi}_{11}\mathbf{x}_s[k]+\mathbf{\Phi}_{12}\mathbf{x}_f[k]+\mathbf{\Gamma}_1\mathbf{u}[k] \\ \mathbf{x}_f[k+1] &= \mathbf{\Phi}_{21}\mathbf{x}_s[k]+\mathbf{\Phi}_{22}\mathbf{x}_f[k]+\mathbf{\Gamma}_2\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{H}_1\mathbf{x}_s[k]+\mathbf{H}_2\mathbf{x}_f[k]+\mathbf{E}\mathbf{u}[k]\end{aligned} \tag{15}$$

The state vector $\mathbf{x}_s[k]$ is slow and the state vector $\mathbf{x}_f[k]$ is fast. The matrix $\mathbf{\Phi}_{22}$ is asymptotically stable and hence invertible.

Since $\mathbf{x}_f[k]$ is fast compared to $\mathbf{x}_s[k]$, the state vector $\mathbf{x}_f[k]$ is residualized by setting $\mathbf{x}_f[k+1] = \mathbf{x}_f[k]$. Solving for $\mathbf{x}_f[k]$ and substituting, we get the slow reduced-order system

$$\begin{aligned}\mathbf{x}_s[k+1] &= \mathbf{\Phi}_s\mathbf{x}_s[k] + \mathbf{\Gamma}_s\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{H}_s\mathbf{x}_s[k] + \mathbf{E}_s\mathbf{u}[k]\end{aligned} \tag{16}$$

where

$$\begin{aligned}\mathbf{\Phi}_s &= \mathbf{\Phi}_{11} + \mathbf{\Phi}_{12}(\mathbf{I}-\mathbf{\Phi}_{22})^{-1}\mathbf{\Phi}_{21} \\ \mathbf{\Gamma}_s &= \mathbf{\Gamma}_1 + \mathbf{\Phi}_{12}(\mathbf{I}-\mathbf{\Phi}_{22})^{-1}\mathbf{\Gamma}_2 \\ \mathbf{H}_s &= \mathbf{H}_1 + \mathbf{H}_2(\mathbf{I}-\mathbf{\Phi}_{22})^{-1}\mathbf{\Phi}_{21} \\ \mathbf{E}_s &= \mathbf{E} + \mathbf{H}_2(\mathbf{I}-\mathbf{\Phi}_{22})^{-1}\mathbf{\Gamma}_2\end{aligned} \tag{17}$$

The system in (16) is called a balanced residualization of the system in (15).

Liu and Anderson (1989) showed that for the minimal linear time-invariant discrete system defined by (15), (which is assumed to be asymptotically stable and internally balanced) its slow singular perturbation approximation defined by (16) is also minimal, internally balanced with Grammian $\mathbf{\Sigma}_1$, and asymptotically stable if $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ have no diagonal elements in common.

The transfer function of the reduced-order system is

$$\mathbf{G}_{br}[z] = \mathbf{H}_s(z\mathbf{I}-\mathbf{\Phi}_s)^{-1}\mathbf{\Gamma}_s + \mathbf{E}_s \tag{18}$$

The residualized reduced-order system has the same H_∞ norm of the truncated reduced-order system

$$\begin{aligned}\|\mathbf{G}[z]-\mathbf{G}_{br}[z]\|_\infty &\le 2trace(\mathbf{\Sigma}_2) \\ &\le 2(\sigma_{r+1}+\sigma_{r+2}+\cdots+\sigma_n)\end{aligned} \tag{19}$$

The DC gain of the reduced-order system is equal to that of the full-order system. This can be proved by using an approach similar to the one used by Samar *et al.* (1995) for the continuous case.

Liu and Anderson (1989) established a relation between continuous and discrete residualized reduced-order systems through a bilinear mapping.

2. DISCRETE SYSTEM ORDER–REDUCTION VIA BALANCING USING THE METHOD OF SINGULAR PERTURBATIONS

The order-reduction methods presented in Section 1 will be implemented in the fast-time scale form of singularly perturbed discrete systems. The generalized balanced residualization method is then presented with three of its variants.

2.1 Singularly Perturbed Discrete Systems

A singularly perturbed discrete system can be represented in two forms depending on the sampling interval. For small sampling intervals the resulting form is the fast-time scale representation given by

$$\begin{aligned}\begin{bmatrix}\mathbf{x}_1[k+1]\\ \mathbf{x}_2[k+1]\end{bmatrix} &= \begin{bmatrix}\mathbf{I}+\varepsilon\mathbf{A}_{11} & \varepsilon\mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1[k]\\ \mathbf{x}_2[k]\end{bmatrix} + \begin{bmatrix}\varepsilon\mathbf{B}_1\\ \mathbf{B}_2\end{bmatrix}\mathbf{u}[k] \\ \mathbf{y}[k] &= \begin{bmatrix}\mathbf{C}_1 & \mathbf{C}_2\end{bmatrix}\begin{bmatrix}\mathbf{x}_1[k]\\ \mathbf{x}_2[k]\end{bmatrix} + \mathbf{D}\mathbf{u}[k]\end{aligned} \tag{20}$$

where ε is a small positive singular perturbation parameter that indicates the separation of the state space variables into slow variables $\mathbf{x}_1[k]$ and fast variables $\mathbf{x}_2[k]$. A comprehensive overview of singular perturbations is given in (Naidu, 2002). Considering the fast-time scale form is more appropriate than considering the slow-time scale form. The reason is that the slow-time scale form assumes the asymptotic stability of the fast-time scale form (Litkouhi and Khalil, 1985).

The discrete system in (7) that is partitioned as in (10) can be expressed in the fast-time scale singularly perturbed form by defining the following matrices

$$\begin{aligned}&\mathbf{A}_{11} = \frac{1}{\varepsilon}(\mathbf{\Phi}_{11}-\mathbf{I}) && \mathbf{B}_1 = \frac{1}{\varepsilon}\mathbf{\Gamma}_1 && \mathbf{C}_1 = \mathbf{H}_1 \\ &\mathbf{A}_{12} = \frac{1}{\varepsilon}\mathbf{\Phi}_{12} && \mathbf{B}_2 = \mathbf{\Gamma}_2 && \mathbf{C}_2 = \mathbf{H}_2 \\ &\mathbf{A}_{21} = \mathbf{\Phi}_{21} && && \mathbf{D} = \mathbf{E} \\ &\mathbf{A}_{22} = \mathbf{\Phi}_{22}\end{aligned} \tag{21}$$

where $\mathbf{I}$ is an identity matrix with compatible dimensions.

2.2 Discrete System Order-Reduction via Direct Truncation and Balanced Residualization

Consider the internally balanced singularly perturbed discrete system in (20). The balancing truncation method described in Section 1.2 can be used to obtain the reduced-order system

$$\begin{aligned}\mathbf{x}_1[k+1] &= (\mathbf{I}+\varepsilon\mathbf{A}_{11})\mathbf{x}_1[k] + \varepsilon\mathbf{B}_1\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{C}_1\mathbf{x}_1[k] + \mathbf{D}\mathbf{u}[k]\end{aligned} \tag{22}$$

The transfer function of the reduced-order system is

$$\mathbf{G}_{tr}[z] = \varepsilon\mathbf{C}_1((z-1)\mathbf{I}-\varepsilon\mathbf{A}_{11})^{-1}\mathbf{B}_1 + \mathbf{D} \tag{23}$$

and the DC gain is

$$\mathbf{g}_{tr} = -\mathbf{C}_1\mathbf{A}_{11}^{-1}\mathbf{B}_1 + \mathbf{D} \tag{24}$$

The discrete corrected truncation can be applied to fix the DC gain of the reduced-order system.

$$\begin{aligned}\mathbf{G}_{tr}^{corr}[z] &= \varepsilon\mathbf{C}_1((z-1)\mathbf{I}-\varepsilon\mathbf{A}_{11})^{-1}\mathbf{B}_1 + \mathbf{C}_1\mathbf{A}_{11}^{-1}\mathbf{B}_1 \\ &+ \mathbf{C}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\end{aligned} \tag{25}$$

Applying the balanced residualization to system (20) results in the slow reduced-order system

$$\begin{aligned}\mathbf{x}_s[k+1] &= (\mathbf{I}+\varepsilon\mathbf{A}_s)\mathbf{x}_s[k] + \varepsilon\mathbf{B}_s\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{C}_s\mathbf{x}_s[k] + \mathbf{D}_s\mathbf{u}[k]\end{aligned} \tag{26}$$

where

$$\begin{aligned}\mathbf{A}_s &= \mathbf{A}_{11} + \mathbf{A}_{12}(\mathbf{I}-\mathbf{A}_{22})^{-1}\mathbf{A}_{21} \\ \mathbf{B}_s &= \mathbf{B}_1 + \mathbf{A}_{12}(\mathbf{I}-\mathbf{A}_{22})^{-1}\mathbf{B}_2 \\ \mathbf{C}_s &= \mathbf{C}_1 + \mathbf{C}_2(\mathbf{I}-\mathbf{A}_{22})^{-1}\mathbf{A}_{21} \\ \mathbf{D}_s &= \mathbf{D} + \mathbf{C}_2(\mathbf{I}-\mathbf{A}_{22})^{-1}\mathbf{B}_2\end{aligned} \tag{27}$$

The transfer function of the reduced-order system is

$$\mathbf{G}_{br}[z] = \varepsilon\mathbf{C}_s\left((z-1)\mathbf{I} - \varepsilon\mathbf{A}_s\right)^{-1}\mathbf{B}_s + \mathbf{D}_s \tag{28}$$

The DC gain is

$$\mathbf{g}_{br} = -\mathbf{C}_s\mathbf{A}_s^{-1}\mathbf{B}_s + \mathbf{D}_s \tag{29}$$

2.3 Discrete System Order-Reduction via Generalized Residualization

The order-reduction technique introduced here is a generalization for the results obtained from the balanced residualization method. The generalized residualization method will be applied to balanced singularly perturbed systems represented in the fast-time scale form as in (20). The next step is to decouple this system by using the Chang transformation (Chang, 1972) defined by

$$\begin{aligned}\begin{bmatrix}\mathbf{z}_1[k]\\ \mathbf{z}_2[k]\end{bmatrix} &= \begin{bmatrix}\mathbf{I}-\varepsilon\mathbf{HL} & -\varepsilon\mathbf{H}\\ \mathbf{L} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1[k]\\ \mathbf{x}_2[k]\end{bmatrix}\\ \begin{bmatrix}\mathbf{x}_1[k]\\ \mathbf{x}_2[k]\end{bmatrix} &= \begin{bmatrix}\mathbf{I} & \varepsilon\mathbf{H}\\ -\mathbf{L} & \mathbf{I}-\varepsilon\mathbf{LH}\end{bmatrix}\begin{bmatrix}\mathbf{z}_1[k]\\ \mathbf{z}_2[k]\end{bmatrix}\end{aligned} \tag{30}$$

The Chang transformation decouples system (20) completely and exactly (up to computer accuracy) into two independent slow and fast subsystems as

$$\begin{aligned}\mathbf{z}_1[k+1] &= (\mathbf{I}+\varepsilon\mathbf{A}_s)\mathbf{z}_1[k] + \varepsilon\mathbf{B}_s\mathbf{u}[k]\\ \mathbf{z}_2[k+1] &= \mathbf{A}_f\mathbf{z}_2[k] + \mathbf{B}_f\mathbf{u}[k]\\ \mathbf{y}[k] &= \mathbf{y}_s[k] + \mathbf{y}_f[k]\end{aligned} \tag{31}$$

where

$$\begin{aligned}\mathbf{A}_s &= \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{L}\\ \mathbf{B}_s &= \mathbf{B}_1 - \mathbf{H}(\mathbf{B}_2 + \varepsilon\mathbf{LB}_1)\\ \mathbf{C}_s &= \mathbf{C}_1 - \mathbf{C}_2\mathbf{L}\\ \mathbf{A}_f &= \mathbf{A}_{22} + \varepsilon\mathbf{LA}_{12}\\ \mathbf{B}_f &= \mathbf{B}_2 + \varepsilon\mathbf{LB}_1\\ \mathbf{C}_f &= \mathbf{C}_2 + \varepsilon(\mathbf{C}_1 - \mathbf{C}_2\mathbf{L})\mathbf{H}\end{aligned} \tag{32}$$

and the matrices $\mathbf{L}$ and $\mathbf{H}$ satisfy the algebraic equations

$$\begin{aligned}(\mathbf{A}_{22}-\mathbf{I})\mathbf{L} - \mathbf{A}_{21} - \varepsilon\mathbf{L}(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{L}) = 0\\ \mathbf{H}(\mathbf{A}_{22}-\mathbf{I}) - \mathbf{A}_{12} + \varepsilon[\mathbf{HLA}_{12} - (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{L})\mathbf{H}] = 0\end{aligned} \tag{33}$$

Matrices $\mathbf{L}$ and $\mathbf{H}$ can be computed by solving equations (33) numerically. For small values of ε, either the fixed-point iterations (Kokotovic, *et al.*, 1980) or the Newton method (Grodt and Gajic, 1988) can be used to solve these equations. For relatively large values of ε, the solutions can be found using the eigenvector method (Kecman, *et al.*, 1999). The zeroth-order approximate solutions of equations (33) are

$$\mathbf{L}^{(0)} = (\mathbf{A}_{22}-\mathbf{I})^{-1}\mathbf{A}_{21},\ \mathbf{H}^{(0)} = \mathbf{A}_{12}(\mathbf{A}_{22}-\mathbf{I})^{-1} \tag{34}$$

The slow subsystem output $\mathbf{y}_s[k]$ and the fast subsystem output $\mathbf{y}_f[k]$ are defined as

$$\begin{aligned}\mathbf{y}_s[k] &= \mathbf{C}_s\mathbf{z}_1[k] + \mathbf{Du}[k]\\ \mathbf{y}_f[k] &= \mathbf{C}_f\mathbf{z}_2[k]\end{aligned} \tag{35}$$

The feedforward loop of the system can be included in either subsystem (slow or fast) depending on the nature of the system input. The transfer functions of the slow and fast subsystems are given by

$$\begin{aligned}\mathbf{G}_s[z] &= \varepsilon\mathbf{C}_s\left((z-1)\mathbf{I} - \varepsilon\mathbf{A}_s\right)^{-1}\mathbf{B}_s + \mathbf{D}\\ \mathbf{G}_f[z] &= \mathbf{C}_f\left(z\mathbf{I} - \mathbf{A}_f\right)^{-1}\mathbf{B}_f\end{aligned} \tag{36}$$

The DC gains are

$$\begin{aligned}\mathbf{g}_s &= -\mathbf{C}_s\mathbf{A}_s^{-1}\mathbf{B}_s + \mathbf{D}\\ \mathbf{g}_f &= \mathbf{C}_f\left(\mathbf{I} - \mathbf{A}_f\right)^{-1}\mathbf{B}_f\end{aligned} \tag{37}$$

The *generalized residualization* approximation can be obtained by considering the slow subsystem while approximating the fast subsystem by its DC gain.

$$\begin{aligned}\mathbf{G}_{gr}[z] &= \mathbf{G}_s[z] + \mathbf{g}_f\\ &= \varepsilon\mathbf{C}_s\left((z-1)\mathbf{I} - \varepsilon\mathbf{A}_s\right)^{-1}\mathbf{B}_s + \mathbf{C}_f\left(\mathbf{I} - \mathbf{A}_f\right)^{-1}\mathbf{B}_f + \mathbf{D}\end{aligned} \tag{38}$$

The reduced-order system preserves the DC gain of the original system, that is

$$\begin{aligned}\mathbf{g}_{gr} &= -\mathbf{C}_s\mathbf{A}_s^{-1}\mathbf{B}_s + \mathbf{C}_f\left(\mathbf{I} - \mathbf{A}_f\right)^{-1}\mathbf{B}_f + \mathbf{D}\\ &= \mathbf{C}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\end{aligned} \tag{39}$$

with $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are as defined in (20). This is true because the decoupled system (31) was obtained by applying a nonsingular transformation. The reduced-order system becomes

$$\begin{aligned}\mathbf{x}_s[k+1] &= (\mathbf{I}+\varepsilon\mathbf{A}_s)\mathbf{x}_s[k] + \varepsilon\mathbf{B}_s\mathbf{u}[k]\\ \mathbf{y}_s[k] &= \mathbf{C}_s\mathbf{z}_1[k] + \mathbf{D}_s\mathbf{u}[k]\end{aligned} \tag{40}$$

where

$$\mathbf{D}_s = \mathbf{D} + \mathbf{C}_f\left(\mathbf{I} - \mathbf{A}_f\right)^{-1}\mathbf{B}_f \tag{41}$$

Note that the zeroth-order approximated system (obtained by setting $\varepsilon = 0$) is the same as the reduced-order system obtained via balanced residualization as discussed in Section 1.3.

2.4 Alternative Variants

Several alternative variants of the slow-fast system decomposition and corresponding system order-reduction can be obtained by introducing different changes of variables. A change of variables results in moving some slow components into the fast subsystem or moving some fast components into the slow subsystem. This changes the DC gain for both subsystems but reserves the DC gain of the original system. In this section three variants are presented.

Variant 1. In this variant, the fast subsystem is decoupled from the slow subsystem by introducing the following change of variables to the system defined by (20)

$$\mathbf{z}_2[k] = \mathbf{x}_2[k] + \mathbf{Lx}_1[k] \tag{42}$$

where $\mathbf{L}$ satisfies the algebraic equation

$$(\mathbf{A}_{22}-\mathbf{I})\mathbf{L} - \mathbf{A}_{21} - \varepsilon\mathbf{L}(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{L}) = 0 \tag{43}$$

Approximating the contribution of the fast elements by its DC gain, the transfer function of the reduced-order system resulting from this method becomes

$$\begin{aligned}&\mathbf{G}_{v1}[z] = \varepsilon\mathbf{C}_s\left((z-1)\mathbf{I} - \varepsilon\mathbf{A}_s\right)^{-1}\\ &\left(\mathbf{A}_{12}\left(\mathbf{I} - \mathbf{A}_f\right)^{-1}\mathbf{B}_f + \mathbf{B}_1\right) + \mathbf{C}_2\left(\mathbf{I} - \mathbf{A}_f\right)^{-1}\mathbf{B}_f + \mathbf{D}\end{aligned} \tag{44}$$

The same transfer function can be obtained by applying balanced residualization after the change of variables instead of approximating the fast elements contribution.

Variant 2. The slow subsystem is decoupled from the fast subsystem. The following change of variables is introduced to the system defined by (20)

$$\mathbf{z}_1[k] = \mathbf{x}_1[k] - \varepsilon\mathbf{H}\mathbf{x}_2[k] \qquad (45)$$

where $\mathbf{H}$ satisfies the algebraic equation

$$\mathbf{H}(\mathbf{A}_{22} - \mathbf{I}) - \mathbf{A}_{12} - \varepsilon(\mathbf{A}_{11} - \mathbf{H}\mathbf{A}_{21})\mathbf{H} = 0 \qquad (46)$$

The fast subsystem is approximated by its DC gain and added to the transfer function of the slow subsystem to get the transfer function of the reduced-order system as

$$\begin{aligned}&\mathbf{G}_{v2}[z] = \varepsilon\mathbf{C}_1\big((z-1)\mathbf{I} - \varepsilon(\mathbf{A}_{11} - \mathbf{H}\mathbf{A}_{21})\big)^{-1}\\ &(\mathbf{B}_1 - \mathbf{H}\mathbf{B}_2) + \mathbf{D} + \mathbf{g}_{sf}\end{aligned} \qquad (47)$$

where

$$\begin{aligned}\mathbf{g}_{sf} &= (\mathbf{C}_2 + \varepsilon\mathbf{C}_1\mathbf{H})(\mathbf{I} - \mathbf{A}_{22} - \varepsilon\mathbf{A}_{21}\mathbf{H})^{-1}\\ &\big(\mathbf{B}_2 - \mathbf{A}_{21}(\mathbf{A}_{11} - \mathbf{H}\mathbf{A}_{21})^{-1}(\mathbf{B}_1 - \mathbf{H}\mathbf{B}_2)\big)\end{aligned} \qquad (48)$$

Variant 3. Here, the same change of variables as in Variant 2 is used. The transfer function of the reduced-order subsystem is obtained by applying the balanced residualization to the system (20) after changing its variables.

$$\begin{aligned}&\mathbf{G}_{v3}[z] = \varepsilon\big(\mathbf{C}_1 + (\mathbf{C}_2 + \varepsilon\mathbf{C}_1\mathbf{H})(\mathbf{I} - \mathbf{A}_{22} - \varepsilon\mathbf{A}_{21}\mathbf{H})^{-1}\\ &\mathbf{A}_{21}\big)\big((z-1)\mathbf{I} - \varepsilon(\mathbf{A}_{11} - \mathbf{H}\mathbf{A}_{21})\big)^{-1}(\mathbf{B}_1 - \mathbf{H}\mathbf{B}_2)\\ &+\big((\mathbf{C}_2 + \varepsilon\mathbf{C}_1\mathbf{H})(\mathbf{I} - \mathbf{A}_{22} - \varepsilon\mathbf{A}_{21}\mathbf{H})^{-1}\mathbf{B}_2 + \mathbf{D}\big)\end{aligned} \qquad (49)$$

2.5 Approximation Based on Fast Subsystems

Order-reduction of flexible structure systems has been studied by many researchers (see for example (Gregory, 1984) and (Gawronski and Williams, 1991)). It was pointed out in (Jonckheere and Silverman, 1983) that some unmodeled high-frequency states can lead to a drastic reduction of feedback performance or even instability (spillover problem). It was also stated in (Jonckheere and Silverman, 1983) that the less important fats states in open-loop analysis become more important in closed-loop analysis. This section presents order-reduction techniques based on the fast subsystems.

Reverse Residualization. In this technique the slow states are residualized by setting $\mathbf{x}_s[k+1] = \mathbf{x}_s[k]$ and solving to get

$$\begin{aligned}\mathbf{x}_f[k+1] &= \mathbf{A}_f\mathbf{x}_f[k] + \mathbf{B}_f\mathbf{u}[k]\\ \mathbf{y}[k] &= \mathbf{C}_f\mathbf{x}_f[k] + \mathbf{D}_f\mathbf{u}[k]\end{aligned} \qquad (50)$$

where

$$\begin{aligned}\mathbf{A}_f &= \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\\ \mathbf{B}_f &= \mathbf{B}_2 - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{B}_1\\ \mathbf{C}_f &= \mathbf{C}_2 - \mathbf{C}_1\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\\ \mathbf{D}_f &= \mathbf{D} - \mathbf{C}_1\mathbf{A}_{11}^{-1}\mathbf{B}_1\end{aligned} \qquad (51)$$

Slow-Fast Decoupling. Starting with the slow-fast decoupled system (31), the *fast generalized residualization approximation* is obtained by approximating the slow subsystem by its DC gain and adding it to the fast subsystem.

$$\mathbf{G}_{fgr}[z] = \mathbf{C}_f(z\mathbf{I} - \mathbf{A}_f)^{-1}\mathbf{B}_f - \mathbf{C}_s\mathbf{A}_s^{-1}\mathbf{B}_s + \mathbf{D} \qquad (52)$$

Variants. To derive the fast approximation version of the first and second variants in section 2.4, the same change of variables are used. Then, the contribution of the slow elements is approximated by its DC gain. The counterpart variant of the third variant in section 2.4 is obtained by using the same change of variables (45) and residualizing the slow state vector $\mathbf{z}_1[k]$.

3. SIMULATION RESULTS

Two numerical examples are presented in (Al-Takrouri, 2004). The first example implements the order-reduction methods for low and medium frequencies. Except for the first few seconds, the generalized residualization gives better approximations of the step and impulse responses than the other methods. The reduced-order systems fail to track the high-frequency response of the full-order system. A closer look of the frequency responses shows the superiority of the generalized resdiualization at low and medium frequencies.

The second example implements approximations based on fast subsystems to a flexible structure system. It is observed that the choice of the order of the reduced system affects the accuracy of the approximation techniques. The absence of the poles responsible for the primary oscillations results with a failure of the approximation.

4. CONCLUSIONS

Employing the method of singular perturbations to order-reduction techniques that are based on system balancing enhances the obtained results. Among the various methods that were discussed, the prpposed generalized residualization technique has the best performance at low and medium frequencies.

The steady-state error comprised in the direct truncation technique was corrected by suggesting the corrected truncation technique, but on the expense of a larger error in the transient response.

Order-reduction of lightly damped systems requires more attention. Whether an approximation is based on the fast subsystem or the slow subsystem, the fast states responsible for the primary oscillations should be included. The states responsible for the secondary (and faster) oscillations can be tolerated provided some of them remain in the approximation to preserve the nature of the oscillatory system. The decision of what states to keep and what states to remove depends on the needed accuracy of the approximation.

One way to keep all the states of the oscillatory system is to use the slow-fast decomposition to divide the system into two parallel subsystems, and then design independent reduced-order controllers for each subsystem. The approximations based either on the fast or slow subsystem can be applied. Order-reduction of lightly damped systems based on subsystems gives good approximations to the step and impulse responses of the full-order system. It also gives good results at low, medium and high frequencies, but fails at very high frequencies.

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