LIMIT SHAPES OF REACHABLE SETS FOR LINEAR IMPULSE CONTROL SYSTEMS

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Abstract: The main purpose of the paper is to study the asymptotic behavior of reachable sets to linear time-invariant impulsive control systems. The issue is analyzed within the framework of *shapes* of reachable sets. This approach enables an exhaustive description of attractors arising in the space of shapes and the related dynamics. The results are compared with (Ovseevich, 1991; Figurina and Ovseevich, 1999).

Keywords: Reachable sets, impulse control, linear dynamic systems, shapes of convex bodies.

1. INTRODUCTION

The importance of the general problem of reachable sets characterization is widely recognized. A detailed discussion can be found, e.g., in (Chernousko, 1994). A topic in its own right is to study the long run reachable set dynamics. Our analysis focuses on linear impulsive control systems. This issue arises naturally, for example, in the optimal impulsive orbit correction during a long time space trip.

The related issues have received a detailed treatment in (Ovseevich, 1991; Figurina and Ovseevich, 1999) for the linear control systems with geometric control constraints. At first glance, the issue looks trivial because there is an explicit integral formula for the support function of the reachable set arising at any given time. In fact, this is not the case. First, in the nonautonomous case, the fundamental matrices involved might be very complicated. Second, the explicit formulas do not give an apparent clue for better understanding of the asymptotic behavior of reachable sets, even in the autonomous case. Still, there is a very simple observation that, in the time independent stable case, reachable sets approach a limit set as time tends to infinity. Here, the set convergence can be understood in many equivalent ways: in the sense of the Hausdorff metric, the Banach-Mazur one, or in terms of convergence of the support functions.

The approach behind the cited results reveals the advantage of study of the *shapes* of reachable sets. The idea is to identify the sets which can be obtained by means of invertible linear maps from one set, the ensuing object being called a *shape*, and study the asymptotic dynamics of shapes. In fact, the notion of shape is rather old and well-known under the disguise of Banach spaces regarded up to an isomorphism (Milman and Schechtman, 1986). Shapes of reachable sets

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seem to have better properties than the original reachable sets in the sense that, in the long run, shapes behave no worse than reachable sets in the stable case. This phenomenon has no exhaustive explanation so far. Perhaps, the basic reason is that the space of shapes of convex sets is compact unlike the space of convex sets. Note that, usually, if the asymptotic behavior of shapes is known, one can find the "lost by definition" matrix multipliers and recover the picture of the asymptotic behavior of reachable sets.

However, the range of application of this approach is definitely wider than the case of time-invariant (as in (Ovseevich, 1991)) or periodic (as in (Figurina and Ovseevich, 1999)) linear control systems with geometric type constraints. The advantage of addressing the dynamics of shapes will be evident in the present paper, where this ideology is applied to linear impulsive control systems. It is shown that, in the generic *hyperbolic case*, shapes of reachable sets have a limit in the sense of the Banach-Mazur metric. In the more general context, similar considerations imply that, in the space of shapes, there arise a finite-dimensional attractor with an explicit and simple description of the related dynamics.

2. THE PROBLEM STATEMENT

The present paper addresses the measure driven linear autonomous systems

$$dx(t) = Ax(t)\,dt + B\,du(t), \quad x(0) \in M, \quad (1)$$

where $x(t) \in V = \mathbf{R}^n$, $u(t) \in W = \mathbf{R}^m$, and A, B are matrices of appropriate dimensions. Consider this system on a time interval $[0,T]$ with the following constraint imposed on the control measure du:

$$\left| \int_0^T f(t)\,du(t) \right| \le 1 \quad (2)$$

for all continuous vector functions f such that $|f(t)| \le 1$, $t \in [0,T]$, which can be rewritten as $\mathrm{Var}_{[0,T]}\, u(t) \le 1$. Let the initial set M be a central symmetric convex compact, and suppose that the Kalman controllability condition is met for (1).

The reachable sets $\mathcal{D}(T)$ of the system (1), (2) are central symmetric convex bodies (convex compacts with a nonvoid interior) defined by

$$\mathcal{D}(T) = \mathcal{D}(T,M) = \{x(T) :$$
$$x(T) = e^{AT}x(0) + \int_0^T e^{A(T-s)} B\,du(s)\},$$

where $x(0) \in M$ and du is subject to (2).

The goal is to study the behavior of the sets $\mathcal{D}(T)$ as $T \to \infty$.

3. PRELIMINARIES

Denote by $\mathbf{B}$ a space of convex central symmetric bodies. Let us associate with any point (convex body) $\Omega \in \mathbf{B}$ its *shape* $\mathrm{Sh}\,\Omega$ which is, by definition, the orbit of Ω under the natural action of the group $GL(V)$ of nonsingular matrices:

$$\mathrm{Sh}\,\Omega = \{C\Omega : \det C \ne 0\}.$$

Note that $\mathbf{B}$ can be thought of as the space of the Banach norms on V, while the space of shapes is just the space of the norms *up to an isomorphism.* The space $\mathbf{B}$ possesses the so-called Banach-Mazur metric (distance) defined by

$$\rho(\Omega_1, \Omega_2) = \log\left(t(\Omega_1,\Omega_2)\,t(\Omega_2,\Omega_1)\right),$$

where $t(\Omega_1,\Omega_2) = \inf\{t \ge 1 : \ t\Omega_1 \supset \Omega_2\}$. This distance is $GL(V)$-invariant: $\rho(\mathrm{Sh}\,\Omega_1, \mathrm{Sh}\,\Omega_2) = \rho(C\Omega_1, C\Omega_2)$ if $C \in GL(V)$, and defines a natural metric in the space $\overline{\mathbf{B}}$ of shapes:

$$\rho(\mathrm{Sh}\,\Omega_1, \mathrm{Sh}\,\Omega_2) = \inf_{C,\, \det C \ne 0} \rho(\Omega_1, C\Omega_2).$$

The convergence of the reachable sets $\mathcal{D}(T)$ and their shapes $\mathrm{Sh}\,\mathcal{D}(T)$ is to be understood in the Banach-Mazur metric. In particular, two $\mathbf{B}$-valued functions are said to be asymptotically equal $\Omega_1(T) \sim \Omega_2(T)$ if $\rho(\Omega_1(T), \Omega_2(T)) \to 0$ as $T \to \infty$. It worth noting that the description of convex bodies given by their support functions is generally more useful in analyzing asymptotically close reachable sets than is a direct appeal to the definition. The first lemma below (Figurina and Ovseevich, 1999) justifies the equivalence of the ways to examine convergence of convex sets.

Lemma 1. A sequence of the points Ω_i in $\mathbf{B}$ converges to $\Omega \in \mathbf{B}$ with respect to the Banach-Mazur metric if and only if the corresponding sequence of the support functions $H_i(\xi) = H_{\Omega_i}(\xi) :=\sup_{x\in\Omega_i}(x,\xi)$ converges to the support function $H_\Omega(\xi)$ *pointwise*, and is uniformly bounded on the unit sphere of the dual space V^*.

The section is closed with a technically simple lemma (Goncharova and Ovseevich, 2004) which provides a ground for all subsequent computations:

Lemma 2. The support function of the reachable set $\mathcal{D}(T)$ to the system (1), (2) is given by

$$H_{\mathcal{D}(T)}(\xi) = H_M(e^{A^*T}\xi) + \sup_{[0,T]} \left| B^* e^{A^*t}\xi \right|. (3)$$

4. MAIN RESULTS

This section aims at exploring the limit behavior of the curve $T \mapsto \mathrm{Sh}\,\mathcal{D}(T)$ as $T \to \infty$ under less (or no) restrictions on the $\mathrm{Spec}\, A$.

It follows immediately from the lemma 1 and (3) that if A is a stable matrix ($\mathrm{Re}\,\mathrm{Spec}\, A < 0$), the reachable set $\mathcal{D}(T)$ has a limit as $T \to \infty$.

Let A be a strictly unstable matrix ($\mathrm{Re}\,\mathrm{Spec}\, A > 0$). Define a matrix multiplier $C(T) = e^{-AT}$ and consider $\widetilde{\mathcal{D}}(T) \stackrel{\text{def}}{=} C(T)\mathcal{D}(T)$ instead of $\mathcal{D}(T)$. It is immediately deduced that $H_{\widetilde{\mathcal{D}}(T)}(\xi) = H_M(\xi) + \sup_{[0,T]} \left|B^* e^{-A^* s}\xi\right|$ tends to $H_{\mathcal{D}_\infty}(\xi) = H_M(\xi) + \sup_{[0,\infty]} \left|B^* e^{-A^* s}\xi\right|$ as $T \to \infty$ uniformly over ξ in a compact set, and, therefore, $\mathrm{Sh}\,\mathcal{D}(T)$ tends to $\mathrm{Sh}\,\mathcal{D}_\infty$ as $T \to \infty$.

This gives the asymptotics of $\mathcal{D}(T)$ of the form $\mathcal{D}(T) \sim e^{AT}\mathcal{D}_\infty$. In the language of shapes of reachable sets, there is no behavioral difference between stable and unstable cases: in both situations the shape $\mathrm{Sh}\,\mathcal{D}(T)$ has a limit as $T \to \infty$.

The general context will be discussed after first explaining a special case.

4.1 The hyperbolic case

In this section, the spectrum of A is assumed to have an empty intersection with the imaginary axis, so that it can be represented as the union of two sets of eigenvalues with positive and negative real parts respectively. Then, the matrix A admits the canonical decomposition as $A = A_+ \oplus A_-$, where $\mathrm{Re}\,\mathrm{Spec}\, A_+ > 0$ and $\mathrm{Re}\,\mathrm{Spec}\, A_- < 0$. This spectrum decomposition induces a pair of complementary (spectral) projectors $P_\pm$ and the corresponding decomposition of phase space V into the direct sum of two subspaces $V_\pm = P_\pm V$.

The following notion will be used shortly. Suppose that sets $\Omega_\pm \subset V_\pm$ are convex. Define the *join* $\Omega = \Omega_+ * \Omega_- \subset V = V_+ \oplus V_-$ by

$$\Omega = \{t\omega_+ \oplus (1-t)\omega_- \,:\, t \in [0,1],\ \omega_\pm \in \Omega_\pm\}\,.$$

Note that the support function of Ω is given by

$$H_\Omega(\xi) = \max\left\{H_{\Omega_+}(\xi_+),\ H_{\Omega_-}(\xi_-)\right\}, \quad (4)$$

where $\xi = \xi_+ \oplus \xi_-$ is the canonical decomposition of ξ in $V^* = V_+^* \oplus V_-^*$, $V_\pm^* = P_\pm^* V^*$.

The system (1) is equivalent to the following one:

$$dx_+(t) = A_+ x_+(t)\,dt + B_+ du(t), \quad (5)$$

$$dx_-(t) = A_- x_-(t)\,dt + B_- du(t), \quad (6)$$

$$x_+(0) \in M_+, \qquad x_-(0) \in M_-, \quad (7)$$

where $x_\pm = P_\pm x$, $A_\pm = P_\pm A$, $B_\pm = P_\pm B$, and $M_\pm = P_\pm M$. Denote by $\mathcal{D}_\pm(T) = \mathcal{D}_\pm(T, M_\pm)$ the reachable sets of the systems (5) and (6) subject to (7) and with the same control constraint (2). The asymptotic behavior of $\mathcal{D}(T)$ and $\mathcal{D}_\pm(T)$ will be related by using the matrix multiplier $C(T) = C_+(T) \oplus C_-(T)$, where $C_+(T) = e^{A_+T}$ and $C_-(T) = I$ (the unit matrix).

Theorem 1. In the hyperbolic case, the reachable set to the system (1), (2) satisfies the asymptotic equality

$$\mathcal{D}(T) \sim C(T)\left(M_+ + \mathcal{D}_\infty\right), \quad T \to \infty, \quad (8)$$

where $\mathcal{D}_\infty = \lim_{T\to\infty} C(-T)\mathcal{D}(T, \{0\})$ is a convex body independent of T.

Furthermore, $\mathcal{D}_\infty$ is the join of the similarly defined convex bodies $\mathcal{D}_{+\infty}$ and $\mathcal{D}_{-\infty}$, $\mathcal{D}_{\pm\infty} = \lim C_\pm(-T)\mathcal{D}_\pm(T, \{0\})$ associated with the systems (5) and (6) subject to (7), (2).

In particular, the shapes $\mathrm{Sh}\,\mathcal{D}(T)$ tend to the limit $\mathrm{Sh}(M_+ + \mathcal{D}_{+\infty} * \mathcal{D}_{-\infty})$ as $T \to \infty$.

Proof. To establish (8), one has to study the support function of the set $\widetilde{\mathcal{D}}(T) = C(-T)\mathcal{D}(T)$:

$$H_{\widetilde{\mathcal{D}}(T)}(\xi) = H_{M_+}(\xi_+) + H_{M_-}(e^{A_-^* T}\xi_-) +$$
$$+ \sup_{t\in[0,T]} \left|B^* e^{A_+^*(t-T)}\xi_+ + B^* e^{A_-^* t}\xi_-\right|. \quad (9)$$

Notice that the term $H_{M_+}(\xi_+)$ does not depend on T, and $H_{M_-}(e^{A_-^* T}\xi_-)$ approaches the zero exponentially fast as $T \to \infty$. The third term in (9) is the support function of the set $C(-T)\mathcal{D}(T, \{0\})$. To examine it separately, assume that $M_\pm = \{0\}$ without loss of generality, so that

$$H_{\widetilde{\mathcal{D}}(T)}(\xi) = \sup_{t\in[0,T]} \left|B^* e^{A_+^*(t-T)}\xi_+ + B^* e^{A_-^* t}\xi_-\right|. (10)$$

Now, divide the interval $\mathcal{I} = [0, T]$ into the three subintervals

$$\mathcal{I} = \mathcal{I}_- \cup \mathcal{I}_0 \cup \mathcal{I}_+ :=$$
$$= [0, \varepsilon T] \cup [\varepsilon T, (1-\varepsilon)T] \cup [(1-\varepsilon)T, T].$$

Here, $\varepsilon = \varepsilon(T)$ is an arbitrary (strictly) positive function of T such that $\varepsilon(T) \to 0$, and $\varepsilon(T)T \to \infty$ as $T \to \infty$. (E.g., one may put $\varepsilon(T) = T^{-1/2}$).

Given the time interval partition and (10), one has that

$$H_{\widetilde{\mathcal{D}}(T)}(\xi) = \max\left(\sup_{t\in\mathcal{I}_-} f_T(t),\ \sup_{t\in\mathcal{I}_0} f_T(t),\ \sup_{t\in\mathcal{I}_+} f_T(t)\right),$$

where $f_T(t) = |B^* e^{A_+^*(t-T)}\xi_+ + B^* e^{A_-^* t}\xi_-|$, $t \in [0, T]$. Due to the estimates $e^{A_-^* t} = O(e^{-\alpha t})$, $e^{-A_+^* t} = O(e^{-\alpha t})$, $t \geq 0$, where a positive number

α is assumed to be less than $\min_{\lambda \in \operatorname{Spec} A} |\operatorname{Re} \lambda|$, it is easy to see that $f_T(t) = |B^* e^{A_-^* t} \xi_-| + O(e^{-\alpha\varepsilon T})$, $f_T(t) = O(e^{-\alpha(1-2\varepsilon)T})$, and $f_T(t) = |B^* e^{A_+^* t} \xi_-| + O(e^{-\alpha\varepsilon T})$, for $t \in \mathcal{I}_i$, $i = -, 0, +$, respectively. Therefore,

$$H_{\widetilde{\mathcal{D}}(T)}(\xi) = O(e^{-\alpha\epsilon T}) +$$

$$+ \max \left(\sup_{t \in \mathcal{I}_-} |B^* e^{A_-^* t} \xi_-|, \sup_{t \in \mathcal{I}_+} |B^* e^{A_+^*(t-T)} \xi_+| \right).$$

The next easy observation is that, if T is sufficiently large, then

$$\sup_{t \in \mathcal{I}_-} |B^* e^{A_-^* t} \xi_-| = \sup_{t \in [0,T]} |B^* e^{A_-^* t} \xi_-|, \quad \text{and}$$

$$\sup_{t \in \mathcal{I}_+} |B^* e^{A_+^*(t-T)} \xi_+| = \sup_{t \in [0,T]} |B^* e^{A_+^*(t-T)} \xi_+| =$$

$$= \sup_{t \in [0,T]} |B^* e^{-A_+^* t} \xi_+|.$$

This implies

$$H_{\widetilde{\mathcal{D}}(T)}(\xi) = O(e^{-\alpha\epsilon T}) +$$

$$+ \max \left(\sup_{t \in [0,T]} |B^* e^{A_-^* t} \xi_-|, \sup_{t \in [0,T]} |B^* e^{-A_+^* t} \xi_+| \right).$$

Passing to the limit, the immediate conclusion is that $\widetilde{\mathcal{D}}(T) \to \mathcal{D}_\infty$ as $T \to \infty$, and the support function of $\mathcal{D}_\infty$ is given by

$$H_{\mathcal{D}_\infty}(\xi) = \max \left(\sup_{t \geq 0} |B^* e^{-A_+^* t} \xi_+|, \sup_{t \geq 0} |B^* e^{A_-^* t} \xi_-| \right).$$

The right-hand side of the last formula is the support function

$$H_{\mathcal{D}_\infty}(\xi) = \max \left\{ H_{\mathcal{D}_{-\infty}}(\xi_-), H_{\mathcal{D}_{+\infty}}(\xi_+) \right\}$$

of the join of $\mathcal{D}_{+\infty} = \lim e^{-A_+ T} \mathcal{D}_+(T)$ and $\mathcal{D}_{-\infty} = \lim \mathcal{D}_-(T)$. By a slight abuse of the language, one can say that the limit shape of reachable sets of (1), (2) is the join of the limit shapes of reachable sets of the systems (5) and (6) subject to (7) and (2).

The generic hyperbolic case, where the neutral component A_0 is absent from the canonical decomposition (11), was already addressed in more details in (Goncharova and Ovseevich, 2004). The previous result was stated separately because it is more similar to the well-established results pertaining to linear systems with geometric bounds on control. Furthermore, the hyperbolicity assumption simplifies the proof greatly.

4.2 Main result: the general case

Consider the canonical decomposition

$$A = A_+ \oplus A_0 \oplus A_-, \tag{11}$$

where $\operatorname{Re} \operatorname{Spec} A_+ > 0$, $\operatorname{Re} \operatorname{Spec} A_0 = 0$, and $\operatorname{Re} \operatorname{Spec} A_- < 0$. The system (1) can be reduced to the following one:

$$dx_i(t) = A_i x_i(t) dt + B_i du(t), \quad x_i(0) \in M_i, \tag{12}$$

where $x_i \in V_i = P_i V$, and $A_i = P_i A P_i$, $i = +, 0, -$.

Here again, we are faced with the subtle issue on proper choosing a matrix multiplier $C(T)$ to ensure convergence of the transformed reachable sets as $T \to \infty$. In fact, a recipe to determine $C(T)$ with the desired properties is already given in (Ovseevich, 1991; Figurina and Ovseevich, 1999).

For the matrix A_0, let us consider the Jordan decomposition $A_0 = D + N$, where D is diagonalizable (semisimple), N is a nilpotent matrix, and $DN = ND$. Let $F(T) = F(N, T)$ be a matrix function such that

$$F(N \oplus M, T) = F(N, T) \oplus F(M, T),$$
$$F(CNC^{-1}, T) = CF(N, T)C^{-1}$$

for any invertible matrix C, and

$$F(N, T) = \begin{pmatrix} 1 & & & 0 \\ & T^{-1} & & \\ & & \ddots & \\ 0 & & & T^{-(n-1)} \end{pmatrix} \quad \text{for}$$

$$N = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix}.$$

Moreover, $F(N, T) N F(N, T)^{-1} = T^{-1} N$ and $\lim_{T \to \infty} F(N, T) = F_\infty$ is a projector on $\ker N$.

Then, the proper matrix multiplier which governs the long term behavior of the reachable sets for (1), (2) is given by

$$C(T) = C_+(T) \oplus C_0(T) \oplus C_-(T), \tag{13}$$

where $C_+(T) = e^{A_+ T}$, $C_0(T) = F(T)^{-1}$, and $C_-(T) = I$.

To compare with the hyperbolic case, note that in the general framework, there is no more convergence of shapes $\operatorname{Sh} \mathcal{D}(T)$ to a single shape $\operatorname{Sh} \mathcal{D}_\infty$. In the space of shapes, there arise an attractor $\mathcal{A}$ of a positive dimension so that $\operatorname{Sh} \mathcal{D}(T) \to \mathcal{A}$ as $T \to \infty$. Let $\mathcal{T} \subset \operatorname{GL}(V_0)$ be a torus generated by the one-parameter group of operators $\{e^{Dt} : V_0 \to V_0\}$ in the neutral canonical subspace $V_0 \subset V$. The attractor $\mathcal{A}$ is an image of the torus $\mathcal{T}$ under a continuous map Φ. Furthermore, in the space of shapes the curve $T \mapsto \operatorname{Sh} \mathcal{D}(T)$ is asymptotically close to the image $T \mapsto \Phi(\gamma(T))$ of a straight geodesic line $\gamma(T) = e^{DT} \mathbf{t}$ on the torus $\mathcal{T}$, where $\mathbf{t} \in \mathcal{T}$ is a fixed element. (See a general discussion on straight geodesic lines on tori in (Arnold, 1989)).

Consider the matrix function $e^{A_0^* t}$ represented as

$$e^{A_0^* t} = \sum_{\omega \in \mathbf{R}} e^{i\omega t} p_\omega(t),$$

where $p_\omega(t)$ is a polynomial with matrix coefficients. One has

$$e^{A_0^* t} F(T)^* = F(T)^* \sum_\omega e^{i\omega t} p_\omega(\frac{t}{T})$$

and

$$e^{A_0^* t} e^{-D^* T} = e^{-D^* T} e^{A_0^* t} = \sum_\omega e^{i\omega(t-T)} p_\omega(t).$$

Now, the attractor $\mathcal{A}$ in the space of shapes and the map $\Phi : \mathcal{T} \to \mathcal{A}$ can be explicitly described. Let an element $\mathbf{t} = \mathbf{t}(\phi_\omega) \in \mathcal{T}$ be given by the sequence $\phi_\omega \in \mathbf{R}/2\pi\mathbf{Z}$ of angles, where ω runs over the set $\frac{1}{i}\{\operatorname{Spec} A_0 \setminus \{0\}\}$. Then, let us define the matrix function

$$g_{+0}(\mathbf{t}, t) = F_\infty^* \sum_\omega e^{i(\phi_\omega - \omega t)} p_\omega(1)$$

and the convex body $\mathcal{D}_\infty^{+0} = \mathcal{D}_\infty^{+0}(\mathbf{t})$ in the space $V_{+0} \stackrel{\text{def}}{=} V_+ \oplus V_0$ via the support function

$$H_{\mathcal{D}_\infty^{+0}}(\xi_+, \xi_0) = \sup_{t \geq 0} |B^* e^{-A_+^* t} \xi_+ + B^* g_{+0}(\mathbf{t}, t)\xi_0|.$$

Similarly, define the matrix function

$$g_{-0}(t) = F_\infty^* \sum_\omega e^{i\omega t} p_\omega(0)$$

and specify the convex body $\mathcal{D}_\infty^{-0} = \mathcal{D}_\infty^{-0}(\mathbf{t})$ in the space $V_{-0} \stackrel{\text{def}}{=} V_- \oplus V_0$ as follows

$$H_{\mathcal{D}_\infty^{-0}}(\xi_-, \xi_0) = \sup_{t \geq 0} |B^* e^{A_-^* t} \xi_- + B^* g_{-0}(t)\xi_0|.$$

The convex body $\mathcal{D}_\infty^0 \subset V_0$ is given by the support function

$$H_{\mathcal{D}_\infty^0}(\xi_0) = \sup_{\substack{\tau \in [0,1] \\ \mathbf{t} \in \mathcal{T}}} |B^* g_0(\mathbf{t}, \tau)\xi_0|,$$

where

$$g_0(\mathbf{t}, \tau) = F_\infty^* \sum_\omega e^{i\phi_\omega} p_\omega(\tau).$$

Finally, define the body

$$\mathcal{D}_\infty = \mathcal{D}_\infty(\mathbf{t}) \subset V = V_+ \oplus V_0 \oplus V_-$$

via the support function $H_{\mathcal{D}_\infty}(\xi)$, which is equal to

$$\max\left(H_{\mathcal{D}_\infty^{+0}}(\xi_+, \xi_0),\ H_{\mathcal{D}_\infty^0}(\xi_0),\ H_{\mathcal{D}_\infty^{-0}}(\xi_-, \xi_0)\right).$$

Note that in the preceding expression only $\mathcal{D}_\infty^{+0}$ depends on $\mathbf{t}$.

Consider a convex compact $\mathcal{M} = \mathcal{M}(\mathbf{t})$,

$$\mathcal{M}(\mathbf{t}) = M_+ \oplus g_0(\mathbf{t}, 1)^* M_0.$$

Notice that it depends only on the initial sets M_i, $i = +, 0$. Its support function has the form

$$H_\mathcal{M}(\xi) = H_{M_+}(\xi_+) + H_{M_0}(g_0(\mathbf{t}, 1)\xi_0).$$

Now, the main result can be formulated as follows:

Theorem 2. The reachable sets of the system (1), (2) satisfy the following asymptotic equality

$$\mathcal{D}(T) \sim C(T)\Omega(\mathbf{t}) \quad \text{as } T \to \infty.$$

Here $\Omega(\mathbf{t}) = \mathcal{M}(\mathbf{t}) + \mathcal{D}_\infty(\mathbf{t})$, $C(T)$ is given in (13), and $\mathbf{t} = \mathbf{t}(\omega T)$ is an element of the torus $\mathcal{T}$ defined by the sequence $\omega \mapsto \omega T \bmod 2\pi$, where $\omega \in \frac{1}{i}\{\operatorname{Spec} A_0 \setminus \{0\}\}$.

In particular, $\operatorname{Sh} \mathcal{D}(T) \sim \operatorname{Sh} \Omega(\mathbf{t})$, and the map $\Phi : \mathcal{T} \to \mathcal{A}$ is given by $\Phi(\mathbf{t}) = \operatorname{Sh} \Omega(\mathbf{t})$.

Geometrically, this means that the attractor arising in the space of limit shapes is parameterized by the (flat) multidimensional torus $\mathcal{T}$, and the limit dynamics is given by the straight motion on the torus.

5. CONCLUSION

The above results give an exhaustive description of the large time dynamics of the reachable sets and their shapes for linear impulse control systems. They prove the advantage of first studying the dynamics of shapes, and then investigating the asymptotics for reachable sets themselves. Note that the hyperbolic framework is much simpler than the general one, because the torus $\mathcal{T}$ and the attractor arising in the general case are reduced to a point. Note also that the studied case of time-invariant impulse systems reveal a greater complexity in comparison with the case of time-invariant systems with geometric constraints investigated in (Ovseevich, 1991). The dimension of the attractor might be arbitrary large and, in this respect, the situation is even more complicated than in the case of periodic systems with geometric constraints (Figurina and Ovseevich, 1999), where there arise a one-dimensional attractor. It would be interesting to incorporate the above results and the ones of (Ovseevich, 1991; Figurina and Ovseevich, 1999) into a unified picture.

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