NUMERICAL COMPUTATION OF PARETO OPTIMAL STRATEGY FOR GENERAL MULTIPARAMETER SINGULARLY PERTURBED SYSTEMS

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Abstract: In this paper, Pareto optimal strategy for general multiparameter singularly perturbed systems is investigated. The main contribution is to propose a new computational method for obtaining the high–order Pareto near–optimal strategy. Newton's method and two fixed point algorithms are combined. As a result, the new iterative algorithm achieves the quadratic convergence property and succeeds in reducing the computing workspace dramatically. It is newly shown that the resulting optimal strategy achieves the cost functional $J_j^* + O(\|\mu\|^{2^i})$.

Keywords: Numerical algorithms, Singular perturbations, Differential games, Optimal control.

1. INTRODUCTION

Multimodeling stability, control and filtering problems have been investigated extensively (see e.g., (Khalil and Kokotović, 1978; Khalil and Kokotović, 1979; Gajić, 1988; Coumarbatch and Gajic, 2000; Wang et al., 1994)). The popular approaches to deal with the multiparameter singularly perturbed systems (MSPS) are the two–time–scale design method (Khalil and Kokotović, 1978; Khalil and Kokotović, 1979; Gajić, 1988) and the descriptor technique (Wang et al., 1994). When the positive parameters $\varepsilon_j,\ j = 1,\ \dots,N$ are very small or unknown the previously used techniques are very efficient. However, when the parameters ε_j are not small enough, it is known from (Coumarbatch and Gajic, 2000) that an $O(\|\mu\|)$ accuracy is very often not sufficient.

In order to avoid the $O(\|\mu\|)$ accuracy of the cost, the exact decomposition method has been studied (Coumarbatch and Gajic, 2000). More recently, the numerical algorithm which is based on Newton's method for solving the multiparameter algebraic Riccati equation (MARE) has been established (Mukaidani et al., 2002). However, these approaches can only be applied to the MSPS which has two fast subsystems. From the viewpoint of application of practical systems, it is very important to study the general MSPS that includes the numerous fast subsystems. Furthermore, the numerical algorithm which require smaller computational dimension for solving the generalized MARE has to be developed.

In this paper, Pareto optimal strategy for the general multiparameter singularly perturbed systems (MSPS) which includes numerous fast subsystems compared with the previous results (Coumarbatch and Gajic, 2000; Mukaidani et al., 2002) is investigated via the numerical computation method. The main contribution of this paper is to propose a new numerical algorithm to obtain Pareto optimal strategy. Our new idea is to combine Newton's method with two fixed point algorithms for solving the generalized multiparameter algebraic Riccati equation (GMARE). As a result, although the general MSPS has numerous fast subsystems, the required workspace for computing the solution is dramatically small. As another important feature, it is newly shown that the proposed strategy achieves the cost functional $J_j^* + O(\|\mu\|^{2^i})$, $\mu = [\varepsilon_1 \ \dots \ \varepsilon_N]$.

2. PARETO OPTIMAL STRATEGY

Let us consider a linear time–invariant general MSPS (Özgüner, 1979; Mukaidani et al., 2003)

$$\dot{x}_0(t) = \sum_{j=0}^{N} A_{0j} x_j(t) + \sum_{j=1}^{N} B_{0j} u_j(t), \quad \text{(1a)}$$

$$\varepsilon_j \dot{x}_j(t) = A_{j0} x_0(t) + A_{jj} x_j(t) + B_{jj} u_j(t), \quad \text{(1b)}$$

$$x_j(0) = x_j^0, \ j = 0, \ 1, \ \dots, N,$$

where $x_j(t) \in \mathbf{R}^{n_j}$, $j = 0, \ 1, \ \cdots, N$ are the state vectors, $u_j(t) \in \mathbf{R}^{m_j}$, $j = 1, \ \cdots, N$ are the control inputs. It is assumed that the ratios of the small positive parameters $\varepsilon_j > 0, j = 1, \ \cdots, N$ are bounded by some positive constants $\underline{k}_{ij}$, $\bar{k}_{ij}$ (Khalil and Kokotović, 1978; Khalil and Kokotović, 1979),

$$0 < \underline{k}_{ij} \le \alpha_{ij} \equiv \frac{\varepsilon_j}{\varepsilon_i} \le \bar{k}_{ij} < \infty. \quad (2)$$

Note that the fast state matrices A_{jj}, $j = 1, \ \dots, N$ may be singular. In Pareto optimal strategy of the above general MSPS (1), the quadratic cost functionals are given by

$$J_j = \frac{1}{2} \int_0^\infty [z_j^T(t) z_j(t) + u_j^T(t) R_j u_j(t)] dt, \quad (3)$$

where $z_j(t) = C_{j0} x_0(t) + C_{jj} x_j(t) \in \mathbf{R}^{r_j}$, $j = 1, \ \dots, N$.

Pareto solution is a set $(u_1, \ \dots, u_N)$ which minimizes

$$J = \sum_{j=1}^{N} \gamma_j J_j, \ 0 < \gamma_j < 1, \ \sum_{j=1}^{N} \gamma_j = 1 \quad (4)$$

for some γ_j, $j = 1, \ \dots, N$. It is well–known that Pareto optimal strategy is given by

$$u_j^*(t) = -\gamma_j^{-1} R_j^{-1} B_j^T P x(t), \ j = 1, \ \dots, N, \quad (5)$$

where P is the solution of the following GMARE such that $\Phi_e P$ is the unique positive semidefinite stabilizing solution.

$$A^T P + P^T A - P^T S P + Q = 0, \quad (6)$$

where

$$\Phi_e := \mathbf{block\ diag} \left(I_{n_0} \ \varepsilon_1 I_{n_1} \ \cdots \ \varepsilon_N I_{n_N} \right),$$

$$A := \begin{bmatrix} A_{00} & A_{0f} \\ A_{f0} & A_f \end{bmatrix}, \ A_{0f} := \begin{bmatrix} A_{01} & \cdots & A_{0N} \end{bmatrix},$$

$$A_{f0} := \begin{bmatrix} A_{10}^T & \cdots & A_{N0}^T \end{bmatrix}^T,$$

$$A_f := \mathbf{block\ diag} \left(A_{11} \ \cdots \ A_{NN} \right),$$

$$B_1 := \begin{bmatrix} B_{01}^T & B_{11}^T & 0 & 0 & \cdots & 0 \end{bmatrix}^T, \ \dots,$$

$$B_N := \begin{bmatrix} B_{0N}^T & 0 & 0 & 0 & \cdots & B_{NN}^T \end{bmatrix}^T,$$

$$C_1 := \begin{bmatrix} C_{10} & C_{11} & 0 & 0 & \cdots & 0 \end{bmatrix}, \ \dots,$$

$$C_N := \begin{bmatrix} C_{N0} & 0 & 0 & 0 & \cdots & C_{NN} \end{bmatrix},$$

$$S = \sum_{j=1}^{N} \gamma_j^{-1} S_j, \ Q = \sum_{j=1}^{N} \gamma_j Q_j,$$

$$S_j := B_j R_j^{-1} B_j^T, \ Q_j := C_j^T C_j, \ j = 1, \ \dots, N,$$

$$S := \begin{bmatrix} S_{00} & S_{0f} \\ S_{0f}^T & S_f \end{bmatrix}, \ S_{00} := \sum_{j=1}^{N} \gamma_j^{-1} B_{0j} R_j^{-1} B_{0j}^T,$$

$$S_{0f} := \begin{bmatrix} S_{01} & \cdots & S_{0N} \end{bmatrix}, \ S_{0j} := \gamma_j^{-1} B_{0j} R_j^{-1} B_{jj}^T,$$

$$S_f := \mathbf{block\ diag} \left(S_{11} \ \cdots \ S_{NN} \right),$$

$$S_{jj} := \gamma_j^{-1} B_{jj} R_j^{-1} B_{jj}^T,$$

$$Q := \begin{bmatrix} Q_{00} & Q_{0f} \\ Q_{0f}^T & Q_f \end{bmatrix}, \ Q_{00} := \sum_{j=0}^{N} C_{j0}^T C_{j0},$$

$$Q_{0f} := \begin{bmatrix} Q_{01} & \cdots & Q_{0N} \end{bmatrix}, \ Q_{0j} := C_{j0}^T C_{jj},$$

$$Q_f := \mathbf{block\ diag} \left(Q_{11} \ \cdots \ Q_{NN} \right),$$

$$Q_{jj} := C_{jj}^T C_{jj}.$$

In order to avoid the ill–conditioned caused by ε_j^{-1}, the GMARE is used instead of the ordinary multiparameter algebraic Riccati equation (MARE). It should be noted that the GMARE is introduced in (Mukaidani et al., 2003). It is assumed that the solution P of the GMARE (6) has the following structure.

$$P := \begin{bmatrix} P_{00} & P_{f0}^T \Pi_e \\ P_{f0} & P_f \end{bmatrix}, \ P_{00} = P_{00}^T,$$

$$P_{f0} := \begin{bmatrix} P_{10}^T & \cdots & P_{N0}^T \end{bmatrix}^T,$$

$$P_f := \begin{bmatrix} P_{11} & \alpha_{12} P_{21}^T & \alpha_{13} P_{31}^T & \cdots & \alpha_{1N} P_{N1}^T \\ P_{21} & P_{22} & \alpha_{23} P_{32}^T & \cdots & \alpha_{2N} P_{N2}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{N1} & P_{N2} & P_{N3} & \cdots & P_{NN} \end{bmatrix},$$

$$\Pi_e P_f = P_f^T \Pi_e,$$

$$\Pi_e := \mathbf{block\ diag} \left(\varepsilon_1 I_{n_1} \ \cdots \ \varepsilon_N I_{n_N} \right).$$

The near–optimal Pareto strategy for the MSPS that has two fast subsystems has been proposed in (Khalil and Kokotović, 1978). However, when the parameters ε_j are not small enough, the previous technique (Khalil and Kokotović, 1978) is very often not sufficient. To improve the $O(\|\mu\|)$, $\mu = [\varepsilon_1 \ \dots \ \varepsilon_N]$ accuracy of the cost for large parameters ε_j, a new numerical method which is different from the existing method (Coumarbatch and Gajic, 2000; Mukaidani et al., 2002) for the MSPS is proposed.

3. ASYMPTOTIC STRUCTURE OF GMARE

Before the design of Pareto strategy, the asymptotic structure of the GMARE (6) is investigated. In the following analysis, some assumptions are needed. These assumptions play an important role in proving the results which will be given later.

Assumption 1. The triples $(A_{jj},\ B_{jj},\ C_{jj}),\ j = 1,\ \dots,N$ are stabilizable and detectable.

Assumption 2.

$$\mathrm{rank}\begin{bmatrix} sI_{n_0} - A_{00} & -A_{0f} & B_0 \\ -A_{f0} & -A_f & B_f \end{bmatrix} = \bar{n}, \quad (7a)$$

$$\mathrm{rank}\begin{bmatrix} sI_{n_0} - A_{00}^T & -A_{f0}^T & C_0^T \\ -A_{0f}^T & -A_f^T & C_f^T \end{bmatrix} = \bar{n}, \quad (7b)$$

where $\forall s \in \mathbf{C}$, $\mathrm{Re}[s] \ge 0$ and

$$\bar{n} := \sum_{j=0}^{N} n_j,\ B_0 := \left[\, B_{01}\ \cdots\ B_{0N} \,\right],$$

$$B_f := \textbf{block diag}\left(B_{11}\ \cdots\ B_{NN} \right),$$

$$C_0 := \left[\, C_{10}^T\ \cdots\ C_{N0}^T \,\right]^T,$$

$$C_f := \textbf{block diag}\left(C_{11}\ \cdots\ C_{NN} \right).$$

Assumption 3. The Hamiltonian matrices

$$T_{jj} := \begin{bmatrix} A_{jj} & -S_{jj} \\ -Q_{jj} & -A_{jj}^T \end{bmatrix},\ j = 1,\ \dots,N$$

are nonsingular.

Using the existing result (Mukaidani et al., 2003), it is easy to derive the following useful lemma.

Lemma 4. Under Assumptions 1–3, there exists a small σ^* such that for all $\|\mu\| \in (0,\ \sigma^*)$, the GMARE (6) admits a symmetric positive semidefinite stabilizing solution $\Phi_e P$ which can be written as

$$P = \begin{bmatrix} \bar{P}_{00} + O(\|\mu\|) & [\bar{P}_{f0} + O(\|\mu\|)]^T \Pi_e \\ \bar{P}_{f0} + O(\|\mu\|) & \bar{P}_f + O(\|\mu\|) \end{bmatrix}, \quad (8)$$

where

$$\bar{P}_{00}\mathcal{A} + \mathcal{A}^T \bar{P}_{00} - \bar{P}_{00}\mathcal{S}\bar{P}_{00} + \mathcal{Q} = 0, \quad (9a)$$

$$\bar{P}_{j0} = \left[\, \bar{P}_{jj}\ -I_{n_j} \,\right] T_{jj}^{-1} T_{j0} \begin{bmatrix} I_{n_0} \\ \bar{P}_{00} \end{bmatrix}, \quad (9b)$$

$$\bar{P}_{jj}A_{jj} + A_{jj}^T \bar{P}_{jj} - \bar{P}_{jj}S_{jj}\bar{P}_{jj} + Q_{jj} = 0, \quad (9c)$$

with

$$\bar{P}_{f0} := \left[\, \bar{P}_{10}^T\ \cdots\ \bar{P}_{N0}^T \,\right]^T,$$

$$\bar{P}_f := \textbf{block diag}\left(\bar{P}_{11}\ \cdots\ \bar{P}_{NN} \right),$$

$$\mathcal{T} := \begin{bmatrix} \mathcal{A} & -\mathcal{S} \\ -\mathcal{Q} & -\mathcal{A}^T \end{bmatrix} = T_{00} - \sum_{j=1}^{N} T_{0j} T_{jj}^{-1} T_{j0},$$

$$T_{00} := \begin{bmatrix} A_{00} & -S_{00} \\ -Q_{00} & -A_{00}^T \end{bmatrix},\ T_{0j} := \begin{bmatrix} A_{0j} & -S_{0j} \\ -Q_{0j} & -A_{j0}^T \end{bmatrix},$$

$$T_{j0} := \begin{bmatrix} A_{j0} & -S_{0j}^T \\ -Q_{0j}^T & -A_{0j}^T \end{bmatrix},\ j = 1,\ \dots,N.$$

Proof : Since the proof can be done by using the implicit function theorem, it is omitted. See detail in (Mukaidani et al., 2003). □

4. A NEW ITERATIVE ALGORITHM

In order to solve the GMARE (6) without the ill–conditioned, the following algorithm is established.

Lemma 5. Consider the iterative algorithm which is based on Newton's method

$$\begin{aligned} &(A - SP^{(i)})^T P^{(i+1)} + P^{(i+1)T}(A - SP^{(i)}) \\ &\quad + P^{(i)T} S P^{(i)} + Q = 0,\ P^{(0)} = \bar{P}, \\ &\quad i = 0,\ \dots, \end{aligned} \quad (10)$$

with

$$\bar{P} = \begin{bmatrix} \bar{P}_{00} & \bar{P}_{f0}^T \Pi_e \\ \bar{P}_{f0} & \bar{P}_f \end{bmatrix},\ P^{(i)} = \begin{bmatrix} P_{00}^{(i)} & P_{f0}^{(i)T} \Pi_e \\ P_{f0}^{(i)} & P_f^{(i)} \end{bmatrix}. \quad (11)$$

Under Assumptions 1–3, there exists a small $\bar{\sigma}$ such that for all $\|\mu\| \in (0,\ \bar{\sigma})$, $\bar{\sigma} \le \sigma^*$, the iterative algorithm (10) converges to the exact solution of P with the rate of quadratic convergence, where $\Phi_e P^{(i)} = P^{(i)T}\Phi_e$ is the positive semidefinite solution. That is, the following condition is satisfied.

$$\|P^{(i)} - P\| = O(\|\mu\|^{2^i}),\ i = 0,\ 1,\ \dots. \quad (12)$$

Proof : Since the proof of Lemma 5 can be done by using Newton–Kantorovich theorem similarly as in (Mukaidani et al., 2001), it is omitted. For Newton–Kantorovich theorem , see e.g. (Yamamoto, 1986). □

One needs to solve the GMALE (10) with the dimension $\bar{n} := \sum_{j=0}^{N} n_j$ larger than the dimension $n_j,\ j = 0,\ \dots,N$ compared with the exact decomposition technique (Coumarbatch and Gajic, 2000). Thus, in order to reduce the dimension of the workspace, the new algorithm for solving the MALE (10) which is based on the fixed point algorithm is established. Let us consider the following GMALE (13), in a general form.

$$\Lambda^T Y + Y^T \Lambda + U = 0, \quad (13)$$

where Y is the solution of the GMALE (13). Moreover, Y, Λ and U have the following forms, respectively.

$$Y := \begin{bmatrix} Y_{00} & Y_{f0}^T \Pi_e \\ Y_{f0} & Y_f \end{bmatrix},\ Y_{00} = Y_{00}^T,$$
$$Y_{f0} := \begin{bmatrix} Y_{10}^T & \cdots & Y_{N0}^T \end{bmatrix}^T,$$
$$Y_f := \begin{bmatrix} Y_{11} & \alpha_{12}\varepsilon E_{21}^T & \alpha_{13}\varepsilon E_{31}^T & \cdots & \alpha_{1N}\varepsilon E_{N1}^T \\ \varepsilon E_{21} & Y_{22} & \alpha_{23}\varepsilon E_{32}^T & \cdots & \alpha_{2N}\varepsilon E_{N2}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon E_{(N-1)1} & \varepsilon E_{(N-1)2} & \varepsilon E_{(N-1)3} & \cdots & \alpha_{(N-1)N}\varepsilon E_{N(N-1)}^T \\ \varepsilon E_{N1} & \varepsilon E_{N2} & \varepsilon E_{N3} & \cdots & Y_{NN} \end{bmatrix},$$
$$\Pi_e Y_f = Y_f^T \Pi_e,\ \varepsilon := \|\mu\| = \sqrt{\varepsilon_1^2 + \cdots + \varepsilon_N^2},$$
$$\Lambda := \begin{bmatrix} \Lambda_{00} & \Lambda_{0f} \\ \Lambda_{f0} & \Lambda_f \end{bmatrix},\ \Lambda_{0f} := \begin{bmatrix} \Lambda_{01} & \cdots & \Lambda_{0N} \end{bmatrix},$$
$$\Lambda_{f0} := \begin{bmatrix} \Lambda_{10}^T & \cdots & \Lambda_{N0}^T \end{bmatrix}^T,$$
$$\Lambda_f := \begin{bmatrix} \Lambda_{11} & \varepsilon\Lambda_{12} & \cdots & \varepsilon\Lambda_{1N} \\ \varepsilon\Lambda_{21} & \Lambda_{22} & \cdots & \varepsilon\Lambda_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon\Lambda_{N1} & \varepsilon\Lambda_{N2} & \cdots & \Lambda_{NN} \end{bmatrix},$$
$$U := \begin{bmatrix} U_{00} & U_{0f} \\ U_{0f}^T & U_f \end{bmatrix},\ U_{0f} := \begin{bmatrix} U_{01} & \cdots & U_{0N} \end{bmatrix},$$
$$U_f := \begin{bmatrix} U_{11} & \varepsilon U_{12} & \cdots & \varepsilon U_{1N} \\ \varepsilon U_{12}^T & U_{22} & \cdots & \varepsilon U_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon U_{1N}^T & \varepsilon U_{2N}^T & \cdots & U_{NN} \end{bmatrix},$$
$$U_{00} = U_{00}^T,\ U_f = U_f^T.$$

It should be noted that

$$P^{(i+1)} \Rightarrow Y,\ A - SP^{(i)} \Rightarrow \Lambda,$$
$$P^{(i)T} S P^{(i)} + Q \Rightarrow U$$

where $\Rightarrow$ stands for the replacement.

Without loss of generality, the following condition for the GMALE (13) is assumed.

Assumption 6. $\Lambda_{jj},\ j = 1,\ \ldots,N$ and $\Lambda_0 := \Lambda_{00} - \sum_{j=1}^{N} \Lambda_{0j}\Lambda_{jj}^{-1}\Lambda_{j0}$ are stable.

The following algorithm (14) for solving the GMALE (13) is given.

$$\Lambda_f^T Y_f^{(l+1)} + Y_f^{(l+1)T}\Lambda_f + (\Lambda_{0f}^T Y_{f0}^{(l)T}\Pi_e$$
$$+\Pi_e Y_{f0}^{(l)}\Lambda_{0f}) + U_f = 0, \tag{14a}$$
$$\Lambda_0^T Y_{00}^{(l+1)} + Y_{00}^{(l+1)}\Lambda_0 - \Lambda_{f0}^T\Lambda_f^{-T}\Xi_{f0}^{(l)}$$
$$-\Xi_{f0}^{(l)T}\Lambda_f^{-1}\Lambda_{f0} + U_{00} = 0, \tag{14b}$$
$$Y_{f0}^{(l+1)} = -\Lambda_f^{-T}(\Lambda_{0f}^T Y_{00}^{(l+1)} + \Xi_{f0}^{(l)}), \tag{14c}$$

where

$$\Lambda_0 = \Lambda_{00} - \Lambda_{0f}\Lambda_f^{-1}\Lambda_{f0},$$
$$\Xi_{f0}^{(l)} = \Pi_e Y_{f0}^{(l)}\Lambda_{00} + Y_f^{(l+1)}\Lambda_{f0} + U_{0f}^T,$$
$$Y_{00}^{(0)} = \bar{Y}_{00},\ Y_{f0}^{(0)} = \bar{Y}_{f0},\ Y_f^{(0)} = \bar{Y}_f,$$
$$\bar{\Lambda}_0^T\bar{Y}_{00} + \bar{Y}_{00}\bar{\Lambda}_0 - \Lambda_{f0}^T\bar{\Lambda}_f^{-T}U_{0f}^T - U_{0f}\bar{\Lambda}_f^{-1}\Lambda_{f0}$$
$$+\Lambda_{f0}^T\bar{\Lambda}_f^{-T}\bar{U}_f\bar{\Lambda}_f^{-1}\Lambda_{f0} + U_{00} = 0,$$
$$\bar{Y}_{f0}^T = -(\bar{Y}_{00}\Lambda_{0f} + \Lambda_{f0}^T\bar{Y}_f + U_{0f})\bar{\Lambda}_f^{-1},$$
$$\bar{\Lambda}_0 = \Lambda_{00} - \Lambda_{0f}\bar{\Lambda}_f^{-1}\Lambda_{f0},$$
$$\bar{Y}_f := \textbf{block diag}\begin{pmatrix} \bar{Y}_{11} & \cdots & \bar{Y}_{NN} \end{pmatrix},$$
$$\bar{\Lambda}_f := \textbf{block diag}\begin{pmatrix} \Lambda_{11} & \cdots & \Lambda_{NN} \end{pmatrix},$$
$$\bar{U}_f := \textbf{block diag}\begin{pmatrix} U_{11} & \cdots & U_{NN} \end{pmatrix},$$
$$\Lambda_{jj}^T\bar{Y}_{jj} + \bar{Y}_{jj}\Lambda_{jj} + U_{jj} = 0,\ j = 1,\ \ldots,N.$$

The following theorem indicates the convergence of the algorithm (14).

Theorem 7. Under Assumption 6, the fixed point algorithm (14) converges to the exact solutions Y_{00}, Y_{f0} and Y_f with the rate of convergence of $O(\|\mu\|^{l+1})$, that is

$$\|Y_f^{(l)} - Y_f\| = O(\|\mu\|^{l+1}),\ l = 0,\ 1,\ \ldots, \tag{15a}$$
$$\|Y_{00}^{(l)} - Y_{00}\| = O(\|\mu\|^{l+1}),\ l = 0,\ 1,\ \ldots, \tag{15b}$$
$$\|Y_{f0}^{(l)} - Y_{f0}\| = O(\|\mu\|^{l+1}),\ l = 0,\ 1,\ \ldots. \tag{15c}$$

Proof : Since the proof is done by applying the mathematical induction and the fixed point theorem, it is omitted. □

In order to solve the ALE (14a), not each dimension $n_i,\ i = 1,\ \ldots,N$ but the very large dimension $\hat{n} := \sum_{i=1}^{N} n_i$ is needed. Thus, the reduction of the dimension of the computing workspace must be needed. Therefore, the new algorithm for solving the ALE (14a) which is based on the fixed point algorithm is established. Let us consider the following ALE (16), in a general form.

$$\Psi_e^T X_e + X_e^T\Psi_e + V_e = 0, \tag{16}$$

where X_e is the solution of the ALE (16). Moreover, X_e, Ψ_e and V_e have the following forms, respectively.

$$X_e = \begin{bmatrix} X_{11} & \alpha_{12}\varepsilon X_{21}^T & \alpha_{13}\varepsilon X_{31}^T & \cdots & \alpha_{1N}\varepsilon X_{N1}^T \\ \varepsilon X_{21} & X_{22} & \alpha_{23}\varepsilon X_{32}^T & \cdots & \alpha_{2N}\varepsilon X_{N2}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon X_{(N-1)1} & \varepsilon X_{(N-1)2} & \varepsilon X_{(N-1)3} & \cdots & \alpha_{(N-1)N}\varepsilon X_{N(N-1)}^T \\ \varepsilon X_{N1} & \varepsilon X_{N2} & \varepsilon X_{N3} & \cdots & X_{NN} \end{bmatrix},$$

$$\Psi_e := \begin{bmatrix} \Psi_{11} & \varepsilon\Psi_{12} & \cdots & \varepsilon\Psi_{1N} \\ \varepsilon\Psi_{21} & \Psi_{22} & \cdots & \varepsilon\Psi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon\Psi_{N1} & \varepsilon\Psi_{N2} & \cdots & \Psi_{NN} \end{bmatrix},$$

$$V_e := \begin{bmatrix} V_{11} & \varepsilon V_{12} & \cdots & \varepsilon V_{1N} \\ \varepsilon V_{12}^T & V_{22} & \cdots & \varepsilon V_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon V_{1N}^T & \varepsilon V_{2N}^T & \cdots & V_{NN} \end{bmatrix}.$$

It should be noted that

$$Y_f^{(l+1)} \Rightarrow X_e,\ \Lambda_f \Rightarrow \Psi_e,$$
$$\Lambda_{0f}^T Y_{f0}^{(l)T}\Pi_e + \Pi_e Y_{f0}^{(l)}\Lambda_{0f} + U_f \Rightarrow V_e$$

where $\Rightarrow$ stands for the replacement. Furthermore, the ALE (16) is a part of the ALE (13).

Without loss of generality, the following condition for the ALE (16) is also assumed.

Assumption 8. $\Psi_{11},\ \dots,\Psi_{NN}$ are stable.

The following algorithms (17) for solving the ALE (16) are newly given.

$$X_{11}^{(m+1)}\Psi_{11} + \Psi_{11}^T X_{11}^{(m+1)} + \varepsilon^2\sum_{l=2}^{N}(X_{1l}^{(m)}\Psi_{l1} + \Psi_{l1}^T X_{l1}^{(m)}) + V_{11} = 0, \quad (17a)$$

$$\vdots$$

$$X_{NN}^{(m+1)}\Psi_{NN} + \Psi_{NN}^T X_{NN}^{(m+1)} + \varepsilon^2\sum_{l=1}^{N-1}(\alpha_{lN}X_{Nl}^{(m)}\Psi_{li} + \alpha_{lN}\Psi_{lN}^T X_{lN}^{(m)}) + V_{NN} = 0, \quad (17b)$$

$$X_{12}^{(m+1)}\Psi_{22} + \alpha_{12}\Psi_{11}^T X_{12}^{(m+1)} + X_{11}^{(m+1)}\Psi_{12} + \Psi_{21}^T X_{22}^{(m+1)} + \varepsilon\sum_{l=3}^{N}(X_{1l}^{(m)}\Psi_{l2} + \Psi_{l1}^T X_{l2}^{(m)}) + V_{12} = 0, \quad (17c)$$

$$\vdots$$

$$X_{(N-1)N}^{(m+1)}\Psi_{NN} + \alpha_{(N-1)N}\Psi_{(N-1)(N-1)}^T X_{(N-1)N}^{(m+1)} + X_{(N-1)(N-1)}^{(m+1)}\Psi_{(N-1)N} + \Psi_{N(N-1)}^T X_{NN}^{(m+1)} + \varepsilon\sum_{l=1}^{N-2}(\alpha_{l(N-1)}X_{(N-1)l}^{(m)}\Psi_{lN} + \alpha_{ln}\Psi_{l(N-1)}^T X_{lN}^{(m)}) + V_{(N-1)N} = 0, \quad (17d)$$

$m = 0,\ 1,\ \cdots,$

where

$$X_{ii}^{(0)} = \bar{X}_{ii},\ X_{ij}^{(0)} = \bar{X}_{ij},\quad i<j,\ \bar{X}_{ij} = \bar{X}_{ji}^T,$$
$$\bar{X}_{ii}\Psi_{ii} + \Psi_{ii}^T\bar{X}_{ii} + V_{ii} = 0,$$
$$\bar{X}_{ij}\Psi_{jj} + \Psi_{ii}^T\bar{X}_{ij} + \bar{X}_{ii}\Psi_{ij} + \Psi_{ji}^T\bar{X}_{jj} + V_{ii} = 0.$$

The following theorem indicates the convergence of the algorithm (17).

Theorem 9. Under Assumption 8, the fixed point algorithm (17) converges to the exact solution X_{ij} with the rate of

$$\|X_{ii}^{(m)} - X_{ii}\| = O(\varepsilon^{m+2}),\ m = 1,\ \dots\ , \quad (18a)$$
$$\|X_{ij}^{(m)} - X_{ij}\| = O(\varepsilon^{m+1}),\ i<j,\ m = 1,\ \dots \quad (18b)$$

Proof : The proof of Theorem 9 can be also done by using mathematical induction and the fixed point theorem. In order to respect the pages limitation, it is omitted. □

An algorithm which solves the GMARE (6) with the small positive parameters ε_j is given below.

Step 1. *Solve the AREs (9) that are given as the initial conditions of the Newton's method (10).*

Step 2. *Partitioning the solution $P^{(i+1)}$ of the purpose into*

$$P^{(i+1)} = \begin{bmatrix} Y_{00} & Y_{f0}^T\Pi_e \\ Y_{f0} & Y_f \end{bmatrix},$$
$$A - SP^{(i)} = \begin{bmatrix} \Lambda_{00} & \Lambda_{0f} \\ \Lambda_{f0} & \Lambda_f \end{bmatrix},$$
$$P^{(i)T}SP^{(i)} + Q = \begin{bmatrix} U_{00} & U_{0f} \\ U_{0f}^T & U_f \end{bmatrix},$$

and do the preparation for solving the following GMALE.

$$\Lambda_f^T Y_f + Y_f\Lambda_f + (\Lambda_{0f}^T Y_{f0}\Pi_e + \Pi_e Y_{f0}\Lambda_{0f}) + U_f = 0, \quad (19a)$$
$$\Lambda_0^T Y_{00} + Y_{00}\Lambda_0 - \Lambda_{f0}^T\Lambda_f^{-T}\Xi_{f0} - \Xi_{f0}\Lambda_f^{-1}\Lambda_{f0} + U_{00} = 0, \quad (19b)$$
$$\Lambda_f^T Y_{f0} + \Lambda_{0f}^T Y_{00} + \Xi_{f0} = 0, \quad (19c)$$

where $\Xi_{f0} = \Pi_e Y_{f0}\Lambda_{00} + Y_f\Lambda_{f0} + U_{0f}^T$.

Step 3. *In order to solve the GMALE (19), apply the new proposed algorithm (14).*

Step 4. *In order to reduce the dimension of the workspace for solving the ALE (14a), apply the new proposed algorithm (17).*

Step 5. *Solve the solutions* $Y_f^{(l+1)}$ *and* $Y_{00}^{(l+1)}$ *of the ALE (14a) and (14b), respectively and compute* $Y_{f0}^{(l+1)}$ *using the relation of (14c). As a result, the sequence of solution of Newton's method (10) is obtained.*

Step 6. *If the new combined algorithm converges, go to Step 7. Otherwise, increment* $i \to i+1$ *and go to Step 3.]*

Step 7. *Calculate the solution* P *of the GMARE (6) by using (11).*

5. HIGH–ORDER APPROXIMATE PARETO OPTIMAL STRATEGY

Our attention is focused on the optimal strategy design. Such a strategy is obtained by using the iterative solutions (10). The high–order approximate Pareto optimal strategy is newly given.

$$u_{\mathrm{app}j} = -\gamma_j^{-1} R_j^{-1} B_j^T P^{(i)} x,\ j = 1,\ \dots ,N. \quad (20)$$

Theorem 10. Under Assumptions 1–3, the use of the high–order approximate Pareto strategy (20) results in $J_j^{(i)}$ satisfying

$$J_j^{(i)} = J_j^* + O(\|\mu\|^{2^i}),\ j = 1,\ \dots ,N, \quad (21)$$

where the value of the actual cost is

$$J_j^{(i)} = \frac{1}{2} x(0)^T \Phi_e Y_j x(0) = \frac{1}{2} x(0)^T Y_{je} x(0) \quad (22)$$

and Y_{je} is a positive semidefinite solution of the multiparameter algebraic Lyapunov equation (MALE)

$$\begin{aligned} &Y_{je}(A_e - S_e P_e^{(i)}) + (A_e - S_e P_e^{(i)})^T Y_{je} \\ &\quad + Q_j + \gamma_j^{-2} P_e^{(i)} S_{je} P_e^{(i)} = 0,\ j = 1,\ \dots ,N. \quad (23) \end{aligned}$$

Proof: When Pareto optimal strategy (5) is given, they result in $J_j^* = \frac{1}{2} x(0)^T X_{je} x(0),\ j = 1,\ \dots ,N$, where X_{je} is a positive semidefinite solution of the MALE: $X_{je}(A_e - S_e P_e) + (A_e - S_e P_e)^T X_{je} + Q_j + \gamma_j^{-2} P_e S_{je} P_e = 0,\ j = 1,\ \dots ,N$. In order to calculate the loss of performance $J_j^{(i)} - J_j^*$, subtracting these equations, $Z_{je} = Y_{je} - X_{je}$ satisfies the MALE: $Z_{je}(A_e - S_e P_e^{(i)}) + (A_e - S_e P_e^{(i)})^T Z_{je} + \gamma_j^{-2} P_e^{(i)} S_{je} P_e^{(i)} - \gamma_j^{-2} P_e S_{je} P_e + X_{je} S_{je}(P_e - P_e^{(i)}) + (P_e - P_e^{(i)})^T S_{je} X_{je} = 0,\ j = 1,\ \dots ,N$. Using the result established in (12), it is easy to verify that $\|P_e^{(i)} - P_e\| = O(\|\mu\|^{2^i})$. It follows from the above relation that $Z_{je}(A_e - S_e P_e^{(i)}) + (A_e - S_e P_e^{(i)})^T Z_{je} + O(\|\mu\|^{2^i}) = 0,\ j = 1,\ \dots ,N$. Since $D_{jj},\ j = 1,\ \dots ,N$ and D_0 are stable, $A_e - S_e P_e^{(i)}$ is also stable (Mukaidani et al., 2003). Hence, using (Zhou, 1998), $Z_{je} = O(\|\mu\|^{2^i}),\ j = 1,\ \dots ,N$, results in (21). □

6. CONCLUSION

In this paper, the high–order Pareto approximate strategy of the genaral MSPS has been studied. The new iterative algorithm that combined Newton's method with two fixed point algorithms has been proposed for solving the GMARE. As a result, the new iterative algorithm has achieved the quadratic convergence property and has succeeded in reducing the dimension of the algebraic manipulation. Moreover, it has been newly shown that the $O(\|\mu\|^{2^i})$ high–order approximate Pareto strategy achieved the cost functional $J_j^* + O(\|\mu\|^{2^i})$.

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