# CONTROLLER SYNTHESIS WITH INPUT AND OUTPUT CONSTRAINTS FOR FUZZY SYSTEMS[1]

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Abstract: This work extends the synthesis of controllers for Takagi-Sugeno fuzzy systems based on a piecewise Lyapunov function to include constraints on the input or the output. Extension follows naturally from the existing results based on a common Lyapunov function and can be implemented via linear matrix inequalities, which are numerically solvable with commercial available software. An illustrative example is provided. 

Keywords: Controller synthesis, fuzzy systems, linear matrix inequality (LMI).

## 1. INTRODUCTION

Fuzzy logic-based control systems have been subjected to a big growth of industrial applications during the recent years as well as to big effort to study their properties and increase their reliability, mainly because of their satisfactory results in dealing with highly nonlinear systems with a good compromise between simplicity and accuracy.

Among fuzzy systems, those defined in (Takagi and Sugeno 1985) have been considered as the most convenient for analysis and design, because of their simplicity and efficiency in modelling nonlinear systems. First attempts on investigation of stability of Takagi-Sugeno fuzzy systems (TSFS) were made employing common Lyapunov functions as in (Tanaka and Sugeno 1990), (Tanaka and Sugeno 1992), (Chen and Ying 1993) and (Farinwata and Vachtsevanos 1993). Controller synthesis under this scheme was achieved, including a lot of performance requirements as decay rate, input or output constraints, robustness and optimality (Tanaka and Sano 1994), (Wang *et al.* 1996), (Tanaka *et al.* 1998) and (Tanaka and Wang 2001).

Nevertheless, analysis and design based on common Lyapunov functions lack flexibility because of their conservativeness. In order to relax this restrictions, a number of recent stability analysis procedures based on piecewise quadratic Lyapunov functions have been developed (Johansson *et al.* 1999), (Rantzer and Johansson 2000), (Feng 2004). In a very recent work (Feng 2003), controller synthesis under this scheme was made possible by constructing controllers in such a way that a piecewise continuous Lyapunov function can be used to establish the global stability with $H_\infty$ performance. Moreover, this synthesis can be achieved by means of linear matrix inequalities

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(LMIs), which can be solved with commercially available software.

In the present paper, two performance characteristics are added to the controller designed in (Feng 2003): constraints on the input and constraints on the output. Both of them can be implemented independently via LMIs and follow naturally from the elementary cases in (Tanaka *et al.* 1998).

This paper is organized as follows: section 2 introduces the dynamical fuzzy systems and piecewise quadratic design this work is based on; section 3 shows new results on input and output constraints; section 4 illustrates the previous results with some examples and, finally, section 5 draws some conclusions.

## 2. FUZZY SYSTEMS AND PIECEWISE QUADRATIC DESIGN

Consider the following Takagi-Sugeno fuzzy system as in (Feng 2003):

$$\begin{aligned} R^l : IF\ x_1\ \ is\ \ F_1^l\ AND\ \cdots\ x_n\ is\ F_n^l\ THEN \\ \dot{x}(t) = A_l x(t) + B_l u(t) + D_l v(t) \\ z_l(t) = H_l x(t), \\ l = 1, 2, \cdots, m \end{aligned} \quad (1)$$

where $R^l$ denotes the $l$th fuzzy rule, $m$ is the number of rules, $F_j^l$ the fuzzy sets, $x(t) \in R^n$ the state vector, $u(t) \in R^p$ the control input, $z(t) \in R^r$ the controlled output and $A_l, B_l, D_l, H_l$ the $l$th local model of the fuzzy system (1).

The previous scheme can be compactly rewritten considering membership functions $\mu_l(x) \geq 0$, as follows:

$$\begin{aligned} \dot{x}(t) = A(\mu)x(t) + B(\mu)u(t) + D(\mu)v(t) \\ z(t) = H(\mu)x(t) \end{aligned} \quad (2)$$

where

$$A(\mu) = \sum_{l=1}^{m} \mu_l A_l \quad B(\mu) = \sum_{l=1}^{m} \mu_l B_l$$
$$D(\mu) = \sum_{l=1}^{m} \mu_l D_l \quad H(\mu) = \sum_{l=1}^{m} \mu_l H_l$$

Piecewise quadratic stability is based on state space partitioning. In (Feng 2003) a partition's method is proposed in order to achieve globally stable controller with disturbance attenuation. Following similar lines, let us divide the state-space as follows:

$$\overline{S}_l = S_l \cup \partial S_l, \quad l = 1, 2, \cdots, m \quad (3)$$

where

$$S_l = \{x \mid \mu_l(x) > \mu_i(x), i = 1, 2, \cdots, m, i \neq l\}$$

and its boundary

$$\partial S_l = \{x \mid \mu_l(x) = \mu_i(x), i = 1, 2, \cdots, m, i \neq l\}.$$

In addition, let us define $L$ as the set of subspace indexes, so we can describe (2) as follows:

$$\begin{aligned} \dot{x}(t) = (A_l + \Delta A_l)x(t) + (B_l + \Delta B_l)u(t) \\ +(D_l + \Delta D_l)v(t) \\ z(t) = (H_l + \Delta H_l)x(t) \end{aligned}$$

for $x(t) \in \overline{S}_l$, where

$$\begin{aligned} \Delta A_l = \sum_{i \in M_l} \mu_i \Delta A_{li} \quad \Delta B_l = \sum_{i \in M_l} \mu_i \Delta B_{li} \\ \Delta D_l = \sum_{i \in M_l} \mu_i \Delta D_{li} \quad \Delta H_l = \sum_{i \in M_l} \mu_i \Delta H_{li} \\ \Delta A_{li} = A_i - A_l \quad \Delta B_{li} = B_i - B_l \\ \Delta D_{li} = D_i - D_l \quad \Delta H_{li} = H_i - H_l \\ M_l = \{i \mid \mu_i \neq 0, \mu_l \geq \mu_i\}. \end{aligned}$$

Piecewise Lyapunov function is constructed as in (Johansson *et al.* 1999):

$$V(t) = x^T R_l x,\ x \in \overline{S}_l, \quad l \in L \quad (4)$$

with

$$\begin{aligned} R_l = F_l^T T F_l, \quad l \in L \\ F_l x = F_j x,\ x \in \overline{S}_l \cap \overline{S}_j,\ l, j \in L. \end{aligned}$$

Employing parallel distributed compensation (PDC)

$$u(t) = Kx(t) = \sum_{l=1}^{m} \mu_l K_l x(t) \quad x \in \overline{S}_l,\ l \in L \quad (5)$$

the system (1) becomes:

$$\begin{aligned} \dot{x}(t) = A_c(\mu)x(t) + D_c(\mu)v(t) \\ z(t) = H_c(\mu)x(t) \end{aligned} \quad (6)$$

where

$$\begin{aligned} A_c(\mu) = A(\mu) + B(\mu)K(x) \quad D_c(\mu) = D(\mu) \\ H_c(t) = H(\mu) \end{aligned}$$

Then, controller synthesis can be achieved according to the following theorem (Feng 2003):

*Theorem 1*: Given a constant $\gamma > 0$, (6) is globally stable with disturbance attenuation $\gamma$, if there exist constants $\epsilon_l > 0$, $l = 1, 2, \cdots, m$, a symmetric matrix $T$ and a set of matrices $Q_l, l \in L$ such that with

$$\begin{aligned} P_l &= (F_l^T F_l)^{-1} F_l^T T F_l (F_l^T F_l)^{-1} \\ P_l &= R_l^{-1}, \quad l \in L \end{aligned} \tag{7}$$

the following LMIs are satisfied:

$$\begin{gathered} 0 < P_l, \quad l \in L \\ \begin{bmatrix} \Omega_l & P_l & Q_l^T \\ P_l & -M_{P_l}^{-1} & 0 \\ Q_l & 0 & -M_{Q_l}^{-1} \end{bmatrix} < 0, \ l \in L \end{gathered} \tag{8}$$

where

$$\begin{gathered} \Omega_l = P_l A_l^T + A_l P_l + Q_l^T B_l^T + B_l Q_l \\ + \epsilon_l (E_{lA} E_{lA}^T + E_{lB} E_{lB}^T) \\ + \gamma^{-2} \left(1 + \frac{1}{\epsilon_l}\right) D_l D_l^T + \gamma^{-2} (1 + \epsilon_l) E_{lD} E_{lD}^T \\ M_{P_l} = \frac{1}{\epsilon_l} I + \left(1 + \frac{3}{\epsilon_l}\right) H_l^T H_l \\ + \left(1 + 2\epsilon_l + \frac{1}{\epsilon_l}\right) E_{lH}^T E_{lH} \\ M_{Q_l} = \frac{1}{\epsilon_l} I \\ [\Delta A_l(\mu)][\Delta A_l(\mu)]^T \leq E_{lA} E_{lA}^T \\ [\Delta B_l(\mu)][\Delta B_l(\mu)]^T \leq E_{lB} E_{lB}^T \\ [\Delta D_l(\mu)][\Delta D_l(\mu)]^T \leq E_{lD} E_{lD}^T \\ [\Delta H_l(\mu)][\Delta H_l(\mu)]^T \leq E_{lH} E_{lH}^T \end{gathered}$$

with matrices $E_{lA}$, $E_{lB}$, $E_{lC}$, $E_{lD}$ and $E_{lH}$ being upper bounds for the uncertainty terms that can be easily calculated (see (Feng 2003)).

The controller gain for each local subsystem is given by

$$K_l = Q_l P_l^{-1}, \quad l \in L. \tag{9}$$

The piecewise Lyapunov function can be then expressed as in (4).

## 3. CONSTRAINTS ON INPUT AND OUTPUT

Results in (Feng 2003), briefly described in the previous section, can be extended by considering constraints on input and output.

*Theorem 2* Assume that the initial condition $x(0)$ in system (6) is known. The constraint $\|u(t)\|_2 < \lambda$ is enforced at all times $t \geq 0$ if the LMIs

$$\begin{bmatrix} 1 & x(0)^T \\ x(0) & P_l \end{bmatrix} \geq 0, \ l \in L \tag{10}$$

$$\begin{bmatrix} P_l & Q_l^T \\ Q_l & \lambda^2 I \end{bmatrix} \geq 0, \ l \in L \tag{11}$$

hold, where $P_l$ and $Q_l$ are defined as in Theorem 1. Then $K_l = Q_l P_l^{-1}, \ l \in L$.

*Proof:* Without loss of generality, suppose that

$$V(0) = x^T(0) R_l x(0) \leq 1, \ l \in L, \ x(0) \in \overline{S}_l. \tag{12}$$

From (7) and (12), we have $1 - x^T(0) P_l^{-1} x(0) \geq 0$, so by Schur complement we arrive to the LMI (10).

Condition $\|u(t)\|_2 < \lambda$ combined with (5) can be rewritten as follows:

$$\begin{aligned} u^T(t)u(t) &= \sum_{l=1}^{m} \sum_{j=1}^{m} \mu_l(x)\mu_j(x) x^T(t) K_l^T K_j x(t) \\ &\leq \lambda^2 \end{aligned}$$

from which

$$\frac{1}{\lambda^2} \sum_{l=1}^{m} \sum_{j=1}^{m} \mu_l(x)\mu_j(x) x^T(t) K_l^T K_j x(t) \leq 1. \tag{13}$$

Notice that since $x^T(t) P_l^{-1} x(t) \leq x^T(0) P_l^{-1} x(0) \leq 1$ for $t > 0$, if

$$\begin{aligned} \frac{1}{\lambda^2} \sum_{l=1}^{m} \sum_{j=1}^{m} \mu_l(x)\mu_j(x) x^T(t) K_l^T K_j x(t) \\ \leq x^T(t) P_l^{-1} x(t) \end{aligned}$$

then (13) holds. Therefore, condition (11) can be obtained from the previous inequality, which can be transformed as follows:

$$\begin{aligned} \sum_{l=1}^{m} \sum_{j=1}^{m} \mu_l(x)\mu_j(x) x^T(t) \left( \frac{1}{\lambda^2} K_l^T K_j - P_l^{-1} \right) x(t) \\ \leq 0 \end{aligned}$$

and by Schur complement

$$\sum_{l=1}^{m} \mu_l(x) \begin{bmatrix} P_l^{-1} & K_l^T \\ K_l & \lambda^2 I \end{bmatrix} \geq 0.$$

Congruence with the full rank matrix

$$\begin{bmatrix} P_l & 0 \\ 0 & I \end{bmatrix}$$

leads to (11), where $K_l = Q_l P_l^{-1}, \ l \in L$, which completes the proof.

*Theorem 3* Assume that the initial condition $x(0)$ in system (6) is known. The constraint $\|z(t)\|_2 < \lambda$ is enforced at all times $t \geq 0$ if the LMIs

$$\begin{bmatrix} 1 & x(0)^T \\ x(0) & P_l \end{bmatrix} \geq 0, \; l \in L \tag{14}$$

$$\begin{bmatrix} P_l & P_l H_l^T \\ H_l P_l & \lambda^2 I \end{bmatrix} \geq 0, \; l \in L \tag{15}$$

hold, where $P_l$ and $Q_l$ are defined as in Theorem 1. Then $K_l = Q_l P_l^{-1}, \; l \in L$.

*Proof:* Proof follows the same lines as that for Theorem 2.

## 4. EXAMPLE

In order to illustrate the influence of the input and output constraints, consider the following example taken from (Feng 2003), corresponding to a ball and beam system:

$$\begin{aligned} &R^1 : IF \;\; x_1 > 0 \;\; THEN \\ &\quad \dot{x}(t) \;=\; A_1 x(t) + B_1 u(t) + D_1 v(t) \\ &\quad z_1(t) \;=\; H_1 x(t) \\ &R^2 : IF \;\; x_1 < 0 \;\; THEN \\ &\quad \dot{x}(t) \;=\; A_2 x(t) + B_2 u(t) + D_2 v(t) \\ &\quad z_2(t) \;=\; H_2 x(t) \end{aligned} \tag{16}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -bg & -2b\beta \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$D_1 = [0 \; 0 \; 0 \; 1]^T, \; H_1 = [1 \; 0 \; 0 \; 0]$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -bg & 2b\beta \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$D_2 = [0 \; 0 \; 0 \; 1]^T, \; H_2 = [1 \; 0 \; 0 \; 0]$$

$$\beta = 0.01, \; b = 0.7143, \; g = 9.81$$

and $x(t) = [x_1 \; x_2 \; x_3 \; x_4]^T$ is the state vector, where $x_1$ represents the ball position, $x_2$ the ball velocity, $x_3$ the beam angle and $x_4$ the beam's angular velocity. Notice also that $z_1 = z_2 = x_1$.

Fig. 1 shows the membership functions employed in the example. According to the state-space partition (3), we will have two subspaces for which the following characterizing and bounding matrices can be taken:

$$F_1 = \begin{bmatrix} -1 \; 0 \; 0 \; 0 \\ I_{4\times 4} \end{bmatrix}, \; F_2 = \begin{bmatrix} -1 \; 0 \; 0 \; 0 \\ I_{4\times 4} \end{bmatrix}$$

$$E_{1A} = E_{2A} = 0.5(A_2 - A_1)$$

$$E_{1B} = E_{2B} = 0.5(B_2 - B_1)$$

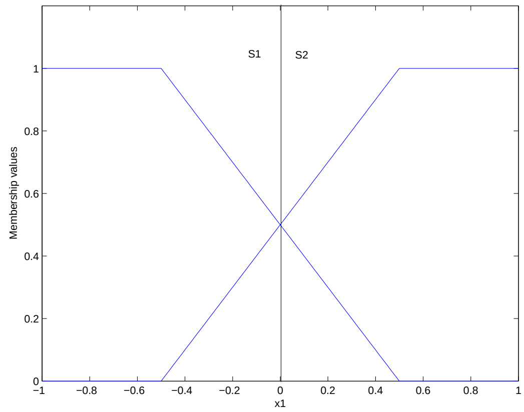


Fig. 1. Membership functions and state-space partition

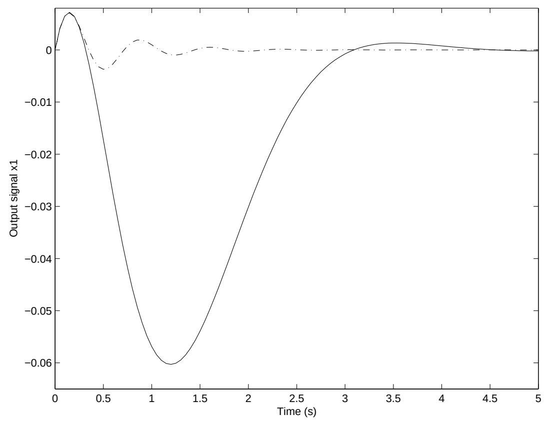


Fig. 2. Comparison of output signal $x_1$ with and without constraint

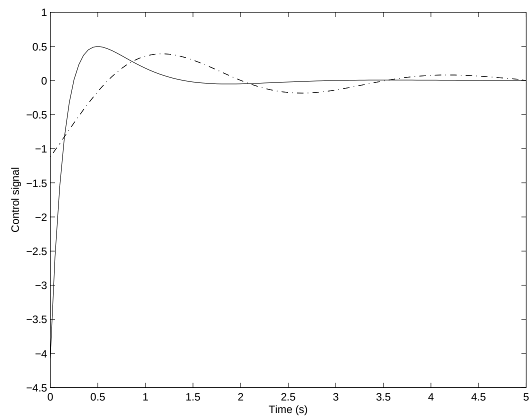


Fig. 3. Comparison of input signal with and without constraint

$$E_{1D} = E_{2D} = 0.5(D_2 - D_1)$$

$$E_{1H} = E_{2H} = 0.5(H_2 - H_1)$$

Employing the synthesis procedure described in Theorem 1 with $\gamma = 100$, $\epsilon_1 = \epsilon_2 = 10$, we have a feasible solution for LMIs (8) giving controller gains $K_1 = [\,8.0334 \;\; 10.4772 \;\; -40.0059 \;\; -11.2510]$ and $K_2 = [\,7.9824 \;\; 10.4272 \;\; -39.8936 \;\; -11.2299]$ which stabilize the system output $x_1$ as is shown

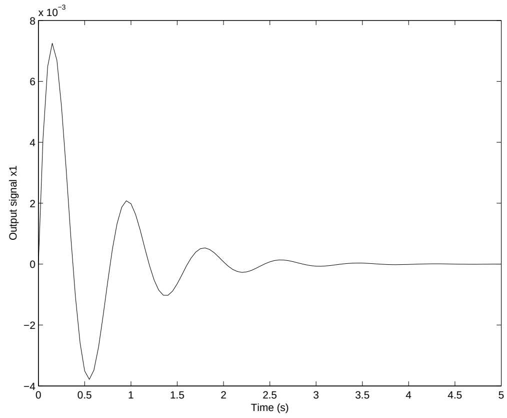


Fig. 4. Constrained output signal $x_1$

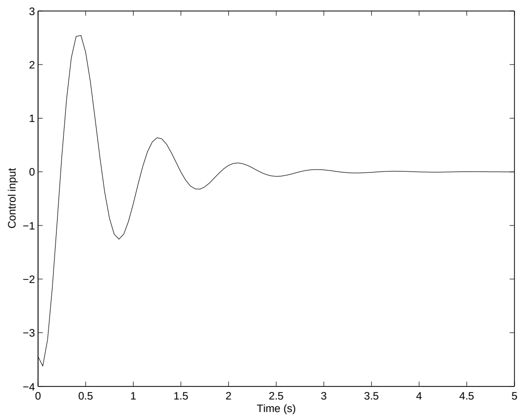


Fig. 5. Constrained input signal

with a solid line in Fig. 2. Corresponding control signal is also shown with a solid line in Fig. 3.

In order to reduce the magnitude of the control input signal, LMIs (10-11) should be added to those of (8). Choosing $\lambda = 3$ under initial conditions $x(0) = [0\ 0.1\ 0.1\ 0.1]^T$, these LMIs proved to be feasible with controller gains $K_1 = [1.3398\quad 1.9901\quad -9.4936\quad -3.6913]$ and $K_2 = [1.3248\quad 1.9847\quad -9.5022\quad -3.6899]$. In Fig. 3, control signal is shown with a dashed line to make clear the difference between non-constrained and constrained case. Constraint $\|u(t)\|_2 \leq \lambda$ has been satisfactory accomplished.

Constraints on the output can be satisfied by adding LMIs (14-15) to the original design in (8). With $\lambda = 0.009$ under initial conditions $x(0) = [0\ 0.1\ 0.1\ 0.1]^T$, these LMIs proved to be feasible with controller gains $K_1 = [\ 5020.3\ \ 710.9\ \ -690.1\ \ -47.3]$ and $K_2 = [\ 6439.1\ \ 878.3\ \ -847.8\ \ -55.2]$. In Fig. 2 output signal $z(t) = x_1$ is shown with a dashed line so can be compared with the original one. Constraint $\|z(t)\|_2 \leq \lambda$ holds.

Finally, combining all the previous schemes under the initial conditions $x(0) = [0\ 0.1\ 0.1\ 0.1]^T$ to achieve $\|z(t)\|_2 \leq 0.009$ and $\|u(t)\|_2 \leq 4.2$, LMIs (10-11), (14-15) and (8) proved to be feasible giving controller gains $K_1 = [\ 284.5527\quad 107.0897\quad -127.2083\quad -14.0323]$ and $K_2 = [347.2602\quad 120.0060\quad -139.7417\quad -15.0352]$. Fig. 4 shows the output signal $z(t) = x_1$ while Fig. 5 exhibits the corresponding control input.

## 5. CONCLUSION

Controller synthesis for Takagi-Sugeno fuzzy systems (TSFS) based on piecewise Lyapunov function remains a challenging task, which has been developed just recently. Extensions from those results available for common Lyapunov function based TSFS are necessary in order to increase the capabilities of this approach.

In this paper, two new results regarding performance requirements have been added to the existing disturbance rejection theorem. Considering constraints on the input and output signals is a typical demand which increases control quality. Both of them have been established in this paper and can be implemented via LMIs, which can be easily solved with commercially available software. The examples provided illustrate the effectiveness of the developed techniques.

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