MINIMIZATION OF MAXIMUM TRANSIENT ENERGY GROWTH BY OUTPUT FEEDBACK

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Abstract: The the problem of minimizing the transient energy of a linear system following a unit energy initial disturbance is considered in this paper. This paper extends previous results on the state feedback case to the output feedback case. Furthermore, it is shown that the problem can be solved by convex optimization of the free parameter following a Q-parametrization. The techniques are illustrated by numerical examples.

Keywords: transient energy growth, transient bound, transient error, flow control, convex optimization, linear systems

1. INTRODUCTION

In some stable linear systems the trajectory of the system states, following an initial perturbation to the system states, may temporarily move a great distance from the origin before returning to approach the origin later. Such behaviour is highly undesirable in some non-linear systems where analysis of eigenvalues of the linearized system at the equilibrium point indicates very good stability, but the behaviour of the trajectories means that even very small perturbations in the state variables can cause the states to leave the domain of attraction and so become unstable.

This phenomenon is particularly prevalent in fluid dynamic systems. For example, it is known that a laminar flow can become turbulent even for Reynolds numbers for which linear stability analysis predicts stable eigenvalues. In fact, the reason for this phenomenon was unknown to fluid dynamicists until fairly recently (Trefethen *et al.*, 1993). In the fluid dynamics community, the distance from the equilibrium is usually measured by the energy of the perturbations, and the maximum transient energy growth is of interest in many fluid systems (e.g. Reddy and Henningson, 1993). For fluid control systems, a useful control objective is the minimization of the maximum transient energy of the flow perturbations (Bewley and Liu, 1998).

The problem of constraining transient trajectory norms has been considered elsewhere (recent results have bee reported in Hinrichsen and Pritchard, 2000; Pritchard, 2000; Hinrichsen *et al.*, 2002; Plischke and Wirth, 2004; Wirth, 2004). An LMI approach to minimization of maximum transient energy growth has been proposed by Boyd *et al.* (1994). The state feedback problem of minimization of maximum transient energy growth has also been considered by Whidborne *et al.* (2004).

This paper extends some of the results of Whidborne *et al.* (2004) to the output feedback case. Conditions for the existence of controllers that restrict the transient energy growth to unity are established along with a characterization of all such controllers. For systems where such controllers do not exist, it is shown that the problem may be solved by convex optimization over the free parameter in a Q-parametrization of the problem. The theory is illustrated with some numerical examples.

Notation

M^T denotes the transpose of a matrix M
$M^\dagger$ denotes the Moore-Penrose inverse of the matrix M
$M^\perp$ denotes the left null space of a matrix M, that is $M^\perp = U_2^T$ where $\left[U_1 \; U_2 \right] \left[\begin{smallmatrix} \Sigma_{11} & 0 \\ 0 & 0 \end{smallmatrix} \right] \left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix} \right] = M$ is the singular value decomposition of M
$\|M\| := \max \left\{ \sqrt{\lambda_i} : \lambda_i \text{ are the eigenvalues of } M^T M \right\}$ denotes the spectral norm of a real matrix M
$\|x\| := \sqrt{x^T x}$ denotes the Euclidian 2-norm of a vector x
$\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ denote, respectively, the largest and smallest eigenvalues of the matrix M
I_n represents the identity matrix of dimension $n \times n$

2. MAXIMUM TRANSIENT ENERGY GROWTH

Consider the stable linear time-invariant system described by

$$\dot{x} = Ax, \quad x(0) = x_0, \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $x(t) \in \mathbb{R}^n$ which has the solution

$$x(t) = \Phi(t)x_0, \tag{2}$$

where $\Phi(t)$ is the state transition matrix given by $\Phi(t) = e^{At} = \sum_{i=0}^{\infty} A^i t^i / i!$.

Definition 1. The transient energy, $\mathcal{E}(t)$, is defined as

$$\mathcal{E}(t) := \max_{\|x(0)\|=1} \|x(t)\|^2 . \tag{3}$$

Definition 2. The maximum transient energy growth, $\widehat{\mathcal{E}}$, is defined as

$$\widehat{\mathcal{E}} := \max_{t\geq 0} \mathcal{E}(t). \tag{4}$$

The following lemma gives the conditions on the state matrix, A, for there to be no transient energy growth. The proof is straightforward, and can be found in Whidborne *et al.* (2004).

Lemma 1. The maximum transient energy growth, $\widehat{\mathcal{E}}$, of the system described by (1) is unity if and only if $A + A^T < 0$.

It is well known that

$$\max_{\|x\|=1} \|Mx\| = \|M\| , \tag{5}$$

so to evaluate $\widehat{\mathcal{E}}$ for cases where $A + A^T \not< 0$, a line search over time, t, is performed on the spectral norm of $\Phi(t)$.

3. OPTIMAL STATIC GAIN FEEDBACK CONTROLLERS

Now consider the linear time-invariant plant

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \end{aligned} \tag{6}$$

with $A \in \mathbb{R}^{n\times n}$, $x(t) \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times \ell}$, $u(t) \in \mathbb{R}^\ell$, $C \in \mathbb{R}^{m\times n}$, $y(t) \in \mathbb{R}^m$. Furthermore, it is assumed that $B^T B > 0$, that is B has full column rank, and $CC^T > 0$, that is C has full row rank, (i.e. all actuators and sensors are independent).

3.1 Unity maximum transient energy growth

In this section, conditions are given for all controllers that obtain unity maximum transient energy growth for static output feedback control.

Theorem 1. For the system of (6), the following are equivalent:

(1) There exists a control $u = Ky$ where K is a constant matrix such that $\widehat{\mathcal{E}} = 1$ where $\widehat{\mathcal{E}}$ is given by Definition 2.
(2) The following two conditions hold

$$B^\perp \left(A + A^T\right) B^{\perp T} < 0 \text{ or } BB^T > 0, \tag{7}$$

$$C^{T\perp} \left(A + A^T\right) C^{T\perp T} < 0 \text{ or } C^T C > 0. \tag{8}$$

Furthermore, if the above statements hold, all controller matrices K are given by

$$K = -R^{-1}B^T \Psi C^T (C\Psi C^T)^{-1} + S^{1/2} L (C\Psi C^T)^{-1/2} \tag{9}$$

where

$$S := R^{-1} - R^{-1}B^T \left[\Psi - \Psi C^T (C\Psi C^T)^{-1} C\Psi\right] BR^{-1} \tag{10}$$

where L is an arbitrary matrix such that $\|L\| < 1$ and $R > 0$ is an arbitrary matrix such that

$$\Psi := \left(BR^{-1}B^T - A - A^T\right)^{-1} > 0. \tag{11}$$

Proof: From Lemma 1, the closed-loop system has unity maximum transient energy growth if and only if

$$(A + BKC) + (A + BKC)^T < 0. \tag{12}$$

The remainder follows directly by application of Theorem 2.3.12 of Skelton *et al.* (1998, p. 29), with the condition that B is full column rank and C has full row rank. □

Remark 1. A matrix R that satisfies (11) can be obtained by $R = I/\rho$. For the case where $BB^T > 0$ (i.e. B is full rank n), ρ is obtained simply by rearranging $BRB^T - A - A^T > 0$ giving the inequality

$$\rho > \lambda_{\max}(B^{-1}(A + A^T)(B^T)^{-1}). \quad (13)$$

For the case where $B^{\perp}(A + A^T)B^{\perp T} < 0$, ρ is obtained by an application of Theorem 2.3.10, of Skelton *et al.* (1998, p. 26), this being an extension to Finsler's Theorem.

Finsler's Theorem is presented below in an appropriate form for use with Theorem 1.

Theorem 2. (Finsler's Theorem). Given $\Gamma = (A + A^T)$, the following statements are equivalent:

(1) There exists a scalar ρ such that

$$\rho BB^T - \Gamma > 0. \quad (14)$$

(2) The following condition holds

$$P := B^{\perp}\Gamma B^{\perp T} < 0. \quad (15)$$

If the above statements hold, then all scalars ρ satisfying (14) are given by

$$\rho > \rho_{\min} := \lambda_{\max}\left\{B^{\dagger}(\Gamma - \Gamma B^{\perp T}P^{-1}B^{\perp}\Gamma)B^{\dagger T}\right\}. \quad (16)$$

4. OPTIMAL DYNAMIC FEEDBACK CONTROLLERS

Consider the linear time-invariant plant

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \end{aligned} \quad (17)$$

with $A \in \mathbb{R}^{n\times n}$, $x(t) \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times \ell}$, $u(t) \in \mathbb{R}^{\ell}$, $C \in \mathbb{R}^{m\times n}$, $y(t) \in \mathbb{R}^m$ with controller

$$\begin{aligned} \dot{x_k} &= A_k x_k + B_k y, \quad x_k(0) = x_{k0}, \\ u &= C_k x_k + D_k y, \end{aligned} \quad (18)$$

with $A_k \in \mathbb{R}^{n_k\times n_k}$, $x_k(t) \in \mathbb{R}^n_k$, $B_k \in \mathbb{R}^{n_k\times m}$, $C \in \mathbb{R}^{n_k\times \ell}$, $D \in \mathbb{R}^{m\times \ell}$. The closed loop system is given by

$$\dot{x}_c = A_c x_c, \quad x_c(0) = x_{c0} \quad (19)$$

where

$$Ac := \begin{bmatrix} A + BD_kC & BC_k \\ B_kC & A_k \end{bmatrix}, \quad x_c := \begin{bmatrix} x \\ x_k \end{bmatrix}. \quad (20)$$

4.1 Unity transient energy growth

Lemma 2. A necessary condition for unity transient energy growth, $\widehat{\mathcal{E}} = 1$, of the plant (17) with a stabilizing feedback controller (18) is that $(A + BD_kC) + (A + BD_kC)^T < 0$.

Proof: From Definition 1, the transient energy of the plant (17) is given by

$$\mathcal{E}(t) := \max_{\|x(0)\|=1} \|x(t)\|^2. \quad (21)$$

Let us replace $\mathcal{E}(t)$ by a modified energy function $\mathcal{E}_\epsilon(t)$ where

$$\mathcal{E}_\epsilon(t) := \max_{\|W_\epsilon^{-1}x_c(0)\|=1} \|W_\epsilon x_c(t)\|^2. \quad (22)$$

where $W_\epsilon := \mathrm{diag}(I_n, \epsilon I_{n_k})$ and $\epsilon \in \mathbb{R}_+$. Clearly as $\epsilon \to 0$, $\mathcal{E}_\epsilon \to \mathcal{E}$. Applying Lemma 1 to (20), $\max_t\{\mathcal{E}_\epsilon(t)\} = 1$ if and only if $W_\epsilon(A_c + A_c^T)W_\epsilon < 0$, that is

$$\begin{bmatrix} A_D + A_D^T & (BC_k + (BC_k)^T)\epsilon \\ (B_kC + C^TB_k^T)\epsilon & (A_k + A_k^T)\epsilon^2 \end{bmatrix} < 0, \quad (23)$$

where $A_D = A + BD_kC$. It is known (e.g. Horn and Johnson, 1985, p. 397) that all the diagonal submatrices of a negative definite matrix are negative definite. Hence $(A+BD_kC)+(A+BD_kC)^T < 0$ is a necessary condition for (23) to hold and for $\widehat{\mathcal{E}} = 1$. □

Remark 2. From the above lemma, it is clear that if no static controller that achieves unity transient energy growth exists, then no dynamic controller exists either.

4.2 Minimal transient energy growth by convex optimization

The operation

$$\max_{t\geq 0} \|\Phi(t)\| \quad (24)$$

represents a norm on the matrix function $\Phi(t)$. By means of a Q-parametrization, control system performance indices that are norms can be minimized by exploiting the convex properties of norms (Boyd and Barratt, 1991). For simplicity, here we just consider the case for an open loop stable system. Details on a parametrization for the unstable case are given in Boyd and Barratt (1991).

Assuming that the system given by (17) is stable, a convex realization of the closed loop system is given by

$$H(s) = U_1(s) + U_2(s)Q(s)U_3(s) \quad (25)$$

where $U_1(s) = (sI - A)^{-1}$, $U_2(s) = (sI - A)^{-1}B$ and $U_3(s) = C(sI - A)^{-1}$, and $Q(s)$ is the free parameter. It is clear that $\Phi(t) = \mathcal{L}^{-1}[H(s)] = 1/2\pi \int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega$.

The problem is then posed as follows

$$\widehat{\mathcal{E}}_{\min} = \min_{\text{stable } Q} \max_{t\geq 0} \|\Phi(t)\| \quad (26)$$

The set of all stable, proper $Q(s)$ can be parameterized by means of a Ritz approximation (Boyd

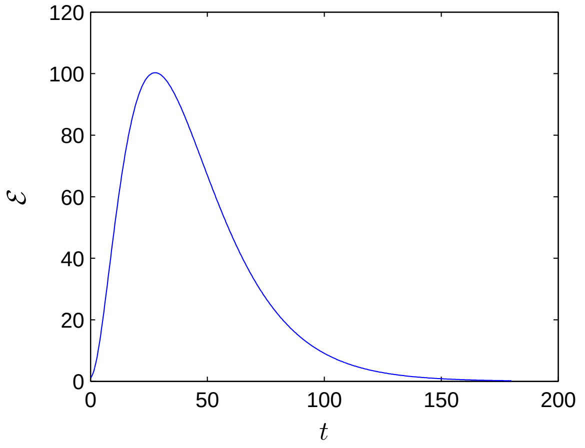

Fig. 1. Example 1: open-loop transient energy growth

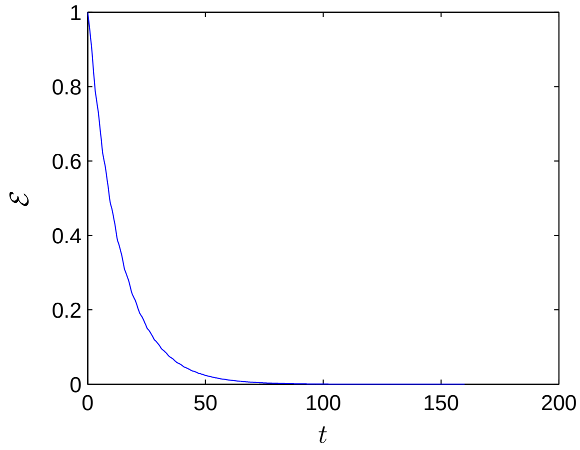

Fig. 2. Example 1: closed-loop transient energy growth

and Barratt, 1991; Linnemann, 1999). The final optimal controller is given by $K_{\mathrm{opt}} = (I + Q_{\mathrm{opt}}G)^{-1}Q_{\mathrm{opt}}$.

5. EXAMPLES

5.1 Example 1

The following example is adapted from Trefethen *et al.* (1993). The system was studied for the state feedback case in Whidborne *et al.* (2004). The linear system is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1/a & 1 \\ 0 & -2/a \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} u \end{aligned} \tag{27}$$

where $a = 40$.

The maximum transient energy growth for the open-loop system is calculated as $\widehat{\mathcal{E}} = 100.313$. The transient energy $\mathcal{E}(t)$ is shown in Figure 1.

From (11), for this example, the matrix R is a scalar with value $R = 1/\rho$ where $\rho > \rho_{\min}$, $\rho_{\min} = a/2 - 4/a = 19.9$. Setting $\rho = 20$ and $L = 0$ and applying Theorem 1 provides a controller $K = -0.99502$ that gives unity transient energy growth. The closed-loop transient energy $\mathcal{E}(t)$ is shown in Figure 2. The closed-loop eigenvalues are $-0.03750 \pm 0.99743j$.

For a simple second order system such as this example, the conditions for unity transient energy growth can be obtained in a straightforward manner. The closed-loop system matrix is

$$A + BKC = \begin{bmatrix} -1/a & (1+K) \\ 0 & -2/a \end{bmatrix}. \tag{28}$$

From condition (12), we deduce that the maximum transient energy growth is unity if and only if $(1+K)^2 < 8/a^2$, that is $(-1 - 2\sqrt{2}/a) < K < (-1 + 2\sqrt{2}/a)$, or $-1.07071 < K < -0.92929$.

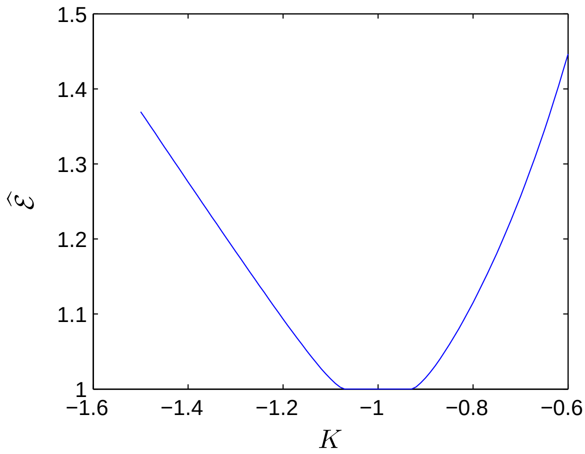

Fig. 3. Example 1: maximum transient energy growth as a function of controller gain K

Figure 3 shows the maximum transient energy growth as a function of the controller gain K. This confirms the bounds on K for unity maximum transient energy growth as well as showing the convex nature of the problem.

5.2 Example 2

The linear system is given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{29}$$

where

$$A = \begin{bmatrix} -2 & 2 & -8.5 & 3 & 3 \\ -2.5 & 1 & -14 & 7.5 & 6 \\ -1 & 0.5 & -6 & 3 & 2 \\ -1 & 1.5 & -9.5 & 4 & 3.5 \\ 0 & -0.5 & 2.5 & -1.5 & -2.5 \end{bmatrix}, \tag{30}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{31}$$

Table 1. Minimal maximum transient energy growth for increasing order of Q

q	$\widehat{\mathcal{E}}_{\min}^{\text{lower}}$	$\widehat{\mathcal{E}}_{\min}^{\text{upper}}$	iterations
0	60.9869	60.9940	79
4	6.9820	7.0341	381
8	6.8020	7.0060	537

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (32)$$

The system was studied for the state feedback case by Whidborne *et al.* (2004). The maximum transient energy growth for the open-loop system is calculated as $\widehat{\mathcal{E}} = 138.572$. The transient energy $\mathcal{E}(t)$ is shown in Figure 4.

From Theorem 1, no unity maximum transient energy growth controller was found to exist. To obtain a minimizing controller, the problem was posed as for (26).

The system is open-loop stable, hence the Q-parametrization is as for (25). The free parameter Q is given the form

$$Q(s) = Q_0 + Q_s(s) \qquad (33)$$

where Q_0 is a constant matrix and $Q_s(s)$ is parameterized using the state-space orthonormal basis suggested by Linnemann (1999). Let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence of real or complex (in conjugate pairs) numbers such that

$$\text{i)}\ \ \mathrm{Re}(\lambda_i) > 0 \text{ for all } i, \qquad (34)$$

$$\text{ii)}\ \ \sum_{i=1}^{\infty} \frac{\mathrm{Re}(\lambda_i)}{|\lambda_i|^2} = \infty, \qquad (35)$$

and any λ_i may be repeated. Then there exists a sequence of functions that provides an orthonormal basis for the space $\mathcal{L}_2$. Thus we can approximate a function in $\mathcal{L}_2$ by a truncated sequence $\{\lambda_i\}_{i=1}^{q} := \Lambda_q$ to an arbitrary accuracy. A state-space realization of the orthonormal basis is provided in Linnemann (1999). A multivariable extension is also provided.

A set of eigenvalues $\Lambda_s = \{1, 10, (1 \pm \sqrt{3}j)\}$ is chosen to provide the basis function sequences, Λ_q, such that $\Lambda_0 = \{\ \}$, $\Lambda_4 = \{\Lambda_s\}$, $\Lambda_8 = \{\Lambda_s, \Lambda_s\}$, $\Lambda_{12} = \{\Lambda_s, \Lambda_s, \Lambda_s\}$, etc. The ellipsoidal algorithm (Boyd and Barratt, 1991) was used to solve the convex optimization problem. The algorithm has proven convergence properties and, at each iteration, provides lower and upper bounds on $\widehat{\mathcal{E}}_{\min}$. The lower and upper bounds, respectively $\widehat{\mathcal{E}}_{\min}^{\text{lower}}$ and $\widehat{\mathcal{E}}_{\min}^{\text{upper}}$, for $q = 0, 4, 8$ are shown in Table 1. It can be seen that the solution has almost converged for $q = 8$.

The minimizing controller for $q = 0$, K_0, is given by

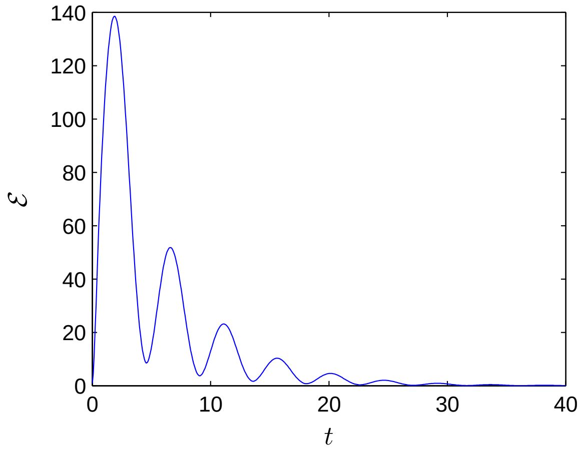

Fig. 4. Example 2: open-loop transient energy growth

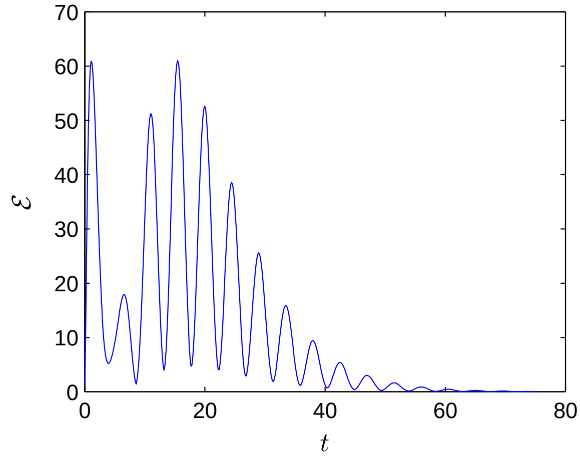

Fig. 5. Example 2: closed-loop transient energy growth for $q = 0$

$$A_k = \begin{bmatrix} -2 & 2 & -8.5 & 3 & 3 \\ -1.489 & 1 & -14 & 7.5 & 6 \\ -1 & 0.5 & -6 & 3 & 2 \\ -0.2541 & 1.5 & -9.5 & 4 & 3.5 \\ -1.462 & -0.5 & 2.5 & -1.5 & -2.5 \end{bmatrix} \qquad (36)$$

$$B_k = \begin{bmatrix} 0 \\ -1.011 \\ 0 \\ -0.7459 \\ 1.462 \end{bmatrix} \qquad (37)$$

$$C_k \begin{bmatrix} -1.462 & 0 & 0 & 0 & 0 \\ 1.011 & 0 & 0 & 0 & 0 \\ 0.7459 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (38)$$

$$D_k \begin{bmatrix} 1.462 \\ -1.011 \\ -0.7459 \end{bmatrix} \qquad (39)$$

and the closed-loop transient energy growth is shown in Figure 5. The minimizing controller for $q = 4$, K_4, is of 9th order and so is not presented. The closed-loop transient energy growth is shown in Figure 6.

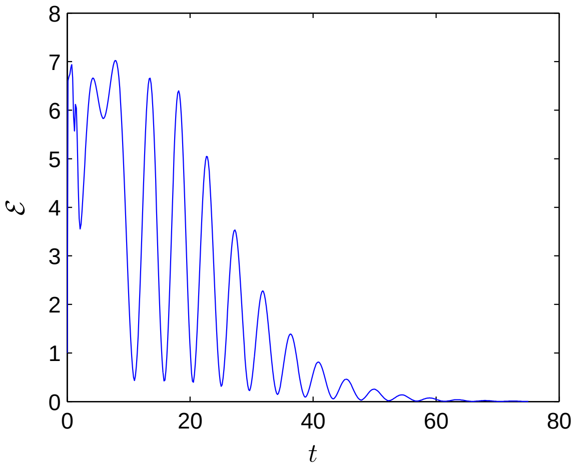

Fig. 6. Example 2: closed-loop transient energy growth for $q = 4$

6. DISCUSSION AND CONCLUSIONS

Methods to calculate the maximum transient energy growth of linear systems by output feedback are provided. All constant gain controllers that restrict the maximum transient energy growth to unity are provided. It is also shown that if no constant gain controller that restricts the maximum transient energy growth to unity exists, then no dynamic controller exists either. It is shown that by a Q-parametrization, the problem of minimizing the maximum transient energy growth is convex in the free parameter Q. Hence, by means of a Ritz approximation, sub-optimal controllers that minimize the maximum transient energy growth can also be obtained by convex programming.

The methods are illustrated by two numerical examples. The energy responses for Example 2 show that the closed-loop systems are very resonant and the controllers clearly do not provide good designs. The intention is not necessarily to design controllers that meet all the required closed-loop requirements, but to provide designers with a means of determining the minimum of the maximum transient energy gain so that the specifications for the controller design can be sensibly set. Alternatively, the convex optimization over Q approach can be used to incorporate other design objectives (Boyd and Barratt, 1991). MATLAB software to achieve this is available (Khaisongkram and Banjerdpongchai, 2003). A weakness of the approach is that it is not clear how to choose the set Λ_s. In the example it was chosen after a small amount of trial and error. Faster convergence could be obtained with another choice.

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