MINIMISATION OF TRANSIENT PERTURBATION GROWTH IN LINEARISED LORENZ EQUATIONS

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Abstract: This paper describes the LMI synthesis of feedback controllers which minimise closed loop transient perturbation growth with limited control effort. Controllers are synthesized for the linearised Lorenz equations, and their performance is compared to that of LQR controllers. At low control effort the controllers behave similarly, but the LMI based controllers are able to produce an almost monotonically falling transient with increasing control effort, whereas LQR controllers have a distinct minimum transient. Evidence is found that controllers which produce the lowest transients do not necessarily have the most orthogonal system eigenvectors, and an explanation in terms of modal and non-modal growth components is presented. Both LMI and LQR controllers are able to stabilise the full Lorenz equations for limited initial conditions.

Keywords: LMI, Lorenz equations, LQR, Optimal Control, Transient Responses

1. INTRODUCTION

The transient growth of a stable linear system may be large, and if the system is the result of linearisation the growth may take the non-linear system outside its domain of attraction. Thus it is relevant to use the transient growth as a performance index in such systems as fluid flow control as derived by Bewley and Liu (1998). Based on Boyd and Barratt (1994), an upper bound for transient growth has been derived by Whidborne *et al.* (2004), along with LMI methods for synthesizing controllers which minimise it. Transient growth is related to the orthogonality of the system eigenvectors - a normal system cannot have transient growth above unity, as shown by Whidborne *et al.* (2004), but that of a non-normal system depends both on the system dynamics and the initial conditions.

The equations derived by Lorenz (1963) are a coupled set of non-linear equations representing a simplified model of fluid convection as ordinary rather than partial differential equations and exhibit deterministic but non-periodic chaotic behaviour. The transition of the system from a steady linearly unstable state to bounded chaotic behaviour may be seen as an analogue of the transition from laminar flow to turbulent flow, the control of which is currently the subject of widespread research, for example Bewley and Liu (1998) and Mckernan *et al.* (2004).

The aim of the work described here is to investigate the minimization of transient growth, to examine the relationship between the non-normality of the system eigenvectors and transient growth and to consider the effectiveness of standard linear quadratic control in reducing transient growth for fluid control systems. Many fluid flow system models are extremely complex and of very high order (Mckernan *et al.*, 2003), but the linearised Lorenz system is simple enough to permit expeditious results.

The organisation of this paper is as follows: section 2 describes a method of extending Whidborne *et al.* (2004) LMI based controllers with minimised transient growth to cover limited control effort. Section 3 introduces and linearises the Lorenz equations and section 4 compares the performance of LMI controllers with LQR ones when applied to the linearised Lorenz system. This section also investigates aspects of system normality and describes the results of simulations of the controllers applied to the full Lorenz equations. Finally section 5 draws conclusions.

2. SYNTHESIS OF LMI CONTROLLERS

2.1 Transient Growth

Minimisation of the transient growth of stable system has recently been investigated by Whidborne *et al.* (2004). The transient growth $\mathcal{E}(t)$ of a stable system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \mathbf{x} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{n\times n}$ from initial conditions $\mathbf{x}(0)$ is defined as

$$\mathcal{E}(t) = \max_{\|\mathbf{x}(0)\|=1} \|\mathbf{x}(t)\|^2 \qquad (1)$$

and the maximum transient growth as

$$\Theta = \max_{t\geq 0} \mathcal{E}(t) \qquad (2)$$

There is a lower bound of unity on Θ, and an upper bound is given by the square of the ratio of the axes of a bounding ellipsoid. A suitable bounding ellipsoid $\mathbf{c}'\mathbf{P}\mathbf{c} = \mathbf{x}(0)'\mathbf{P}\mathbf{x}(0) = 1, \mathbf{c} \in \mathbb{R}^n$ is given by the condition for $\mathbf{x}'\mathbf{P}\mathbf{x}$ being a Lyapunov function i.e. $\mathbf{P} = \mathbf{P}' > 0$ and $\mathbf{PA} + \mathbf{A}'\mathbf{P} < 0$. Since the length of the ith semi-axis of an ellipse is $1/\sqrt{\lambda_i(\mathbf{P})}$, an upper bound on the maximum transient growth is given by $\Theta_u = \lambda_{max}(\mathbf{P})/\lambda_{min}(\mathbf{P})$.

2.2 Closed Loop Transient Growth

Whidborne *et al.* (2004) have established conditions for the existence of feedback controllers which produce unit Θ, and characterised them. The work also proposes a linear matrix inequality (LMI) method to find controllers which minimise the upper bound on the transient growth, Θ_u. The linearised Lorenz equations fail to meet the unit Θ conditions, and thus only controllers which minimise Θ_u are considered here.

Expanding $\mathbf{A}$ as $\mathbf{A}+\mathbf{BK}$, to represent closed loop feedback control $\mathbf{u} = \mathbf{Kx}$ of $\mathbf{A}$ via input matrix $\mathbf{B}$, and substituting $\mathbf{Q} = \mathbf{P}^{-1}, \mathbf{Y} = \mathbf{KQ}$ results in an equivalent equation for $\mathbf{Q}$ and $\mathbf{Y}$

$$\mathbf{AQ} + \mathbf{QA}' + \mathbf{BY} + \mathbf{Y}'\mathbf{B}' < 0 \qquad (3)$$

Since $\lambda(\mathbf{P}) = 1/\lambda(\mathbf{Q})$, the upper bound becomes $\Theta_u = \lambda_{max}(\mathbf{Q})/\lambda_{min}(\mathbf{Q})$ and a controller that minimises it is given by a solution to the LMI generalized eigenvalue problem

$$\begin{aligned} &\min \gamma \\ &s.t. \\ &\mathbf{I} \leq \mathbf{Q} \leq \gamma\mathbf{I}, \mathbf{AQ} + \mathbf{QA}' + \mathbf{BY} + \mathbf{Y}'\mathbf{B}' < 0 \end{aligned} \qquad (4)$$

where $\mathbf{I} \leq \mathbf{Q} \leq \gamma\mathbf{I}$ ensures that $\lambda_{min}(\mathbf{Q}) \geq 1$ and $\lambda_{max}(\mathbf{Q}) \leq \gamma$, so $\Theta_u \leq \gamma$.

2.3 Limited Control Effort

In addition, a limit on the expenditure of control effort can be set by simultaneously solving the LMI described by Boyd *et al.* (1994, p 103) and recommended by Hinrichsen *et al.* (2002).

A norm on the control input $u(t) = \mathbf{Kx}(t)$ is

$$\max_{t\geq 0} \|u(t)\|^2 = \max_{t\geq 0} \left\|\mathbf{YQ}^{-1}\mathbf{x}(t)\right\|^2 \qquad (5)$$

If $\mathbf{Q}$ and $\mathbf{Y}$ satisfy (4) and $\mathbf{x}(0)'\mathbf{Q}^{-1}\mathbf{x}(0) \leq 1$, then $\mathbf{x}$ remains inside $\mathbf{c}'\mathbf{Q}^{-1}\mathbf{c} \leq 1$ for all $t \geq 0$, so

$$\max_{t\geq 0} \|u(t)\|^2 \leq \max_{\mathbf{x}'\mathbf{Q}^{-1/2}\mathbf{Q}^{-1/2}\mathbf{x}<1} \left\|\mathbf{YQ}^{-1/2}\mathbf{Q}^{-1/2}\mathbf{x}\right\|^2 \qquad (6)$$

Since the induced 2-norm is equal to the largest singular value

$$\max_{t\geq 0} \|u(t)\|^2 \leq \lambda_{max}\left(\mathbf{Q}^{-1/2}\mathbf{Y}'\mathbf{YQ}^{-1/2}\right) \qquad (7)$$

So a constraint $\max_{t\geq 0} \|u(t)\|^2 \leq \mu^2$ can be obtained by a solution of the LMIs

$$\begin{bmatrix} 1 & \mathbf{x}(0)' \\ \mathbf{x}(0) & \mathbf{Q} \end{bmatrix} \geq 0, \begin{bmatrix} \mathbf{Q} & \mathbf{Y'} \\ \mathbf{Y} & \mu^2\mathbf{I} \end{bmatrix} \geq 0 \qquad (8)$$

Also, the constraint on the initial conditions $\mathbf{x}(0)'\mathbf{Q}^{-1}\mathbf{x}(0) \leq 1$ can be replaced by the constraint $\mathbf{x}(0)'\mathbf{x}(0) \leq 1$, providing it is more restrictive. The sphere $\mathbf{c}'\mathbf{c} = 1$ lies within the ellipse $\mathbf{c}'\mathbf{Q}^{-1}\mathbf{c} = 1$ if the longest ellipse semi-axis $1/\sqrt{\lambda_{min}(\mathbf{Q}^{-1})}$ is smaller than one. Thus the system of LMIs to be solved to restrict the control effort to μ^2 from initial conditions $\mathbf{x}(0)'\mathbf{x}(0) \leq 1$ becomes

$$\mathbf{Q} \geq \mathbf{I}, \begin{bmatrix} \mathbf{Q} & \mathbf{Y}' \\ \mathbf{Y} & \mu^2\mathbf{I} \end{bmatrix} \geq 0 \qquad (9)$$

and the complete LMI to stabilise the system, minimise the upper bound on the transient growth and limit the control effort becomes

$$
\begin{aligned}
&\min \gamma \\
&s.t. \\
&\mathbf{I} \leq \mathbf{Q} \leq \gamma \mathbf{I}, \mathbf{AQ} + \mathbf{QA}' + \mathbf{BY} + \mathbf{Y}'\mathbf{B}' < 0 \\
&\begin{bmatrix} \mathbf{Q} & \mathbf{Y}' \\ \mathbf{Y} & \mu^2 \mathbf{I} \end{bmatrix} \geq 0
\end{aligned} \tag{10}
$$

Hinrichsen *et al.* (2002) have derived a constraint on the rate of transient decay, which could be simultaneously incorporated into this expression.

2.4 Synthesis of LQR Controllers

LQR controllers are also synthesized, to enable assessment of the performance of the LMI controllers. The standard LQR control problem states that given the system $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$, the feedback control signal which minimizes;-

$$
\int_0^\infty (\mathbf{x}(t)'\mathbf{S}\mathbf{x}(t) + \mathbf{u}(t)'\mathbf{R}\mathbf{u}(t))\, dt \tag{11}
$$

is given by $\mathbf{u} = -\mathbf{Kx}$ where $\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}'\mathbf{X}$ and $\mathbf{X} = \mathbf{X}' \geq 0$ is the solution of the algebraic Riccati equation $\mathbf{A}'\mathbf{X} + \mathbf{XA} - \mathbf{XBR}^{-1}\mathbf{B}'\mathbf{X} + \mathbf{S} = 0$, where $\mathbf{S}$ and $\mathbf{R}$ are weighting matrices. For the present purposes $\mathbf{S}$ is an identity matrix and $\mathbf{R}$ the scalar control weight parameter r.

3. LORENZ EQUATIONS

The Lorenz equations may be cast (Tritton, 1988)

$$
\begin{aligned}
\dot{X}_1 &= -pX_1 + pX_2 \\
\dot{X}_2 &= UX_1 - X_2 - X_1X_3 \\
\dot{X}_3 &= -bX_3 + X_1X_2
\end{aligned} \tag{12}
$$

where state X_1 represents fluid velocity, and X_2 and X_3 horizontal and vertical temperature gradients respectively. Parameter p is related to the fluid properties, b to the geometry and U is related to the heat source. The equations have three steady state solutions: $X_{1s} = X_{2s} = X_{3s} = 0$ (no convection) and $X_{1s} = X_{2s} = \pm (U_s - 1)^{1/2}, X_{3s} = U_s - 1$ (steady clockwise and anticlockwise convection). After linearisation, the equations for small perturbations $\mathbf{x} = (x_1, x_2, x_3)'$ about a non-zero steady state solution are

$$
\begin{aligned}
\dot{x}_1 &= -px_1 + px_2 \\
\dot{x}_2 &= (U_s - X_{3s})\, x_1 - x_2 - X_{1s}x_3 + X_{1s}u \\
\dot{x}_3 &= X_{2s}x_1 + X_{1s}x_2 - bx_3
\end{aligned} \tag{13}
$$

where u is a small perturbation in the steady heat source U_s. The control problem is to determine a state feedback controller $u = \mathbf{kx}$, which will stabilise the plant, and minimise its worst transient growth, subject to a limit on control effort $u'u$.

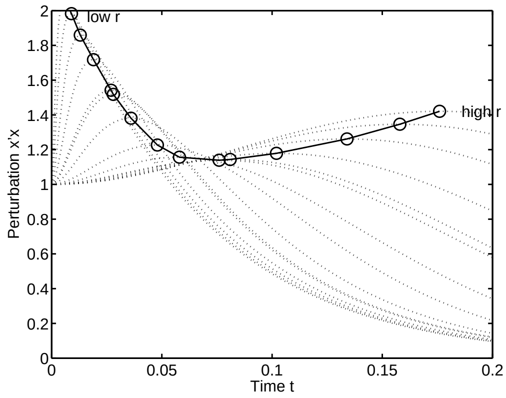

Fig. 1. Transient behaviour of linear LQR system for a range of control weights (dotted) with peak transients (solid).

4. SIMULATIONS, RESULTS AND DISCUSSION

4.1 Simulations

The chaotic regime Lorenz parameters $p = 4, U_s = 48, b = 1$ as used by Bewley (1999) are employed, and yield the three eigenvalues $-6.66, 0.33 \pm 7.50i$, upon linearisation of the system about steady clockwise convection.

LQR and LMI controllers are synthesized for a range of controller weights r and control effort limits μ, using the Matlab Control and LMI toolboxes, respectively. Linear and non-linear simulations are performed using the `lsim` and `ode15s` functions.

The initial conditions for each simulation are those that produce the worst transient growth as used by Mckernan *et al.* (2004).

4.2 Results of Linear Simulations

Figure 1 shows the transient behaviour of the linearised Lorenz equations controlled by LQR controllers with varying control weight r. At low control weight, i.e. high control effort there is a large fast initial transient. At high control weight, i.e. low control effort there is a slower and moderately sized transient. In between there is an optimum control weight which results in the smallest transient.

This is apparent in figure 2 which shows that the maximum transient Θ plotted against LQR control weight r is not monotonic. The lowest transient occurs at a control weight $r \approx 1$. The upper bound on transient Θ_u is also plotted, and is a reasonably close bound at high control weights (low control effort), but poor at low control weights.

Figure 3 shows the transient behaviour of the linearised Lorenz equations controlled by LMI controllers with varying control effort μ. At high con-

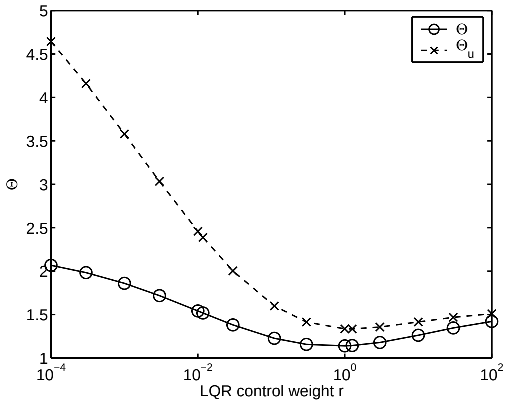

Fig. 2. Maximum transient Θ vs LQR control weight r

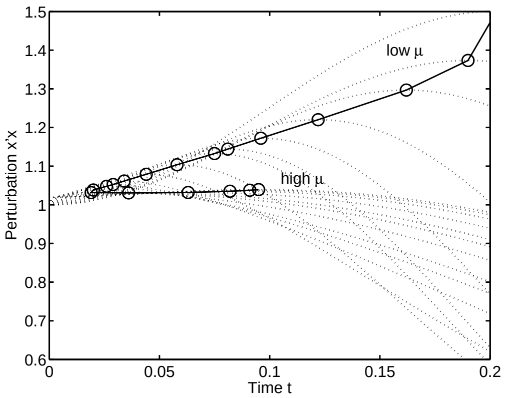

Fig. 3. Transient behaviour of linear LMI system for a range of control limits (dotted) with peak transients shown solid.

trol effort there is a very fast but small magnitude initial transient. At low control effort there is a slow transient with a moderately sized maximum perturbation.

Figure 4 shows the maximum transient plotted against LMI control effort. The lowest transient occurs at a control limit $\mu^2 \approx 10^4$. The upper bound on transient Θ_u is also plotted, and, as for the LQR case, is a close bound only at low control effort.

Figure 5 shows peak transient versus peak control effort for both LMI and LQR controllers. At low control effort both controllers have a similar peak transient. The LQR controller reaches its minimum transient near $\max_{t\geq 0}(u'u) = 1$, and then has an increasing peak transient with control effort, as the control is causing the peak. However the LMI controller continues to produce smaller transients for increasing control effort, until a shallow minimum near $\max_{t\geq 0}(u'u) = 10^2$.

Figure 6 shows the very similar transient behaviour of LQR and LMI controllers with $\max_{t\geq 0}(u'u) = 1$.

Figure 7 shows the transient behaviour of LQR and LMI controllers with $\max_{t\geq 0}(u'u) = 10^2$. The

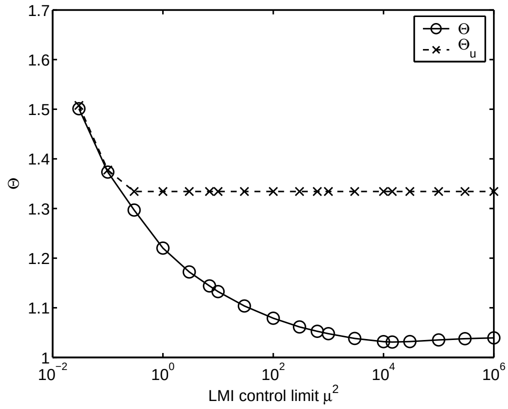

Fig. 4. Maximum transient Θ vs LMI control limit μ^2

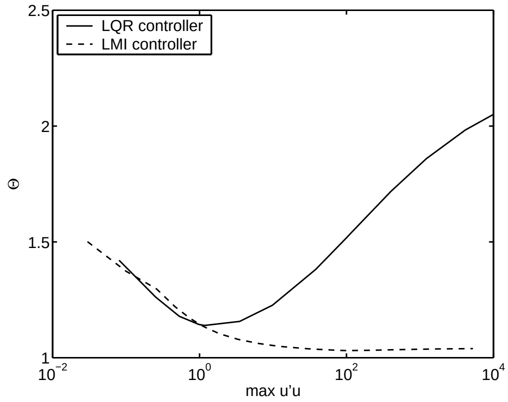

Fig. 5. Maximum transient Θ vs maximum control effort for LQR and LMI controllers

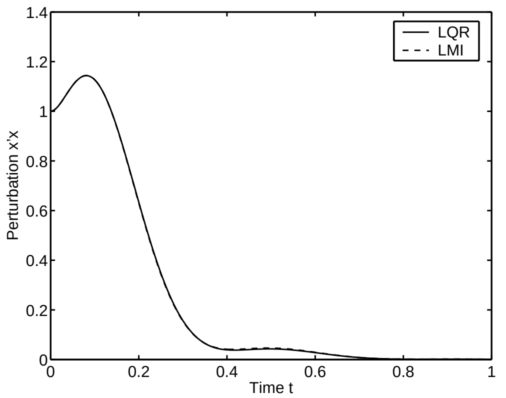

Fig. 6. Transient behaviour of linear LQR and LMI systems at low control effort, $\max_t(u'u) = 1$

LMI controller produces a transient perturbation around 3% above unity, much less that of the LQR controller (52%), although the overall perturbation lasts at least twice as long.

How the low LMI transient is achieved is shown in figure 8, where for the same initial negative peak control effort ($u(0) = -10$), the LMI controller is able to deliver a faster positive control effort.

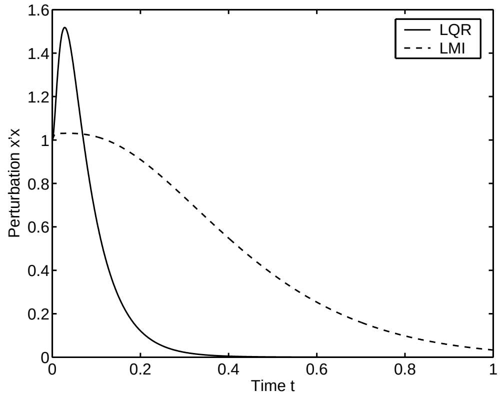

Fig. 7. Transient behaviour of linear LQR and LMI systems at high control effort, $\max_t(u'u) = 100$

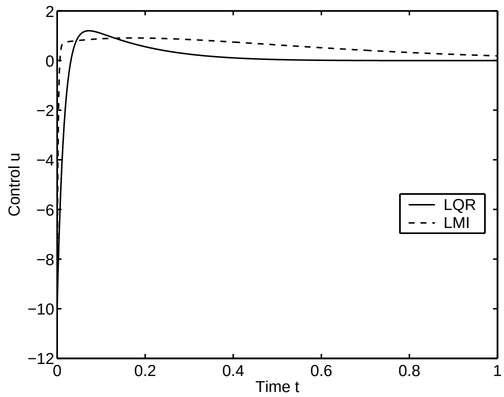

Fig. 8. Control on linear LQR and LMI systems at high control effort, $\max_t(u'u) = 100$

To investigate the relationship between the non-normality of the system eigenvectors and the transient growth, the eigenvector dot products and maximum transient are plotted against LMI control limit in figure 9. For the current system, two dot products are identical. It is evident that the lowest transient does not correspond with the lowest eigenvector dot products, and the same is true of the LQR controller. The reason can be inferred from the expression for transient in modal components. If $\mathbf{A}+\mathbf{BK}$ can be diagonalised $\mathbf{A}+\mathbf{BK} = \mathbf{V}\Lambda\mathbf{V}^{-1}$ where $\mathbf{V}$ is the matrix of right normalised eigenvectors $\mathbf{v}_i \in \mathbb{C}^n$, and Λ is a diagonal matrix containing the stable eigenvalues $\lambda_i \in \mathbb{C}^-$, then

$$\mathbf{x}'\mathbf{x}(t) = \mathbf{a}^* e^{\Lambda^* t}\mathbf{V}^*\mathbf{V}e^{\Lambda t}\mathbf{a} \qquad (14)$$

where $\mathbf{a}$ is a vector of worst initial modal amplitudes $a_i \in \mathbb{C}$ such that $\mathbf{x}'\mathbf{x}(0) = \mathbf{a}^*\mathbf{V}^*\mathbf{V}\mathbf{a} = 1$. As a summation of dot products, $(\mathbf{v}_i \cdot \mathbf{v}_j) = \mathbf{v}_i^*\mathbf{v}_j$

$$\mathbf{x}'\mathbf{x}(t) = \sum_{i=1}^{N}\sum_{j=1}^{N} \bar{a}_i a_j (\mathbf{v}_i \cdot \mathbf{v}_j) e^{\bar{\lambda}_i t} e^{\lambda_j t} \qquad (15)$$

The expression may be recast as monotonically decaying positive terms in one mode, and decaying possibly negative and/or oscillatory cross coupling or non-modal terms

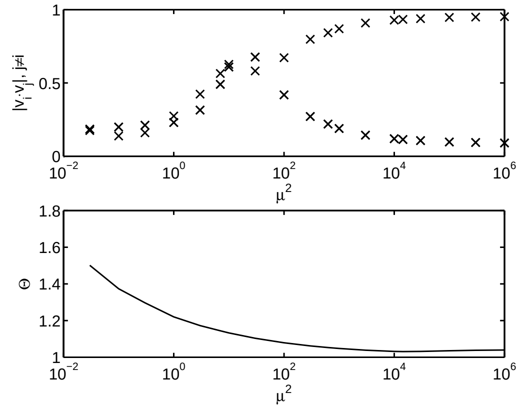

Fig. 9. Eigenvector dot product and Θ vs LMI control limit μ^2

$$\mathbf{x}'\mathbf{x}(t) = \sum_{i=1}^{N} \bar{a}_i a_i e^{(\bar{\lambda}_i+\lambda_i)t} + 2\sum_{i=1}^{N}\sum_{j=i+1}^{N} \Re\left(\bar{a}_i a_j (\mathbf{v}_i \cdot \mathbf{v}_j) e^{\left(\bar{\lambda}_i+\lambda_j\right)t}\right) \qquad (16)$$

Thus if all $(\mathbf{v}_i \cdot \mathbf{v}_{j\neq i}) = 0$, there will be no cross coupling terms, and thus no maximum transient greater than unity. The transient growth arises from the decaying cross coupling terms being negative when $\bar{a}_i a_j (\mathbf{v}_i \cdot \mathbf{v}_j) < 0$ or oscillating when $\Im\left(\bar{\lambda}_i + \lambda_j\right) \neq 0$, and the effect of these terms being greater than that of the modal decay terms. If the system eigenvectors cannot be made orthogonal, selecting the system with the lowest $(\mathbf{v}_i \cdot \mathbf{v}_{j\neq i})$ terms will not necessarily lead to the lowest peak transient due to the presence of the other factors $\bar{a}_i, a_j$ and $e^{\left(\bar{\lambda}_i+\lambda_j\right)t}$ in (16), and since, for any particular system, $\bar{a}_i, a_j$ are selected to maximise the transient energy growth, within the overall constraint $\mathbf{a}^*\mathbf{V}^*\mathbf{V}\mathbf{a} = 1$.

4.3 Results of Non-Linear Simulations

Figure 10 shows the transient perturbation growth of the full Lorenz equations from an arbitrary initial condition $\mathbf{x} = (10, 0, 0)'$, with respect to the linearisation point of stable clockwise convection, and figure (11) the trajectories in phase space. For the first 3 seconds the state spirals in towards one attractor, and then commences to orbit non periodically about both. LMI and LQR controllers (both $\max_{t\geq 0, \|x(0)\|=1}(u'u) = 10^2$ in section 4.2) are switched on at $t = 3s$ and stabilise the system. In this instance, the LMI controller is able to do this with a lower transient, and a more direct trajectory, albeit with a greater control effort.

Neither controller is able to stabilise the Lorenz system if switched on at $t = 3.1s$, rather they cause the trajectory to be expelled from the ball of attraction as described by Bewley (1999).

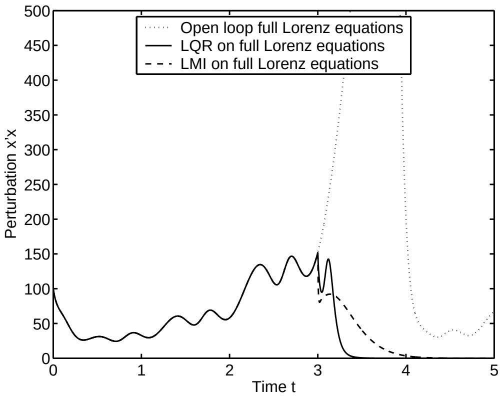

Fig. 10. Effect of LQR and LMI controllers on non-linear Lorenz equation perturbation. Controllers switched on at $t = 3.0$

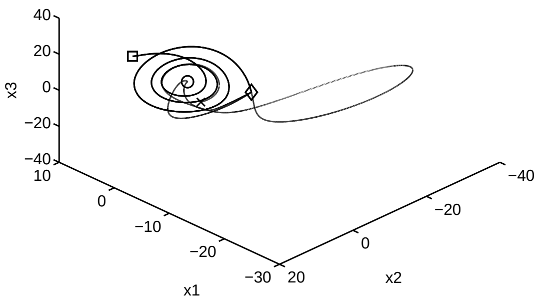

Fig. 11. Effect of LQR and LMI controllers on non-linear Lorenz equation perturbation in phase space. Initial conditions (□), controllers switched on at $t = 3.0$ (◇), and clockwise convection equilibrium at origin (○)

5. CONCLUSIONS

Constraints on peak controller effort have been incorporated in LMIs for the synthesis of controllers aimed at producing systems with minimised transient perturbations. As an illustration, both LMI and LQR controllers have been synthesized for the linearised Lorenz equations. Whereas the LQR controllers have a pronounced minimum achievable peak transient over control effort, since high control effort contributes to the peak transient, the LMI controllers deliver ever smaller peak transients for a long range of controller effort, until a shallow minimum is reached. It is also seen that for high-effort controllers, the upper bound on the peak transient can be very conservative. Research to develop less conservative peak-transient control methods is ongoing (Whidborne *et al.*, 2005).

Whilst the non-orthogonality of system eigenvectors has been shown to be an important factor in transient growth by Trefethen *et al.* (1993), unless the eigenvectors can be made orthogonal, evidence is presented that simply selecting controllers which reduce the non-orthogonality will not necessarily lead to the lowest transient growth. An explanation is presented in terms of the modal and non-modal components of growth.

Both LMI and LQR controllers are able to stabilise a simulation of the full non-linear Lorenz equations from limited initial conditions.

Not unexpectedly, the LMI controllers lead to relatively large settling times compared to the LQR controllers. An exponential time weighting could be incorporated into the LMI to improve the convergence rate, as proposed by Hinrichsen *et al.* (2002) and Boyd *et al.* (1994, p. 89). Furthermore, to obtain more practical controllers, the LMI problem could be augmented by additional convex criteria in a multiobjective approach.

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